SYLLABUS

1 Summary

- Integration in $\mathbb{R}^n$:
  (A) Multiple integral
  Jordan measurable sets. Fubini theorem. Change of variables.
  (B) Line integral
  Line integral of scalar and vector fields. Green's theorem. Conservative vector field.
  (C) Surface integral
  Surface integral of scalar and vector fields. Gauss's theorem. Stokes' theorem. Conservative and solenoidal vector fields.
- Complex analysis:
  (A) Complex derivative
  Cauchy-Riemann conditions. Holomorphic function.
  (B) Complex integration
  Cauchy theorem. Cauchy integral formula. Cauchy integral formula for derivatives. Taylor's series.
  (C) Isolated singularities
  Removable singularity. Pole. Essential isolated singularity. Laurent series.
  (D) Residues
  Residue theorem. Evaluation of definite integrals using residues.
- Partial differential equations:
  (A) First order PDEs. Method of characteristics.
  (B) Classification of second order PDEs. Canonical form.
  (C) Wave equation
  Initial-boundary value problem. D'Alembert's formula. Fourier method.
  (D) Heat equation
  Initial-boundary value problem. Fourier method.
  (E) Laplace and Poisson equations.

2 Multiple integral

- *Jordan measure.* Inner and outer Jordan measures, their basic properties. Jordan measurable set. Jordan measure. Set of measure zero. Examples of sets of measure zero: graph of continuous function, rectifiable curve.

Theorem: A bounded set is Jordan measurable iff its boundary has measure zero.

Properties of measurable sets.

- *Multiple integral.* Partition. Riemann sum. Multiple integral.

  Theorem: If a function f is Riemann integrable on a closed measurable set S, then it is bounded on S.

  Lower and upper Darboux sums. Fluctuation $\omega(f; S) = \sup_{x,y\in S} |f(x) - f(y)|$.

  Theorem: A function f is Riemann integrable on a measurable set S iff f is bounded on S and $\lim_{\delta_\tau \to 0} \sum_{i=1}^{\ell_\tau} \omega(f; \overline{S_i})\mu(S_i) = 0$.

  Theorem: If f is continuous on a closed measurable S, then f is integrable on S.

  Theorem: If f is integrable on S, $E \subset S$ with $\mu(E) = 0$, and g is a bounded function on $\overline{S}$ such that $g(x) = f(x)$ for all $x \notin E$, then g is integrable on S and $\int_S g(x)dx = \int_S f(x)dx$. Thus, the value of the integral $\int_S f(x)dx$ does not depend on values of f on sets of measure 0.

  Corollary: If f is bounded on E and $\mu(E) = 0$, then $\int_E f(x)dx = 0$.

- *Properties of multiple integral.*

  For any measurable S, $\int_S 1dx = \mu(S)$.

  If $S' \subset S$ are measurable and f is integrable on S, then f is integrable on S' and $\int_S f(x)dx = \int_{S'} f(x)dx + \int_{S\setminus S'} f(x)dx$.

  If f and g are integrable on S, then $\alpha f + \beta g$ is also integrable on S and $\int_S (\alpha f + \beta g)(x)dx = \alpha \int_S f(x)dx + \beta \int_S g(x)dx$.

  If f and g are integrable on S, then fg is integrable on S.

  If $f \le g$ on S, then $\int_S f(x)dx \le \int_S g(x)dx$.

  If f is integrable on S, then $|f|$ is integrable on S and $|\int_S f(x)dx| \le \int_S |f(x)|dx$.

  If $S' \subseteq S$ measurable, $f(x) \ge 0$ on S, then $\int_S f(x)dx \ge \int_{S'} f(x)dx$.

