Homework 12 (submit by 29.06.2017)

1. Consider a surface $r = r(u^1, u^2)$ in $\mathbb{R}^3$. Let $r_i = \frac{\partial r}{\partial u^i}$. Then r_1 and r_2 is a basis of the tangent plane to the surface. For vector fields v with coordinates (v^1, v^2) in this basis and w with coordinates (w^1, w^2), tangent to the surface, define the covariant derivative of v in direction of w by

$$\nabla_w v = \left(\frac{\partial v^i}{\partial u^k} + \Gamma^i_{jk} v^j \right) w^k r_i \,,$$

where the Einstein summation convention is used and Γ^i_{jk} are Christoffel symbols. Further, for a function $f = f(u^1, u^2)$, define the derivative of f in direction of w by

$$D_w f = \frac{\partial f}{\partial u^j} w^j \,.$$

Prove that for any vector fields w, a, b tangent to the surface,

$$D_w \langle a, b \rangle = \langle \nabla_w a, b \rangle + \langle a, \nabla_w b \rangle.$$

2. Let $f : X \to Y$ be a mapping of topological spaces X, Y. Prove that the following statements are equivalent:

   (a) f is continuous (i.e., for each $x \in X$ and any neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$)

   (b) for every open set V in Y, the set $f^{-1}(V) = \{x \in X \,:\, f(x) \in V\}$ is open in X.

3. Prove that the graph of a continuous function

$$x_{n+1} = f(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

is a manifold and define on it a smooth structure.

4. Consider a torus T, the set in $\mathbb{R}^3$ generated by revolving the circle

$$(x-2)^2 + z^2 = 1, \quad y = 0$$

about Oz axis. Prove that T is a two-dimensional manifold. Define a smooth structure on T.