Homework 9 (submit by 08.06.2017)

1. Let X be a Banach space and $A \in \mathcal{L}(X)$ such that for some $p \in \mathbb{N}$, $\|A^p\| < 1$. Prove that the operator $I - A$ has bounded inverse.

   [Hint: Consider operators $B_i = \sum_{n=0}^{\infty} A^{pk+i}$.]

2. Let X be a Banach space and $A \in \mathcal{L}(X)$. Prove that A has a bounded inverse if and only if A^2 has a bounded inverse.

   [Hint: Use Banach's theorem.]

3. Let $C[0,1]$ be the Banach space of continuous functions with the supremum norm. Let $a \in C[0,1]$ and consider the operator $A : C[0,1] \to C[0,1]$ defined by

   $$Ax(t) = a(t)x(t).$$

   Prove that A has a bounded inverse if and only if $a(t) \neq 0$ for all $t \in [0,1]$.

4. Let $A \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. Assume that there exists a sequence $x_n \in X$ such that $\|x_n\| = 1$ and $(\lambda I - A)x_n \to 0$ as $n \to \infty$. Prove that $\lambda \in \sigma(A)$. Give an example that the inverse statement is not true.

5. Let $A \in \mathcal{L}(\ell_2)$ be defined by $Ax = (x_2, x_1, x_3, x_4, \ldots)$ (permutation of the first two components). Find and classify the spectrum of A.

6. Prove Hilbert's identity

   $$R_\lambda(A) - R_\mu(A) = (\mu - \lambda)R_\lambda(A)R_\mu(A).$$