**RETAKE SOLUTIONS, 9 October, 17:00 – 19:00**

1. Consider the function $f$ on the interval $[0,1]$ defined by

$$f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ \sin x & x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Prove that $f$ is Lebesgue measurable and compute the Lebesgue integral $\int_0^1 f(x)dx$.

*Answer:* $1 - \cos 1$.

*Solution.* Consider the function $g(x) = \sin x$. $g$ is continuous, thus Lebesgue measurable. Since the set of rational numbers $\mathbb{Q}$ has zero Lebesgue measure, $f = g$ almost everywhere. In particular, $f$ is also Lebesgue measurable and $\int_0^1 f(x)dx = \int_0^1 g(x)dx = \int_0^1 \sin x dx = 1 - \cos 1$. □

2. Consider the $\sigma$-algebra $\mathcal{A}$ generated by all singletons on the real line, i.e., $\mathcal{A} = \sigma(\{x\},\, x \in \mathbb{R})$. Which of the following statements (a)-(c) is true? Justify your answer.

(a) $\mathcal{A} \subsetneq \mathcal{B}(\mathbb{R})$ (b) $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{A}$ (c) $\mathcal{A} = \mathcal{B}(\mathbb{R})$

*Answer:* (a).

*Solution.* First note that for any $x \in \mathbb{R}$, $\{x\} \in \mathcal{B}(\mathbb{R})$. Indeed, $\{x\}$ is the countable intersection of open sets $(x - \frac{1}{n}, x + \frac{1}{n})$. Thus, $\mathcal{A} \subseteq \mathcal{B}(\mathbb{R})$. In particular, (b) is wrong.

We will show that $\mathcal{A} \neq \mathcal{B}(\mathbb{R})$, which implies that (c) is also wrong. Since $\mathcal{A}$ contains any real number, it also contains any countable union of real numbers. Let $\mathcal{A}' = \{B \subseteq \mathbb{R} \,:\, B \text{ or } B^c \text{ is a countable union of real numbers}\}$. Note that $\mathcal{A}'$ is a $\sigma$-algebra. Indeed, (a) it contains $\emptyset$, (b) if $A \in \mathcal{A}'$ then $A^c \in \mathcal{A}'$ by definition of $\mathcal{A}'$, (c) if $A_i \in \mathcal{A}'$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{A}'$. Thus, $\mathcal{A} \subseteq \mathcal{A}'$. (In fact, $\mathcal{A} = \mathcal{A}'$.) Now, any open interval, for instance $(0,1)$, is not in $\mathcal{A}'$, thus $\mathcal{A} \neq \mathcal{B}(\mathbb{R})$.

We conclude that $\mathcal{A} \subsetneq \mathcal{B}(\mathbb{R})$, i.e., (a) is correct. □

3. (5 points) Let $f$ be a linear functional on the space of continuous functions $C[0,2]$ with supremum norm, defined by $f(x) = \int_0^1 x(t)dt - \int_1^2 x(t)dt$. Find the norm of $f$.

*Answer:* 2.

*Solution.* For any $x \in C[0,2]$,

$$|f(x)| = \left| \int_0^1 x(t)dt - \int_1^2 x(t)dt \right| \le \int_0^2 |x(t)|dt \le 2 \sup_{t \in [0,2]} |x(t)| = 2\|x\|.$$

Thus, $\|f\| \le 2$.

Consider continuous functions $x_n$ defined by

$$x_n(t) = \begin{cases} 1 & 0 \le t \le 1 - \frac{1}{n} \\ n(1-t) & 1 - \frac{1}{n} \le t \le 1 + \frac{1}{n} \\ -1 & 1 + \frac{1}{n} \le t \le 2. \end{cases}$$

Note that $\|x_n\| = 1$ and $f(x_n) = 2 - \frac{1}{n}$. Thus, for all $n$, $\|f\| \ge 2 - \frac{1}{n}$, and we conclude that $\|f\| = 2$. $\square$

4. (5 points) Let $f$ be a real-valued bounded measurable function on $[0,1]$ and $A$ a linear operator on $L^2[0,1]$ defined by $Ax(t) = f(t)\,x(t)$. Find the point spectrum $\sigma_p(A)$.