- *Iterated integral.* Elementary sets in $\mathbb{R}^2$. The main tool to compute multiple integrals is the Fubini theorem:

  Theorem (Fubini): If $S = \{(x, y) \mid a \le x \le b, \varphi(x) \le y \le \psi(x)\}$ and f is continuous on S, then

  $$\iint_S f(x, y)dxdy = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x, y)dy \right] dx.$$

  Similarly, if $S = \{(x, y) \mid c \le y \le d, \alpha(y) \le x \le \beta(y)\}$ and f is continuous on S, then

  $$\iint_S f(x, y)dxdy = \int_c^d \left[\int_{\alpha(y)}^{\beta(y)} f(x, y)dx \right] dy.$$

In particular, if S is elementary with respect to both $0x$- and $0y$-axes, then the order of integration in the iterated integral can be changed:

$$\int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right] dx = \int_c^d \left[\int_{\alpha(y)}^{\beta(y)} f(x,y)dx\right] dy.$$

Remark: If S is not elementary, but can be written as the union of elementary sets S_i, then the multiple integral over S can be computed by applying the Fubini theorem to each S_i.

Elementary sets in $\mathbb{R}^3$.

Theorem (Fubini): If $S = \{(x,y,z) \mid (x,y) \in S', \varphi_2(x,y) \le z \le \psi_2(x,y)\}$ and f is continuous in S, then

$$\iiint_S f(x,y,z)dxdydz = \iint_{S'} \left[\int_{\varphi_2(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right] dxdy.$$

For $z_0 \in \mathbb{R}$, let $S(z_0) = \{(x,y,z) \mid (x,y,z) \in S, z = z_0\}$. Then

$$\iiint_S f(x,y,z)dxdydz = \int_a^b \left[\iint_{S(z)} f(x,y,z)dxdy\right] dz.$$

Remark: If S' and $S(z)$ above are elementary, then the multiple integrals over S' and $S(z)$ can be computed using the Fubini theorem (with two variables).

Fubini theorem in $\mathbb{R}^n$.

- *Change of variables.* Change of variables in $\mathbb{R}^2$:

  Theorem: Let $S, S' \subset \mathbb{R}^2$ be measurable. Let $\varphi : S' \to S$ be a bijective and continuously differentiable map on $\overline{S'}$ with non-zero Jacobian on S'. (For every $(u,v) \in S'$, $\varphi(u,v) = (x(u,v), y(u,v)) \in S$, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \ne 0$ in S'.) Then

  $$\iint_S f(x,y)dxdy = \iint_{S'} f(\varphi(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

  Geometric interpretation of the Jacobian.

  Change of variables in $\mathbb{R}^n$. Examples: cylindrical coordinates, spherical coordinates.

- *Improper integral.* Exhaustion of an open S. Improper integral $\int_S f(x)dx$ (cases of unbounded S or unbounded f). Example: $\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$.

  Comparison criterion: Let S be open. $0 \le f(x) \le g(x)$ on S. If $\int_S g(x)dx$ converges, then $\int_S f(x)dx$ converges. Examples:

  $\underset{x_1^2+\ldots+x_n^2\le 1}{\int\cdots\int} \frac{dx_1\ldots dx_n}{\left(\sqrt{x_1^2+\ldots+x_n^2}\right)^\alpha} < \infty$ iff $\alpha < n$, $\underset{x_1^2+\ldots+x_n^2\ge 1}{\int\cdots\int} \frac{dx_1\ldots dx_n}{\left(\sqrt{x_1^2+\ldots+x_n^2}\right)^\alpha} < \infty$ iff $\alpha > n$.

3 Line integrals

- *Line integral of scalar field.* Rectifiable curve parametrized by its length.

$$\int_\gamma f ds = \int_0^L f(x(s), y(s), z(s)) ds.$$

Properties:

$\int_\gamma ds = L =$ length of γ

If f is continuous, then $\int_\gamma f ds$ exists.

If γ^R is the reversal of γ, then $\int_{\gamma^R} f ds = \int_\gamma f ds$.

$\int_\gamma f ds$ as limit of Riemann sums.

If γ is parametrized by $(x(t), y(t), z(t))$, $a \le t \le b$, then

$$\int_\gamma f ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

- *Line integral of vector field.* Work of a force.

If $F = (P, Q, R)$, $\gamma = \{r(t) = (x(t), y(t), z(t)), a \le t \le b\}$, then

$$\int_\gamma P dx + Q dy + R dz = \int_a^b (F \cdot r'(t))\, dt = \int_a^b (P(r(t))x'(t) + Q(r(t))y'(t) + R(r(t))z'(t))\, dt.$$

Remark: the definition is independent of the parametrization of γ.

