SYLLABUS

1 Measure theory

- *Outer Lebesgue measure on* $\mathbb{R}^d$, μ^*. Properties:

  1. $\mu^*(\emptyset) = 0$,
  2. $E \subset F$ implies $\mu^*(E) \leq \mu^*(F)$,
  3. $\mu^*(\cup_{i=1}^\infty E_i) \leq \sum_{i=1}^\infty \mu^*(E_i)$,
  4. if $\mathrm{dist}(E, F) > 0$ then $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$
  5. for pairwise almost disjoint elementary sets B_i, $\mu^*(\cup_{i=1}^\infty B_i) = \sum_{i=1}^\infty m(B_i)$.

  Null set. Any countable set is a null set. Cantor set is an uncountable null set.

  Proposition (outer regularity of μ^*): $\mu^*(E) = \inf_{O \supseteq E, \mathrm{open}} \mu^*(O)$.

- *Lebesgue measure.* Lebesgue measurable set. Properties:

  1. open/closed/null sets are Lebesgue measurable
  2. a complement of a Lebesgue measurable set is Lebesgue measurable
  3. a countable union of Lebesgue measurable sets is Lebesgue measurable
  4. every Jordan measurable set is Lebesgue measurable.

  Lebesgue measure μ. Properties:

  1. (countable additivity) if E_i are pairwise disjoint Lebesgue measurable sets, then $\mu(\cup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i)$
  2. if $E_1 \subseteq E_2 \subseteq \ldots$ are Lebesgue measurable, then $\mu(\cup_{i=1}^\infty E_i) = \lim_{n\to\infty} \mu(E_n)$
  3. if $E_1 \supseteq E_2 \supseteq \ldots$ are Lebesgue measurable and $\mu(E_1) < \infty$, then $\mu(\cap_{i=1}^\infty E_i) = \lim_{n\to\infty} \mu(E_n)$
  4. (translation invariance) if E is Lebesgue measurable then for any $x \in \mathbb{R}^d$, $x+E$ is Lebesgue measurable and $\mu(x + E) = \mu(E)$.

  Example of a non-measurable set.

- *Lebesgue measurable function.*

  Lemma: $f : \mathbb{R}^d \to \mathbb{R}$ is Lebesgue measurable if and only if the sets $\{x \in \mathbb{R}^d : f(x) < \lambda\}$ are Lebesgue measurable for all $\lambda \in \mathbb{R}$.

  Properties of Lebesgue measurable functions:

  1. any continuous function is Lebesgue measurable
  2. if f_n are Lebesgue measurable then $\limsup_{n\to\infty} f_n(x)$ is Lebesgue measurable
  3. if f is Lebesgue measurable and g is continuous then $g(f(x))$ is Lebesgue measurable

4. if f, g are Lebesgue measurable, then $f+g$, fg are Lebesgue measurable

Construction of Lebesgue integral on $\mathbb{R}^d$ (simple/non-negative/real/complex measurable functions).

- *Almost everywhere.* Properties:

  1. if $P(x)$ holds a.e. and $P(x)$ implies $Q(x)$ then $Q(x)$ holds a.e.
  2. if $P_i(x)$ hold a.e. for each i, then $P(x) = \{\text{all } P_i(x) \text{ hold simultaneously for } i = 1, 2, \ldots\}$ holds a.e.
  3. if $f = g$ a.e., then f is Lebesgue measurable if and only if g is Lebesgue measurable
  4. Lebesgue integral does not depend on values of function on a set of measure 0
  5. if $f : \mathbb{R}^d \to [0, \infty)$ is Lebesgue measurable, then $\int_{\mathbb{R}^d} f(x)dx = 0$ if and only if $f(x) = 0$ a.e.
  6. if $0 \le f(x) \le g(x)$ a.e., then $0 \le \int_{\mathbb{R}^d} f(x)dx \le \int_{\mathbb{R}^d} g(x)dx$.

- *σ-algebra.* Measurable space.

  Proposition: If $\mathcal{B}_\alpha$, $\alpha \in I$, are σ-algebras on X, then $\cap_{\alpha \in I} \mathcal{B}_\alpha$ is a σ-algebra on X.

  σ-algebra generated by a family $\mathcal{A}$ of subsets of X, $\sigma(\mathcal{A})$

  Borel σ-algebra, $\mathcal{B}(X)$. Borel measurable set.

