EXAM QUESTIONS

1. Define the outer Lebesgue measure on $\mathbb{R}^d$.
2. Prove that the outer Lebesgue measure μ^* satisfies:
   For all $E_i \subseteq \mathbb{R}^d$, $\mu^*(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.
3. Prove that for any elementary set E in $\mathbb{R}^d$ (finite union of boxes), $\mu^*(E) = m(E)$, where $m(E)$ is the volume of E.
4. Define Lebesgue measurable set and Lebesgue measure.
5. Prove that for any pairwise disjoint Lebesgue measurable sets E_i,
   $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.
6. Prove that for any Lebesgue measurable functions f, g, $f+g$ is Lebesgue measurable.
7. Define σ-algebra and measurable space.
8. Define σ-algebra generated by a family $\mathcal{A}$ of subsets of X, $\sigma(\mathcal{A})$. Explain why $\sigma(\mathcal{A})$ exists.
9. Define Borel σ-algebra, $\mathcal{B}(X)$.
10. Prove that the σ-algebra of Lebesgue measurable sets is generated by Borel measurable sets and null sets.
11. Define product σ-algebra.
12. Define a measure on a measurable space.
13. Prove the following: If $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ and $\mathcal{B}' = \sigma(\mathcal{A}')$, then f is measurable if and only if $f^{-1}(A') \in \mathcal{B}$ for all $A' \in \mathcal{A}'$.
14. Describe construction of the Lebesgue integral of a real measurable function on abstract measure space.
15. Prove the monotone convergence theorem.
16. Prove Fatou's lemma.
17. State the dominated convergence theorem.
18. State the Fubini theorem.
19. Define the space $L^p(X, \mu)$.
20. State Parseval's identity.
21. State Bessel's inequality.
22. State the Gram-Schmidt orthogonalization theorem.

23. Prove the following: Let X be a separable Hilbert space. Then
    (a) if $\dim(X) = N < \infty$, then X is isomorphic to $\mathbb{C}^N$ (or $\mathbb{R}^N$)
    (b) if $\dim(X) = \infty$, then X is isomorphic to ℓ_2.
24. State the projection theorem.
25. State the theorem about the expansion of a function in Fourier series in $L^2[0, 2\pi]$.
26. Prove that in a normed space X, $f : X \to \mathbb{C}$ is a continous linear functional if and only if f is a linear functional continous at 0.
27. Prove that if X is a finite dimensional normed space, then any linear functional on X is continous.
28. Prove that in a normed space X, $f : X \to \mathbb{C}$ is a continous linear functional if and only if f is a bounded linear functional.
29. Define the norm of a linear functional on a normed space.
30. Prove that $(X^*, \|\cdot\|)$ is a Banach space.
31. Prove that $\ell_p^* = \ell_q$, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.
32. State the Riesz lemma.
33. State the Hahn-Banach theorem.
34. Define reflexive space.
35. Prove the following: Let X, Y be normed spaces and $A : X \to Y$ a linear operator.
    (a) A is continuous if and only if A is continuous at $0 \in X$.
    (b) A is continuous if and only if A is bounded.
    (c) If $\dim(X) < \infty$ then A is continuous.
36. Prove Neumann's theorem: Let X be Banach, $A \in \mathcal{L}(X)$ such that $\|A\| < 1$. Then there exists $(I - A)^{-1}$ and $\|(I - A)^{-1}\| \leq \frac{1}{1-\|A\|}$.
37. State the Banach theorem about the inverse operator.
38. Prove the following: Let $(X, \|\cdot\|_0)$ and $(X, \|\cdot\|_1)$ be Banach spaces such that for some $M < \infty$ and all $x \in X$, $\|x\|_1 \leq M\|x\|_0$. Then the two norms are equivalent.
39. Define the resolvent set and the spectrum of an operator.
40. Prove the following: if X is a Banach space and $A \in \mathcal{L}(X)$, then
    (a) $\sigma(A)$ is closed
    (b) $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$.

41. Define point spectrum, residual spectrum and continuous spectrum of an operator.
42. Define the spectral radius of an operator.
43. Define the adjoint operator.
44. Prove the following: If X is a Hilbert space and $A \in \mathcal{L}(X)$, then there exists a unique A^* and $\|A^*\| = \|A\|$.
45. Define self-adjoint operator.
46. Define unitary operator.
47. Prove the following properties of compact operator: Let X, Y be normed spaces, $A : X \to Y$ a linear operator.
    (a) If A is compact, then A is bounded.
    (b) If A is bounded and finite rank, then A is compact.
    (c) If A_n are compact and $A_n \to A$, then A is compact.
48. Define the first and second fundamental forms of a surface in $\mathbb{R}^3$.
49. Define the normal curvature of a surface in direction v.
50. Define principal curvatures k_1, k_2 and principal directions e_1, e_2.
51. Define the Gaussian and mean curvatures of a surface in $\mathbb{R}^3$.
52. Define Christoffel symbols (of the second kind) of a surface in $\mathbb{R}^3$.
53. Define covariant derivative of a vector field v along vector field w on surface in $\mathbb{R}^3$.
54. Define topological space.
55. Define a continuous mapping from one topological space to another.
56. Define Hausdorff second countable space.
57. Define homeomorphism of topological spaces.
58. Define a manifold.
59. Define local charts and an atlas for a manifold.
60. Define smooth manifold.
61. Define smooth mapping $f : M \to N$ (M, N smooth manifolds).
62. Define a Lie group.
63. Define tangent space to a manifold.
64. Define Riemannian manifold.

65. Define affine connection on a smooth manifold.
66. Define parallel transport of a tangent vector along a smooth curve and write the equations of parralel transport in local coordinates.
67. Define Levi-Civita connection.
68. Define Riemann curvature tensor.
69. Define Ricci tensor.