If γ^R is the reversal of γ, then $\int_{\gamma^R} P dx + Q dy + R dz = -\int_\gamma P dx + Q dy + R dz$.

- *Green's formula.* Positively and negatively oriented contours in $\mathbb{R}^2$.

Theorem: Let S be a measurable set in $\mathbb{R}^2$ and its boundary is the union of finitely many continously differentiable curves. Let γ^+ be the positively oriented boundary of S. Let $P(x, y)$, $Q(x, y)$ be continuous on $\overline{S}$ such that $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on $\overline{S}$. Then

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\gamma^+} P dx + Q dy.$$

Corollary: If S contains holes with boundaries $\gamma_1, \dots, \gamma_k$, then

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\gamma^+} P dx + Q dy + \sum_{i=1}^k \int_{\gamma_i^-} P dx + Q dy.$$

Area of a region surrounded by a curve: $\mu(S) = \frac{1}{2} \int_{\gamma^+} x dy - y dx$.

Sign of Jacobian: $J > 0$ if orientation is preserved, $J < 0$ if orientation is reverted.

- *Conservative vector field in* $\mathbb{R}^2$*.* Scalar potential.

  Theorem: Let (P, Q) be continuous vector field on $S \subset \mathbb{R}^2$. The following are equivalent:

  (a) For any contour γ in S, $\int_\gamma P dx + Q dy = 0$,

  (b) For any $A, B \in S$, the integral $\int_\gamma P dx + Q dy$ does not depend on the choice of γ in S from A to B.

  (c) There exists φ such that $(P, Q) = \nabla\varphi$. In this case $\int_\gamma P dx + Q dy = \varphi(B) - \varphi(A)$ for any γ in S from A to B.

  Irrotational (curl-free) vector field in $\mathbb{R}^2$.

  Remark: Conservative implies irrotational, but not vice versa.

  Theorem: If S is simply connected, then (P, Q) on S is conservative iff irrotational.

4 Surface integrals

- *Surfaces.* Continuously differentiable surface. Curvilinear coordinates. $r_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$, $r_v = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v})$. Regular and singular points of a surface. Smooth surface. Tangent plane. Normal line and normal vector to a surface. First fundamental form for a surface:

  $$Q(a, b) = Ea^2 + 2Fab + Gb^2, \quad E = r_u^2,\ F = r_u r_v,\ G = r_v^2.$$

  If $\widetilde{u} = \varphi(u, v)$, $\widetilde{v} = \psi(u, v)$, then $EG - F^2 = (\widetilde{E}\widetilde{G} - \widetilde{F}^2) \left| \frac{\partial(\varphi,\psi)}{\partial(u,v)} \right|^2$.

  Length of a curve on a surface, $r(u(t), v(t))$, $a \le t \le b$,

  $$L = \int_a^b \sqrt{Eu'(t)^2 + 2Fu'(t)v'(t) + Gv'(t)^2}\, dt.$$

  Surface area, $S = \{r(u, v) = (x(u, v), y(u, v), z(u, v)),\ (u, v) \in D\}$,

  $$\text{Area } (S) = \iint_D \|r_u \times r_v\|\, dudv = \iint_D \sqrt{EG - F^2}\, dudv.$$

- *Surface integral of scalar field.*

  $$\iint_S f dS = \iint_D f(x(u, v), y(u, v), z(u, v)) \sqrt{EG - F^2}\, dudv.$$

  Remark: the definition is independent of the parametrization of S.

- *Surface integral of vector field.* Orientable surface. Orientation by the (continous) unit normal $n = \frac{r_u \times r_v}{\|r_u \times r_v\|}$. Flow of fluid through a surface.