  Coarser and finer σ-algebras.

  Proposition: The σ-algebra of Lebesgue measurable sets is generated by Borel measurable sets and null sets.

  Proposition: $\mathcal{B}(R) = \sigma((-\infty, a],\ a \in \mathbb{R}) = \sigma((a, b],\ a, b \in \mathbb{R}) = \sigma([a, b],\ a, b \in \mathbb{R})$.

  Product σ-algebra.

  Proposition: $\mathcal{B}(\mathbb{R}^{m+n}) = \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n)$.

- *Measure.* Definition of measure. Measure space. Examples.

  Properties: if $(X, \mathcal{B}, \mu)$ is a measure space and $A_i \in \mathcal{B}$, then

  1. $\mu(\cup_{i=1}^\infty A_i) \le \sum_{i=1}^\infty \mu(A_i)$
  2. if $A_1 \subseteq A_2 \subseteq \ldots$, then $\mu(\cup_{i=1}^\infty A_i) = \lim_{n\to\infty} \mu(A_n)$
  3. if $A_1 \supseteq A_2 \supseteq \ldots$ and $\mu(A_1) < \infty$, then $\mu(\cap_{i=1}^\infty A_i) = \lim_{n\to\infty} \mu(A_n)$

  Carathéodory extension theorem: For any set X, an algebra $\mathcal{A}$ on X and a premeasure μ_0 on $\mathcal{A}$, there exists an extension of μ_0 to a measure on $\sigma(\mathcal{A})$. Furthermore, if μ_0 is sigma-finite, then the extension is unique.

  Lebesgue-Stiltjes measure. Examples.

  Product measure.

- *Measurable function.* Definition.

  Proposition: If $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ and $\mathcal{B}' = \sigma(\mathcal{A}')$, then f is measurable if and only if $f^{-1}(A') \in \mathcal{B}$ for all $A' \in \mathcal{A}'$.

  Borel function.

  Properties of measurable functions:

  1. f is a Borel function on $(X, \mathcal{B})$ if and only if $\{x \in X : f(x) > \lambda\} \in \mathcal{B}$ for all $\lambda \in \mathbb{R}$
  2. $\chi_E(x)$ is Borel if and only if $E \in \mathcal{B}$
  3. $f : X \to \mathbb{R}$ is measurable if and only if f_+ and f_- are measurable
  4. $f, g : X \to \mathbb{R}$ are measurable, then $f + g, fg$ are measurable.

- *Lebesgue integral.* Construction of Lebesgue integral on a (complete) measure space $(X, \mathcal{B}, \mu)$.

  Properties:

  1. if $f = g$ μ-a.e., then $\int_X f d\mu = \int_X g d\mu$
  2. if $f \leq g$ μ-a.e., then $\int_X f d\mu \leq \int_X g d\mu$
  3. for $f \geq 0$ μ-a.e., $\int_X f d\mu = 0$ if and only if $f = 0$ μ-a.e.
  4. (linearity) $\int_X (af + bg) d\mu = a \int_X f d\mu + b \int_X g d\mu$.

  Monotone convergence theorem: If $0 \leq f_1 \leq f_2 \leq \ldots$ are measurable functions on $(X, \mathcal{B}, \mu)$, then

  $$\lim_{n\to\infty} \int_X f_n d\mu = \int_X (\lim_{n\to\infty} f_n) d\mu.$$

  Fatou's lemma: If $f_1, f_2, \ldots : X \to [0, \infty)$ are measurable functions on $(X, \mathcal{B}, \mu)$, then

  $$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu.$$

  Dominated convergence theorem: If $f_1, f_2, \ldots$ is a sequence of measurable functions on $(X, \mathcal{B}, \mu)$ converging to f and such that for some measurable function $F : X \to [0, \infty)$ (1) $|f_n| \leq F$ μ-a.e. and (2) $\int_X F d\mu < \infty$, then

  $$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

  Fubini theorem: Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be measure spaces and $f : X \times Y \to \mathbb{R}$ a measurable function with respect to $\mathcal{B}_X \otimes \mathcal{B}_Y$. If $\int_{X\times Y} |f(x,y)| d\mu_X \otimes \mu_Y < \infty$, then

  $$\int_{X\times Y} f(x,y) d\mu_X \otimes \mu_Y = \int_X \left(\int_Y f(x,y) d\mu_Y \right) d\mu_X = \int_Y \left(\int_X f(x,y) d\mu_X \right) d\mu_Y.$$

- L^p *spaces.* Space $\mathcal{L}^p(X,\mu)$. Seminorm $\|\cdot\|_p$ on $\mathcal{L}^p(X,\mu)$.