   *Answer:* All $\lambda \in \mathbb{R}$ such that $\mu\{t : f(t) = \lambda\} > 0$.

   *Solution.* Point spectrum of $A$ consists of those $\lambda \in \mathbb{C}$ for which there exists $x \in L^2[0,1]$ such that $x \ne 0$ almost everywhere and $Ax = \lambda x$ almost everywhere. By the definition of $A$, the last equality states that $(f(t) - \lambda)x(t) = 0$ for almost every $t$. We distinguish two cases:

   (a) If $f(t) \ne \lambda$ almost everywhere, i.e., $\mu\{t : f(t) = \lambda\} = 0$, then $x(t) = 0$ almost everywhere, in which case $\lambda \notin \sigma_p(A)$.

   (b) If $\mu\{t : f(t) = \lambda\} > 0$, let $x(t)$ be the characteristic function of the set $\{t : f(t) = \lambda\}$. Then, $x$ is measurable, $x \ne 0$ almost everywhere, and $(f(t) - \lambda)x(t) = 0$ almost everywhere. Thus, $\lambda \in \sigma_p(A)$. (Since $f$ is real-valued, any such $\lambda$ is real.) $\square$

5. (5 points) Let $A$ be a self-adjoint operator on a Hilbert space $X$. Prove that $\ker(A^2) = \ker(A)$. (Here ker is the kernel of operator.)

   *Solution.* Let $x \in \ker(A)$, then $Ax = 0$. By linearity of $A$, $A^2x = A(Ax) = A(0) = 0$. Thus, $x \in \ker(A^2)$. (Note, here we only use that $A$ is linear.)

   Let $x \in \ker(A^2)$, then $A^2x = 0$ and $\langle A^2x, x\rangle = 0$. Since $A$ is self-adjoint, $0 = \langle A^2x, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2$. Thus, $Ax = 0$ and $x \in \ker(A)$. $\square$

6. Compute the first and the second fundamental forms of the surface $r(u,v)$ in $\mathbb{R}^3$ defined by $r(u,v) = (u, v, uv)$, $(u,v) \in \mathbb{R}^2$.

   *Answer:* $E = 1 + v^2$, $F = uv$, $G = 1 + u^2$, $L = 0$, $M = \frac{1}{\sqrt{1+u^2+v^2}}$, $N = 0$.

   *Solution.* First compute $r_u = (1, 0, v)$, $r_v = (0, 1, u)$. The coefficients of the first fundamental form are $E = r_u \cdot r_u = 1 + v^2$, $F = r_u \cdot r_v = uv$, $G = r_v \cdot r_v = 1 + u^2$.

   To compute the second fundamental form, we need $r_{uu} = (0,0,0)$, $r_{uv} = (0,0,1)$, $r_{vv} = (0,0,0)$, $r_u \times r_v = (-v, -u, 1)$, $n = \frac{r_u \times r_v}{\|r_u \times r_v\|} = (\frac{-v}{\sqrt{1+u^2+v^2}}, \frac{-u}{\sqrt{1+u^2+v^2}}, \frac{1}{\sqrt{1+u^2+v^2}})$. The coefficients of the second fundamental form are $L = r_{uu} \cdot n = 0$, $M = r_{uv} \cdot n = \frac{1}{\sqrt{1+u^2+v^2}}$, $N = r_{vv} \cdot n = 0$.

   [Gaussian curvature of the surface is $K = \frac{LN - M^2}{EG - F^2} = \frac{-1}{\sqrt{1+u^2+v^2}}$ ($< 0$ everywhere).] $\square$