Integral of a vector field $F = (P, Q, R)$ over a smooth surface oriented by normal n:

$$\iint_S F \cdot dS = \iint_S (F \cdot n) dS$$

If the normal is taken as $n = \frac{r_u \times r_v}{\|r_u \times r_v\|}$, then

$$\iint_S F \cdot dS = \iint_D F \cdot (r_u \times r_v)\, dudv = \iint_D \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} dudv.$$

In this case, one also uses the notation $\iint_S Pdydz + Qdzdx + Rdxdy$ with convention $dydz = -dzdy, \ldots$

Note: Different parametrizations give rise to different orientations of the same surface. A particular care is needed if a surface orientation is specified in advance (e.g., a flow of fluid). If a chosen parametrization leads to the opposite orientation, the change of sign of the integral is necessary.

- *Gauss divergence theorem.*

  Theorem: Let S be a smooth surface surrounding a solid V, positively oriented by the outgoing normal. Let $F = (P, Q, R)$ be a continously differentiable vector field in $\overline{V}$. Then

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz = \iint_{S^+} Pdydz + Qdzdx + Rdxdy.$$

  Divergence, $\text{div}F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

  Divergence is independ of coordinate system: $\text{div}F = \lim_{\varepsilon \to 0} \frac{1}{\text{Vol}(B_\varepsilon)} \iint_{S_\varepsilon} F \cdot dS$.

  Corollary: If V contains holes with boundaries $S_1, \ldots, S_k$ oriented by outgoing normals, then

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz = \iint_S F \cdot dS + \sum_{i=1}^{k} \iint_{S_i} F \cdot dS.$$

- *Stokes' theorem.*

  Theorem: Let S be a smooth surface oriented by the normal $\frac{r_u \times r_v}{\|r_u \times r_v\|}$. Let γ be positively oriented boundary of S. Let $F = (P, Q, R)$ be continuously differentiable vector field in a neighborhood of S. Then

$$\int_\gamma Pdx + Qdy + Rdz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

  Curl of F, $\text{curl}F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$.

Remark: The orientations of the surface and its boundary in Stoke's theorem agree by the right-hand rule.

Remark: If S is a subset of $\{z = 0\}$ and $R = 0$, then Stokes' theorem implies Green's theorem.

Curl is independent of coordinate system: $\operatorname{curl} F \cdot \ell = \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(S_\varepsilon)} \int_{\gamma^+} Pdx + Qdy + Rdz$.

- *Special vector fields.* Conservative vector field. Irrotational (curl-free) vector field. Solenoidal (divergence free) vector field.

  Theorem: Let V be simply connected. F in V is solenoidal iff for any smooth surface S in V surrounding a solid, the flux of F through S is 0.

  $\operatorname{curl} \nabla\varphi = \overrightarrow{0}$, $\operatorname{div} \operatorname{curl} F = 0$.

  Theorem: If V is simply connected, then F is solenoidal iff there exists a vector field A such that $F = \operatorname{curl} A$. A is called vector potential of F.

  Helmholtz theorem (the fundamental theorem of vector calculus): Any smooth vector field F is the sum of conservative and solenoidal vector fields, $F = \nabla\varphi + \operatorname{curl} A$.

  Remark: If $\operatorname{div} F = 0$ and $\operatorname{curl} F = \overrightarrow{0}$, then $F = \nabla\varphi$ for some φ satisfying $\Delta\varphi = 0$ (Laplace equation). Solutions to the Laplace equation are harmonic functions.

5 Complex analysis

- *Recall.* Algebraic form. Geometric interpretation. Trigonometric form.

  Limit of sequence of complex numbers.

  Functions of complex variable. Limit of function at a point. Properties of the limit.

  Series of complex numbers. Absolutely convergent series.

  Series of functions. Uniformly convergent series. M-test for uniform convergence.

  Power series. Abel's theorem: If $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ converges at some $z_1 \in \mathbb{C}$, then it converges absolutely in $B(z_0, |z_1 - z_0|)$ and uniformly in $B(z_0, \gamma|z_1 - z_0|)$, for all $\gamma < 1$. Radius of convergence. Cauchy-Hadamard formula.

- *Elementary functions.* Exponential function. Properties: $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$, $e^z \neq 0$ for all z, $e^{z+2\pi i} = e^z$ (periodicity).

