Homework 3 (submit by 27.04.2017)

1. For $E \subseteq \mathbb{R}^2$ and $x \in \mathbb{R}$, define

$$E_x = \{y \in \mathbb{R} \,:\, (x,y) \in E\}.$$

Prove that if $E \in \mathcal{B}(\mathbb{R}^2)$, then $E_x \in \mathcal{B}(\mathbb{R})$.

[Hint: Fix $x \in \mathbb{R}$ and consider $\mathcal{E} = \{E \subseteq \mathbb{R}^2 \,:\, E_x \in \mathcal{B}(\mathbb{R})\}$. Prove that (a) $\mathcal{E}$ is a σ-algebra and (b) $\mathcal{E}$ contains all boxes $[a,b] \times [c,d] \subset \mathbb{R}^2$.]

2. Find a Lebesgue measurable set in $\mathbb{R}^2$ that is not Borel measurable.

[Hint: Use previous exercise.]

3. Let f be a Borel function on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Fix $x \in \mathbb{R}$ and consider the function $g(y) = f(x,y)$. Prove that g is a Borel function on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

4. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate, i.e., $\pi(x,y) = x$ for all $x, y \in \mathbb{R}$. Prove that π is a Borel function.

5. Let $(X, \mathcal{B})$ be a measurable space and $f, g : X \to \mathbb{R}$ measurable functions. Prove that $\{x \in X \,:\, f(x) \geq g(x)\} \in \mathcal{B}$.