  Normed space $L^p(X,\mu)$.

  Proposition: $L^p(X,\mu)$ is a Banach space.

  Theorem: Let X be a metric space, μ a σ-finite measure on $(X,\mathcal{B}(X))$ such that for every Borel set A and every $\varepsilon > 0$, there exist closed C and open O such that $C \subseteq A \subseteq O$ and $\mu(O \setminus C) < \varepsilon$. Then continuous functions on X are dense in $L^p(X,\mu)$, i.e., for any $\varepsilon > 0$ and $f \in L^p$ there exists a continuous g such that $\|f-g\|_p < \varepsilon$.

2 Functional analysis

- Review of metric, normed, inner product spaces

- *Hilbert space.* Orthogonality. Orthogonal system, orthonormal system, orthonormal basis.

  Theorem: Any Hilbert space has an orthonormal basis.

  Theorem: Let X be a Hilbert space, $\{x_\alpha\}_{\alpha\in I}$ an orthonormal basis. Then for $x \in X$,

  1. only countably many $\alpha \in I$ satisfy $\langle x, x_\alpha\rangle \neq 0$,
  2. $x = \sum_{\alpha\in I}\langle x, x_\alpha\rangle\, x_\alpha$
  3. $\|x\|^2 = \sum_{\alpha\in I}|\langle x, x_\alpha\rangle|^2$ (Parseval's identity).

  Lemma (Bessel's inequality): Let X be an inner product space and $\{x_1,\dots,x_N\}$ an orthonormal system, then for each $x \in X$, $\|x\|^2 \geq \sum_{i=1}^N |\langle x, x_i\rangle|^2$.

  Theorem: A Hilbert space is separable if and only if it has a *countable* orthonormal basis.

  Lemma (Gram-Schmidt orthogonalization): Let $y_1,\dots$ be a linearly independent system. Then there exists an *orthonormal* system $x_1,\dots$ such that for each $k \geq 1$,

  $$\operatorname{span}(y_1,\dots,y_k) = \operatorname{span}(x_1,\dots,x_k).$$

  Isomorphism of inner product spaces.

  Theorem: Let X be a separable Hilbert space. Then

  1. if $\dim(X) = N < \infty$, then X is isomorphic to $\mathbb{C}^N$ (or $\mathbb{R}^N$)
  2. if $\dim(X) = \infty$, then X is isomorphic to ℓ_2.

  Orthogonal complement $Y^\perp$.

  Theorem (projection theorem): Let X be a Hilbert space and Y a *closed* subspace. Then for each $x \in X$ there exist unique $y \in Y$ and $y' \in Y^\perp$ such that $x = y + y'$.

  Lemma: Let X be a Hilbert space, Y a *closed* subspace and $x \in X$. Then there exists a unique $y_0 \in Y$ *closest* to x.

Direct sum of Hilbert spaces, $X \oplus Y$.

Theorem (Fourier series): The functions $\{\frac{1}{\sqrt{2\pi}} e^{inx}, n \in \mathbb{Z}\}$ form an orthonormal basis of $L^2[0, 2\pi]$ and for any $f \in L^2[0, 2\pi]$, $f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \frac{1}{\sqrt{2\pi}} e^{inx}$ in $L^2[0, 2\pi]$, where $c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)e^{-inx}dx$.

- *Continous linear functionals.*

  Proposition: Let X be a normed space. $f : X \to \mathbb{C}$ is a continous linear functional if and only if f is continous at 0.

  Proposition: If $\dim(X) < \infty$ then any linear functional on X is continous.

  Dual space X^*.

  Bounded linear functional.

  Proposition: Let X be a normed space. $f : X \to \mathbb{C}$ is a continous linear functional if and only if f is bounded.

  The norm of f, $\|f\|$.

  Proposition: $(X^*, \|\cdot\|)$ is a Banach space.

  Examples of bounded linear functionals. Examples of X^*.

  Proposition: $\ell_p^* = \ell_q$, $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

  $\ell_\infty^* \neq \ell_1$.

