Homework 1 (submit by 13.04.2017)

1. Prove the countable subadditivity of the outer Lebesgue measure: for any $E_i \subseteq \mathbb{R}^d$,

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

2. Prove that every Jordan measurable set is Lebesgue measurable.

3. Let $E_i \subseteq \mathbb{R}^d$ be Lebesgue measurable sets. Prove that

   (a) if $E_1 \subseteq E_2 \subseteq \dots$ then
   $$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n),$$

   (b) if $E_1 \supseteq E_2 \supseteq \dots$ and $\mu(E_k) < \infty$ for some k, then
   $$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n),$$

   (c) the assumption "$\mu(E_k) = \infty$ for some k" in (b) cannot be dropped.

4. Prove the inner regularity of the Lebesgue measure: for any Lebesgue measurable set $E \subset \mathbb{R}^d$,
$$\mu(E) = \sup_{\substack{K \subseteq E \\ K \text{ compact}}} \mu(K).$$

5. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the coordinate projection, $\pi(x, y) = x$ for all $(x, y) \in \mathbb{R}^2$. Prove that there exists a set $E \subseteq \mathbb{R}^2$ such that E is Lebesgue measurable in $\mathbb{R}^2$, but $\pi(E)$ is not Lebesgue measurable in $\mathbb{R}$.