Homework 7-8 (submit by 01.06.2017)

1. Find the norms of the following linear functionals.
   (a) $f \in \ell_2^*$, $f(x) = x_1 + x_2$
   (b) $f \in \ell_\infty^*$, $f(x) = x_1 + x_2$
   (c) $f \in (C[-1,1])^*$, $f(x) = x(0) - \int_0^1 x(t)dt$
   ($C[-1,1]$ is the space of continuous functions on $[-1,1]$ with the supremum norm $\|x\| = \sup_{t\in[-1,1]} |x(t)|$)
   (d) $f \in (C^1[-1,1])^*$, $f(x) = x'(0)$
   ($C^1[-1,1]$ is the space of continuously differentiable functions on $[-1,1]$ with the norm $\|x\| = \sup_{t\in[-1,1]} |x(t)| + \sup_{t\in[-1,1]} |x'(t)|$)

2. Let X be a normed space. Prove that for any $x \in X$,
$$\|x\| = \sup_{f\in X^*,\, \|f\|=1} |f(x)|.$$

3. Let X be the vector space of real-valued bounded functions on $[0,1]$ with the supremum norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Prove that there exists $f \in X^*$ such that $f \not\equiv 0$ and $f(x) = 0$ for all $x \in C[0,1]$.

4. Let X be a Hilbert space and $A : X \to X$ a bounded linear operator. Prove that
$$\|A\| = \sup_{\|x\|\le 1,\, \|y\|\le 1} |\langle Ax, y\rangle| .$$

5. Let $A : \ell_2 \to \ell_1$ be the linear operator defined by $Ax = (x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots)$. Find the norm of A.

6. Let X be the subspace of $C[0,1]$ consisting of all infinitely differentiable functions. (Here for $x \in X$, $\|x\| = \sup_{t\in[0,1]} |x(t)|$.) Let $A : X \to X$ be the linear functional defined by $(Ax)(t) = x'(t)$. Is A bounded? If yes, find its norm.