Homework 4 (submit by 04.05.2017)

1. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to [1, \infty)$ a measurable function. Prove that
$$\int_X f(x)\, d\mu < \infty \qquad \text{if and only if} \qquad \sum_{n=1}^{\infty} \mu(x : f(x) \geq n) < \infty.$$

2. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to [0, \infty)$ a measurable function with $\int_X f(x)\, d\mu < \infty$. Prove that for any $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\int_E f(x)\, d\mu < \varepsilon \qquad \text{for all } E \in \mathcal{B} \text{ with } \mu(E) < \delta.$$

3. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ an integrable function. Prove that
$$\left| \int_X f(x)\, d\mu \right| \leq \int_X |f(x)|\, d\mu.$$

4. Let $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ be measurable spaces, μ a measure on $(X, \mathcal{B})$, and $\varphi : X \to X'$ a measurable map (i.e., for all $B' \in \mathcal{B}'$, $\varphi^{-1}(B') \in \mathcal{B}$). Define the *pushforward measure* $\varphi \circ \mu$ on $(X', \mathcal{B}')$ by
$$(\varphi \circ \mu)(B') = \mu(\varphi^{-1}(B')), \quad B' \in \mathcal{B}'.$$
Prove that for any measurable function $f : X' \to [0, \infty)$,
$$\int_X f(\varphi(x))\, d\mu = \int_{X'} f(x')\, d(\varphi \circ \mu).$$

5. Let $E = [-1, 1] \times [-1, 1]$ and the function $f : E \to \mathbb{R}$ is defined by
$$f(x, y) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & (x, y) \neq (0, 0) \\ 0 & x = y = 0. \end{cases}$$
   (a) Prove that
$$\int_{-1}^{1} \left(\int_{-1}^{1} f(x, y) dx \right) dy = \int_{-1}^{1} \left(\int_{-1}^{1} f(x, y) dy \right) dx = 0.$$
   (b) Prove that f is not Lebesgue integrable on E.