EXAM SOLUTIONS, 18 July, 10:00 – 12:00

1. Let A be a *not* Lebesgue measurable set in $\mathbb{R}$. Define the function

$$f(x) = \begin{cases} x^2 & x \in A \\ 0 & x \notin A. \end{cases}$$

Prove that f is not measurable.

Solution. Assume that f is measurable. Then the set $B = \{x : f(x) > 0\}$ is measurable. Note that $f(x) > 0$ if and only if $x \in A$ and $x \neq 0$, that is $B = A \setminus \{0\}$.

Consider two cases. If $0 \notin A$, then $A = B$ is measurable, a contradiction with non-measurability of A. If $0 \in A$, then $A = B \cup \{0\}$ is measurable as a union of two measurable sets (B and $\{0\}$), again a contradiction. Thus, f is non-measurable. $\square$

2. Find the σ-algebra on $\mathbb{R}$ generated by the two sets $\{\{1,2\}, \{2,3\}\}$.

Answer: All subsets of $\{1,2,3\}$ and their complements in $\mathbb{R}$.

Solution. Let $A = \{1,2\}$, $B = \{2,3\}$, and denote the σ-algebra generated by A and B by $\mathcal{A}$. From the axioms of σ-algebra it follows that $A \setminus B = \{1\}$, $B \setminus A = \{3\}$ and $A \cap B = \{2\}$ are in $\mathcal{A}$. Therefore, any subset of $\{1,2,3\}$ (all possible unions of these three singleton sets) is in $\mathcal{A}$. Since $\mathcal{A}$ is closed under complements, it contains complements of all subsets of $\{1,2,3\}$.

Denote the set of all subsets of $\{1,2,3\}$ by $\mathcal{A}_1$ and the set $\{\mathbb{R} \setminus C : C \in \mathcal{A}_1\}$ of their complements by $\mathcal{A}_2$. Then $\mathcal{A}_1 \cup \mathcal{A}_2$ is a σ-algebra. (Indeed, the union of any elements from $\mathcal{A}_i$ is in $\mathcal{A}_i$, and the union of $C_1 \in \mathcal{A}_1$ and $C_2 \in \mathcal{A}_2$ is in $\mathcal{A}_2$, the complement of any $C \in \mathcal{A}_i$ is in $\mathcal{A}_{3-i}$.) Furthermore, by the above considerations, it is the smallest σ-algebra that contains A and B. $\square$

3. Find the norm of the linear operator A on the space $C[0,1]$ (with supremum norm), defined by

$$Ax(t) = x(0) - x(t).$$

Answer: 2.

Solution. Let $x \in C[0,1]$. Denote the norm in $C[0,1]$ by $\|\cdot\|$, that is $\|x\| = \sup_{t\in[0,1]} |x(t)|$.

First note that A is indeed a linear operator on $C[0,1]$: If $x(t)$ is a continuous function on $[0,1]$, then $x(0) - x(t)$ is also continuous on $[0,1]$, and for all $t \in [0,1]$, $A(ax+by)(t) = (ax+by)(0) - (ax+by)(t) = ax(0) + by(0) - ax(t) - by(t) = aAx(t) + bAy(t) = (aAx + bAy)(t)$.

To find the norm of A, we estimate

$$\|Ax\| = \sup_{t\in[0,1]} |x(0) - x(t)| \leq |x(0)| + \sup_{t\in[0,1]} |x(t)| \leq 2\|x\|.$$

Thus, A is a bounded operator and $\|A\| \leq 2$.

Finally, consider the continuous function $x(t) = 1 - 2t$. Note that $\|x\| = 1$ and $\|Ax\| = \sup_{t\in[0,1]} |1 - (1 - 2t)| = 2 = 2\|x\|$. Thus, $\|A\| = 2$. □

4. Let f be a real-valued bounded measurable function on $[0, 1]$ and A a linear operator on $L^2[0, 1]$ defined by
$$Ax(t) = f(t)\,x(t).$$
Find $\sigma_r(A)$.

Answer: $\emptyset$.

Solution. Note that $L^2[0, 1]$ is a Hilbert space with inner product $\langle x, y\rangle = \int_{[0,1]} x(t)\overline{y(t)}\,dt$. Furthermore, for any $x, y \in L^2[0, 1]$, $\langle Ax, y\rangle = \langle x, Ay\rangle$. That is, A is self-adjoint. It follows that $\sigma_r(A) = \emptyset$. □

5. Let X be a Banach space and $A \in \mathcal{L}(X)$. Can the resolvent of A be compact? Justify your answer.

Answer: Yes if and only if $\dim(X) < \infty$.

Solution. Let $\lambda \in \rho(A)$ and consider $R_\lambda(A) = (\lambda I - A)^{-1}$, the resolvent of A. By definition, $R_\lambda(A)$ is a bounded operator. In particular, if $\dim(X) < \infty$, then $R_\lambda(A)$ is compact. On the other hand, if $\dim(X) = \infty$, then $R_\lambda(A)$ cannot be compact, since its inverse $R_\lambda(A)^{-1} = \lambda I - A$ is a bounded operator, and we know that the inverse of any compact operator in an infinite dimensional space is not bounded. □

6. Let π_1 and π_2 be two lines in $\mathbb{R}^2$. Find all pairs of lines (π_1, π_2) such that $\pi_1 \cup \pi_2$ is a manifold. Justify your answer.

Answer: The union of two lines in $\mathbb{R}^2$ is a manifold if and only if they are parallel or coincide.

Solution. If the lines coincide $\pi_1 = \pi_2 = \pi$, then their union is a 1-dimensional manifold. Indeed, for any $p \in \pi$, take its neighborhood $U = \pi$, then U is homeomorphic to $\mathbb{R}$, e.g., map p to 0 and p_1, p_2 at distance x from p to $-x$ and x on $\mathbb{R}$.

If the lines are parallel, then their union is also a 1-dimensional manifold. Indeed, for any $p \in \pi_i$, take its neighborhood $U = \pi_i$. As before, it is homeomorphic to $\mathbb{R}$.

If the lines intersect, then their union is not a manifold. Indeed, let p be the unique point of intersection. Any point on $M = \pi_1 \cup \pi_2$ different from p is at positive distance from p, thus has an open neighborhood homeomorphic to an open interval of $\mathbb{R}$. Thus, if M is a manifold, then its dimension is 1. However, any open neighborhood of p is not homeomorphic to an open set in $\mathbb{R}$. Indeed, any such neighborhood U contains two open intervals $U_1 \subseteq \pi_1$ and $U_2 \subseteq \pi_2$ that contain p. If there exists a homeomorphism $f : U \to f(U) \subseteq \mathbb{R}$, then $f(U_1)$ and $f(U_2)$ are open intervals in $\mathbb{R}$ that contain $f(p)$. Thus, they must intersect, which contradicts injectivity of f. We have shown that p does not have a neighborhood homeomorphic to an open subset of $\mathbb{R}$, thus M is not a manifold. □