  Theorem (Riesz lemma): Let X be a Hilbert space. For any $f \in X^*$ there exists a unique $y_f \in X$ such that $f(x) = \langle x, y_f\rangle$ for all $x \in X$ and $\|f\|_{X^*} = \|y_f\|_X$.

  Theorem (Hahn-Banach): Let X be a normed space, Y a subspace of X and $f_0 \in Y^*$. Then there exists $f \in X^*$ such that (1) $f(y) = f_0(y)$ for $y \in Y$ and (2) $\|f\|_{X^*} = \|f_0\|_{Y^*}$.

  Corollaries:

  1. For any $x_0 \neq 0$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$.
     In particular, if $f(x) = 0$ for all $x \in X^*$, then $x = 0$.
  2. Let L be a subspace of X, $x_0 \in X$ such that the distance from x_0 to L is $d > 0$. Then there exists $f \in X^*$ such that $f(x_0) = 1$, $f(x) = 0$ for all $x \in L$ and $\|f\| = \frac{1}{d}$.
     In particular, if L is a vector subspace of X, then $\overline{L} = X$ if and only if the only $f \in X^*$ such that $f(x) = 0$ for all $x \in L$ is the zero functional.

  Theorem: Let X be a Banach space. If X^* is separable, then X is separable.

  Reflexive spaces. Examples.

- *Continuous operator.* Bounded operator.

  Proposition: Let X, Y be normed spaces and $A : X \to Y$ a linear operator.

  1. A is continuous if and only if A is continuous at $0 \in X$.

2. A is continuous if and only if A is bounded.
3. If $\dim(X) < \infty$ then A is continuous.

Operator norm.

Theorem: If Y is a Banach space, then $(\mathcal{L}(X,Y), \|\cdot\|)$ is also Banach.

Examples: integral operators, convolution operators, inclusion, projection.

Composition of operators. Series of operators. Operator exponent e^A.

Inverse operator.

Proposition: If $A : X \to Y$ is linear invertible, then A^{-1} is also linear.

Theorem: Let X, Y be normed spaces and $A \in \mathcal{L}(X,Y)$. If $\mathrm{Im}(A) = Y$ and there exists $m > 0$ such that for all $x \in X$, $\|Ax\|_Y \geq m\|x\|_X$, then A is invertible and $A^{-1} \in \mathcal{L}(Y,X)$.

Theorem (Neumann): Let X be Banach, $A \in \mathcal{L}(X)$. If $\sum_{n=0}^{\infty} A^n$ converges, then there exists $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n \in \mathcal{L}(X)$. In particular, if $\|A\| < 1$, then there exists $(I-A)^{-1}$ and $\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$.

Theorem (Banach): Let X, Y be *Banach* spaces, $A \in \mathcal{L}(X,Y)$. If A is bijective, then $A^{-1} \in \mathcal{L}(Y,X)$.

Corollary: Let $(X, \|\cdot\|_0)$ and $(X, \|\cdot\|_1)$ be *Banach* spaces such that for some $M < \infty$ and all $x \in X$, $\|x\|_1 \leq M\|x\|_0$. Then the two norms are equivalent.

Proposition: Let X be a Banach space. Then $G = \{A \in \mathcal{L}(X) : A^{-1} \in \mathcal{L}(X)\}$ is open in $\mathcal{L}(X)$.

- *Spectrum.*

  Resolvent set $\rho(A)$. Spectrum $\sigma(A)$.

  Proposition: Let X be a Banach space and $A \in \mathcal{L}(X)$. Then

  1. $\rho(A)$ is open
  2. $\rho(A) \supseteq \{\lambda \in \mathbb{C} : |\lambda| > \|A\|\}$.

  Proposition: Let X be a Banach space and $A \in \mathcal{L}(X)$. Then

  1. $\sigma(A) \neq \emptyset$
  2. $\sigma(A)$ is closed
  3. $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$
  4. if $\lambda \in \sigma(A)$ then $\lambda^n \in \sigma(A^n)$.

  Point spectrum $\sigma_p(A)$, residual spectrum $\sigma_r(A)$, continuous spectrum $\sigma_c(A)$.

  Spectral radius $r_\sigma(A)$.

  Theorem: Let X be a Banach space and $A \in \mathcal{L}(X)$. Then $r_\sigma(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$.

- *Adjoint operators.*

  Theorem: Let X be a Hilbert space and $A \in \mathcal{L}(X)$. Then there exists a unique A^* and $\|A^*\| = \|A\|$.

