Homework 6 (submit by 18.05.2017)

1. Consider the space $C[a,b]$ of continuous functions on the interval $[a,b]$ and the two norms on it:
$$\|f\|_1 = \sup_{t\in[a,b]} |f(t)| \qquad \|f\|_2 = \int_a^b |f(t)|dt\,.$$
Prove that these norms are not equivalent.

2. Let $(X, \|\cdot\|)$ be a normed space. Prove that the following two statements are equivalent:
   (a) X is a Banach space
   (b) for any sequence of elements $x_i \in X$ such that $\sum_{i=1}^{\infty} \|x_i\| < \infty$, the series $\sum_{i=1}^{\infty} x_i$ converges in X (i.e., the sequence of partial sums $\sum_{i=1}^{n} x_i$ converges to an element in X).

3. Consider the Banach space $C[a,b]$ of continuous functions with the norm $\|f\| = \sup_{t\in[a,b]} |f(t)|$. Does there exist an inner product $\langle\cdot,\cdot\rangle$ on $C[a,b]$ which induces the norm (i.e., for any $f \in C[a,b]$, $\|f\| = \langle f,f\rangle^{\frac{1}{2}}$)?

4. Prove that in any euclidean space X, two vectors x, y are orthogonal if and only if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Does the same statement hold in unitary spaces?

5. Let X be a Hilbert space and $\{x_i\}_{i=1}^{\infty}$ an orthogonal system in X. Prove that the following two statements are equivalent:
   (a) the series $\sum_{i=1}^{\infty} x_i$ converges in X
   (b) the series $\sum_{i=1}^{\infty} \|x_i\|^2$ converges.