  Logarithm. $\log w = \log |w| + i\arg w$ (multi-valued function). Continous branch of complex logarithm.

  Remark: If D does not contain a contour around 0, then log admits a continuous branch in D.

  Power function. $a, b \in \mathbb{C}$, $a^b = e^{b \log a}$ (in general, multi-valued). If $b \in \mathbb{Z}$, a^b is single-valued. If $b = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $(m, n) = 1$, then a^b has n values.

  Remark: all elementary functions of complex variable can be expressed with the exponential and logarithmic functions.

  Complex functions as maps.

- *Complex derivative.* Definition.

  Theorem: A function $f = u + iv$ is differentiable at $z_0 = x_0 + iy_0$ iff

  (1) u, v are differentiable at (x_0, y_0),

  (2) Cauchy-Riemann conditions hold: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

  In this case, $u_x = v_y = \mathrm{Re} f'(z_0)$, $u_y = -v_x = -\mathrm{Im} f'(z_0)$. Thus, $f' = u_x + iv_x = u_x - iu_y = v_y + iv_x = v_y - iu_y$.

  Holomoriphic function in an open set. Holomorphic function at a point.

  Geometric interpretation of the derivative of holomorphic function. Conformal map.

  Theorem: Let $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$. Let $D = B(z_0, R)$ be the disk of convergence. Then f is holomorphic in D.

  Corollary: e^z, $\cos z$, $\sin z$, z^n $(n \in \mathbb{N})$ are holomorphic in $\mathbb{C}$.

- *Complex integration.* Integral as limit of Riemann sums, $\int_\gamma f(z)dz$. Properties:

  (Linearity) If f_1 and f_2 are integrable functions, then $\alpha f_1 + \beta f_2$ is integrable and $\int_\gamma (\alpha f_1 + \beta f_2)(z)dz = \alpha \int_\gamma f_1(z)dz + \beta \int_\gamma f_2(z)dz$.

  If γ^R is the reverals of γ, then $\int_{\gamma^R} f(z)dz = -\int_\gamma f(z)dz$.

  (Additivity) $\int_{\gamma_1 \cup \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$.

  $\left| \int_\gamma f(z)dz \right| \leq \int_\gamma |f(z)|ds$ (The RHS is the line integral of scalar field in $\mathbb{R}^2$.)

  Computations of the integral:

  (1) If $f : \mathbb{R} \to \mathbb{C}$, $f(t) = u(t) + iv(t)$, then define $\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$. (The two integrals on the RHS are usual Riemann integrals.)

  Let $\gamma = \{z(t) = x(t) + iy(t),\ a \leq t \leq b\}$ such that there exists $z'(t) = x'(t) + iy'(t)$. Then
  $$\int_\gamma f(z)dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

  (2) If $z = x + iy$, $f = u + iv$, then
  $$\int_\gamma f(z)dz = \int_\gamma udx - vdy + i \int_\gamma vdx + udy$$

  (The two integrals on the RHS are line integrals of vector fields.)

  Antiderivative.

  Theorem (Newton-Leibnitz): If F is an antiderivative of f in D, then for all paths γ in D from a to b,
  $$\int_\gamma f(z)dz = F(b) - F(a).$$

  In particular, for any contour γ in D, $\int_\gamma f(z)dz = 0$.

Theorem: If D is simply connected, then any holomorphic function f in D has antiderivative.

Corollary (Cauchy's theorem): If f is holomorphic in the domain surrounded by γ, then $\oint_\gamma f(z)dz = 0$.

Corollary: If f is holomorphic in D with the outer boundary γ and boundaries of holes $\gamma_1, \ldots, \gamma_k$, then

$$\oint_\gamma f(z)dz + \sum_{i=1}^{k} \oint_{\gamma_i} f(z)dz = 0.$$

Theorem (integral Cauchy formula): Let D be open simply connected, f holomorphic in D. Then for all $z_0 \in D$,

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(t)}{t - z_0} dt.$$

In particular, if f, g are holomorphic in D and $f(t) = g(t)$ for all $t \in \partial D$, then $f(z) = g(z)$ for all $z \in D$.

