Homework 5 (submit by 11.05.2017)

1. Let p_k be non-negative numbers such that

$$\sup_{0<s<1} \sum_{k=1}^{\infty} \frac{\sin^2(sk)}{s^2}\, p_k < \infty.$$

Prove that $\sum_{k=1}^{\infty} k^2 p_k < \infty$.

[Hint: Consider $X = \mathbb{N}$ and $\mu(\{k\}) = p_k$, $k \geq 1$, and apply Fatou's lemma.]

2. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f, g : X \to [0, \infty)$ measurable functions. *Hölder's inequality* states that for any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_X fg\, d\mu \leq \left(\int_X f^p\, d\mu\right)^{\frac{1}{p}} \left(\int_X g^q\, d\mu\right)^{\frac{1}{q}}.$$

Derive Hölder's inequality from *Young's inequality*:
for any $a, b > 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

3. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to [0, \infty)$ a measurable function. Prove that for any $1 \leq a \leq b$,

$$\left(\int_X f^a\, d\mu\right)^{\frac{1}{a}} \leq \mu(X)^{\frac{1}{a} - \frac{1}{b}} \left(\int_X f^b\, d\mu\right)^{\frac{1}{b}}.$$

4. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f, g : X \to \mathbb{R}$ measurable functions. Prove *Minkowski's inequality*:

$$\left(\int_X |f+g|^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p\, d\mu\right)^{\frac{1}{p}}, \qquad \text{for all } p \geq 1.$$

[Hint: Use Hölder's inequality.]