  Properties of A^*.

  Proposition: $\lambda \in \sigma(A)$ if and only if $\overline{\lambda} \in \sigma(A^*)$.

  Theorem: Let X be a Hilbert space and $A \in \mathcal{L}(X)$. Then

  1. if $\lambda \in \sigma_r(A)$ then $\overline{\lambda} \in \sigma_p(A^*)$
  2. if $\lambda \in \sigma_p(A)$ then $\overline{\lambda} \in \sigma_p(A^*) \cup \sigma_r(A^*)$
  3. $\lambda \in \sigma_c(A)$ if and only if $\overline{\lambda} \in \sigma_c(A^*)$.

  Spectrum of a shift operator in ℓ_2.

  Self-adjoint operator. Unitary operator.

  Proposition: If $A^* = A$, then

  1. $\langle Ax, x\rangle \in \mathbb{R}$
  2. $\sigma(A) \subset \mathbb{R}$
  3. $\sigma_r(A) = \emptyset$
  4. $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$
  5. $r_\sigma(A) = \|A\|$.

  Proposition: If A is unitary, then

  1. $\|A\| = 1$
  2. $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

  Theorem (Hellinger-Toeplitz): Let X be a Hilbert space and $A : X \to X$ linear operator such that $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in X$. Then $A \in \mathcal{L}(X)$.

- *Compact operator.*

  Proposition: Let X, Y be normed spaces, $A : X \to Y$ a linear operator.

  1. If A is compact, then A is bounded.
  2. If A is bounded and $\operatorname{rank}(A) < \infty$, then A is compact.
  3. If A, B are compact, then $A + B$ is compact.
  4. If A is compact and B bounded, then AB and BA are compact.
  5. $\operatorname{id} : X \to X$ is compact if and only if $\dim(X) < \infty$.
  6. If $\dim(X) = \infty$ and A is compact, then A does not have bounded inverse.
  7. If A_n are compact and $A_n \to A$, then A is compact.

  Theorem (Riesz-Schauder): Let X be a Hilbert space and A a compact operator. Then

1. $\sigma(A)$ is at most countable, its set of accumulation points is contained in $\{0\}$.
2. if $\lambda \in \sigma(A) \setminus \{0\}$, then $\lambda \in \sigma_p(A)$ and $\dim\ker(\lambda I - A) < \infty$.

Theorem (Hilbert-Schmidt): Let X be a *separable* Hilbert space, A a compact *self-adjoint* operator. Then there exists an orthonormal basis of X, $e_1, e_2, \ldots$ and $\lambda_i \in \mathbb{R}$ such that $Ae_i = \lambda e_i$.

3 Differential geometry

- *Surfaces in* $\mathbb{R}^3$. Local coordinates. Tangent plane.

  First fundamental form. Second fundamental form.

  Normal curvature in direction v.

  Lemma: If A, B are symmetric bilinear forms on a vector space V and A is positive definite, then there exists a basis of V in which the matrix of A is the identity matrix and the matrix of B is diagonal.

  Principal curvatures k_1, k_2. Principal directions e_1, e_2.

  Remark: e_1, e_2 is an orthonormal basis of the tangent plane and k_i is a normal curvature in direction e_i.

  Theorem (Euler): For any tangent vector v, $\frac{\mathrm{II}(v,v)}{\mathrm{I}(v,v)} = k_1 \cos^2\varphi + k_2 \sin^2\varphi$, where φ is an angle between v and e_1.

  Corollary: k_1, k_2 are *extremal* curvatures.

  Gaussian curvature K. Mean curvature H.

  Gauss equations. Christoffel symbols (of the second kind).

  Einstein summation convention.

  Theorem:
  $$\Gamma^k_{ij} = \frac{1}{2} g^{lk} \left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right).$$

  That is, the Christoffel symbols are determined by the first fundamental form.

  Theorem (Gauss):
  $$K = \frac{1}{g_{11}g_{22} - g_{12}^2} \left((\Gamma^k_{12}\Gamma^l_{12} - \Gamma^k_{11}\Gamma^l_{22}) g_{kl} + \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u^2 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u^1 \partial u^1} \right).$$

  That is, the Gaussian curvature is expressed only through the coefficients of the first fundamental form and their derivatives.

  Covariant derivative of a vector field v along a vector field w, $\nabla_w v$.