Remark: If $z_0 \notin D$, then $\frac{f(z)}{z-z_0}$ is holomorphic in D, hence $\oint_{\partial D} \frac{f(t)}{t-z_0} dt = 0$.

Remark: If γ winds around z_0 k times, then $f(z_0) = \frac{1}{2\pi k i} \oint_\gamma \frac{f(t)}{t-z_0} dt$.

Theorem: If f is holomorphic in D and continuous in $\overline{D}$, then f is infinitely many times differentiable in D and

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\partial D} \frac{f(t)}{(t - z_0)^{k+1}} dt.$$

- *Taylor series.*

  Theorem: If f is holomorphic in $D = B(z_0, r)$, then for all $z \in D$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \qquad \text{where } c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{S(z_0,r)} \frac{f(t)}{(t - z_0)^{n+1}} dt.$$

  Analytic function. Analytic = holomorphic.

  Corollary: Let f, g be holomorphic in $B(z_0, r)$ and $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all n, then $f(z) = g(z)$ for all $z \in B(z_0, r)$.

  Theorem: Let $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ in $D = B(z_0, R)$ and $\sup_{z \in D} |f(z)| \le M$. Then for all n, $|c_n| \le \frac{M}{R^n}$.

  Corollary (Liouville theorem): If f is holomorphic and bounded on $\mathbb{C}$, it is constant.

  Corollary (Fundamental theorem of algebra): Every polynomial of degree ≥ 1 has a root in $\mathbb{C}$.

- *Zeros of holomorphic functions.*

  Theorem: If $a \in \mathbb{C}$ is a zero of f, then there exists a neighborhood $B(a, \delta)$ of a such that in this neighborhood either $f \equiv 0$ or a is the unique zero of f. In the second case, there exists g such that $g(z) \neq 0$ for all $z \in B(a, \delta)$ and for some $k \geq 1$, $f(z) = g(z)(z-a)^k$ in $B(a, \delta)$. k is the order of a.

  Theorem: Let f be holomorphic in an open set D. Then either $f \equiv 0$ in D or all zeros of f in D are isolated.

  Corollary: If f, g are holomorphic in D and $\{z : f(z) = g(z)\}$ has a cluster point in D, then $f \equiv g$ in D.

- *Isolated singularities.* $U(a, \varepsilon) = \{z : 0 < |z-a| < \varepsilon\}$. Removable singularity ($\lim_{z \to a} f(z) \in \mathbb{C}$). Pole ($\lim_{z \to a} f(z) = \infty$). Essential isolated singularity.

  Theorem: If a is an isolated singularity for f and f is bounded on $U(a, \varepsilon)$, then a is a removable singularity.

  Theorem: If a is a pole for f, then there exists $\varepsilon > 0$ and a function f_1 holomorphic and not equal to zero everywhere in $B(a, \varepsilon)$ such that for some $k \geq 1$, $f(z) = \frac{f_1(z)}{(z-a)^k}$. k is the order of the pole.

  Remark: Poles are isolated by definition.

  Theorem (Casorati-Sokhotski-Weierstrass): Let a be an essential isolated singularity for f. Then for every $w \in \mathbb{C} \cup \{\infty\}$, there exists $z_n \to a$ such that $f(z_n) \to w$.

  Theorem (Picard's great theorem): Let a be an essential isolated singularity for f. For any $\varepsilon > 0$, f takes any $\mathbb{C}$-value in $U(a, \varepsilon)$, except for maybe one, infinitely often.

- *Laurent series.* Ring of convergence.

  Theorem: Let f be holomorphic in $D = \{z : r < |z - z_0| < R\}$ ($r \geq 0$, $R \leq +\infty$). Then for all $z \in D$ and $\rho \in (r, R)$,

  $$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n, \qquad \text{where } c_n = \frac{1}{2\pi i} \oint_{S(z_0, \rho)} \frac{f(t)}{(t - z_0)^{n+1}} \, dt.$$

  Moreover, the series converges absolutely and uniformly on compact subsets of D.