  Proposition:

  1. $(v, w) \mapsto \nabla_w v$ is bilinear

2. for any smooth $f : U \to \mathbb{R}$,

$$\nabla_{fw} v = f \nabla_w v \qquad \nabla_w (fv) = D_w f\, v + f \nabla_w v,$$

where $D_w f = \frac{\partial f}{\partial u^j} w^j$ is the derivative of f in direction w

3. $\nabla_{r_j} r_i = \Gamma^k_{ij}\, r_k$
4. $\Gamma^k_{ij} = \Gamma^k_{ji}$
5. for any smooth vector fields a, b, w,

$$D_w \langle a, b \rangle = \langle \nabla_w a, b \rangle + \langle a, \nabla_w b \rangle.$$

Remarks:

1. Any bilinear function that satisfies 2. above is an affine connection.
2. Christoffel symbols determine $\nabla_w v$
3. 4. states that $\nabla_w v$ is symmetric
4. 5. states that $\nabla_w v$ is compatible with the metric.

- *Topological spaces.* Definition.

  Base of topology. Neighborhood. Continuous function.

  Hausdorff space. Second countable space.

  Homeomorphism of topological spaces.

- *Manifolds.* Definition.

  Local chart (U, φ). Atlas $\{(U_\alpha, \varphi_\alpha),\, \alpha \in I\}$. Local coordinates $(x^1_\alpha, \dots x^n_\alpha)$. Transition mappings.

  Lie group.

  Smooth structure on a manifold. Smooth manifold. Smooth mapping $f : M \to N$ (M, N smooth manifolds). Smooth function, smooth path.

  Equivalence of atlases.

  Theorem (level set theorem): Let $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ a smooth mapping, $y \in \mathbb{R}^m$. If for each $x \in M = f^{-1}(y)$, $\operatorname{rank}(\frac{\partial f^j}{\partial x^i}) = m$, then M is a manifold.

  Example: $SL(n, \mathbb{R})$.

- *Tangent space.*

  Tangent vector (as an equivalence class of curves, in local coordinates and as a derivation). Tangent space $T_p M$.

  The basis $\frac{\partial}{\partial x^1_\alpha}, \dots, \frac{\partial}{\partial x^n_\alpha}$.

  Tangent bundle TM.

  Theorem: Let M be a smooth manifold. There exists a smooth structure on TM such that

1. the projection $\pi : TM \to M$, $\pi(p, v) = p$ is a smooth mapping
2. for each $p \in M$, there exists a neighborhood U and a diffeomorphism $f : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that $\pi(f^{-1}(p, v)) = p$.

Examples: $T\mathbb{R}$, TS^1.

Covector. Tensor. Metric tensor. Riemannian manifold.

- *Affine connection.* Vector field. Covariant derivative. Christoffel symbols.

  Torsion. Symmetric connection.

  Affine connection for tensors.

  Parallel transport of a tangent vector along a smooth curve.

  Lemma: Let g be a (pseudo)metric on a smooth manifold M. The following claims are equivalent:

  1. for any vector field w, $\nabla_w g_{ij} = 0$
  2. for any smooth curve γ and v, w vector fields parallel along γ, $\frac{d}{dt}\langle v(t), w(t)\rangle = 0$
  3. for any smooth curve γ and v, w vector fields on γ,

  $$\frac{d}{dt}\langle v(t), w(t)\rangle = \langle \nabla_{\dot\gamma}\, v(t), w(t)\rangle + \langle v(t), \nabla_{\dot\gamma}\, w(t)\rangle.$$

  Connection compartible with metric. Levi-Civita connection.

  Theorem (Levi-Civita): For any Riemannian manifold, there exists a unique symmetric connection compartible with the metric. Furthermore,

  $$\Gamma^k_{ij} = \frac{1}{2}\, g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

- *Curvature.* Non-commutativity of parallel transport. Curvature tensor (in local coordinates, using notation of affine connection).

  Proposition:

  1. $R(u, v)w = -R(v, u)w$
  2. for *symmetric* connection,

  $$R(u, v)w + R(v, w)u + R(w, u)v = 0$$

  3. for connection *compartible with metric*, $\langle R(u, v)w, z\rangle = -\langle w, R(u, v)z\rangle$
  4. for *Levi-Civita* connection, $\langle R(u, v)w, z\rangle = \langle R(w, z)u, v\rangle$.

  Sectional curvature.

  Ricci tensor.