  Characterization of isolated singularities:

  (1) z_0 is a removable singularity iff $c_n = 0$ for all $n < 0$.

  (2) z_0 is a pole of order k iff $c_n = 0$ for all $n < -k$.

  (3) z_0 is an essential isolated singularity iff infinitely many c_n's ($n < 0$) are non-zero.

- *Residues.*

  $$\operatorname{Res}_{z_0} f = \frac{1}{2\pi i} \oint_{S(z_0, \rho)} f(t) \, dt = c_{-1}.$$

  Computation of residues:

  (1) If z_0 is not a singularity or removable singularity, then $\operatorname{Res}_{z_0} f = 0$.

(2) If z_0 is a pole of order 1 (simple pole), then $\text{Res}_{z_0} f = \lim_{z\to z_0}(z-z_0)f(z)$.

(3) If z_0 is a pole of order k, then $\text{Res}_{z_0} f = \frac{1}{(k-1)!}\lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k f(z)\right)$.

(4) If $f(z) = \frac{p(z)}{q(z)}$, where p, q are holomorphic at z_0 and z_0 is a zero of q of order 1, then

$$\text{Res}_{z_0} f = \frac{p(z_0)}{q'(z_0)}.$$

Theorem (Residue theorem): Let D be a bounded subset of $\mathbb{C}$ and f a holomorphic function in $D \setminus \{z_1, \dots, z_n\}$. Then

$$\oint_{\partial D} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z_k} f.$$

Evaluation of definite integrals with residues:

(1) Let $R(x, y)$ be a real function of two variables such that $R(\cos t, \sin t)$ is defined on $[0, 2\pi]$. Then

$$\int_0^{2\pi} R(\cos t, \sin t)dt = 2\pi i \sum_{a\in B(0,1)} \text{Res}_a f,$$

where $f(z) = \frac{1}{iz} R(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z}))$ and the sum is over all isolated singularities for f in $B(0, 1)$.

(2) Let P, Q be polynomials such that $\deg Q \geq \deg P + 2$, Q has no real roots, and $a_1, \dots, a_k$ are all the roots of Q with positive imaginary part. Then

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}dx = 2\pi i \sum_{j=1}^{k} \text{Res}_{a_j} \frac{P}{Q}.$$

(3) Let P, Q be polynomials such that $\deg Q \geq \deg P + 1$, Q has no real roots, and $a_1, \dots, a_k$ are all the roots of Q with positive imaginary part. Then for any $\alpha > 0$,

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}e^{i\alpha x}dx = 2\pi i \sum_{j=1}^{k} \text{Res}_{a_j} \left(\frac{P}{Q}e^{i\alpha z}\right).$$

- *Conformal maps.*

Theorem (Riemann): For any open simply connected D, D' not equal to $\mathbb{C}$, for all $z_0 \in D$, $w_0 \in D'$ and $\alpha_0 \in \mathbb{R}$, there exists a unique $f : D \to D'$ holomorphic and bijective such that $f(z_0) = w_0$, $\arg f'(z_0) = \alpha_0$.

Extended complex plane $\overline{\mathbb{C}}$. Riemann theorem for subsets of $\overline{\mathbb{C}}$. Stereographic projection and Riemann sphere.

Linear fractional transformation, $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$. Holomorphic on $\overline{\mathbb{C}} \setminus \{-\frac{d}{c}\}$, bijective. Maps circles of $\overline{\mathbb{C}}$ to circles of $\overline{\mathbb{C}}$. Maps symmetric points to symmetric points. Uniquely determined by images of three points.

Upper half plane to the unit disk, $f(z) = e^{i\alpha}\frac{z-a}{z-\bar{a}}$.

Unit disk to unit disk, $f(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z}$.

Conformal maps with exponential and power functions.

6 Partial differential equations

- *Introduction.* Derivation of the wave and heat equations.
- *First order PDEs.* Method of characteristics.

$$\begin{cases} a(x,t,u)u_t + b(x,t,u)u_x = c(x,t,u) \\ u(x,0) = f(x) \end{cases}$$

Solving the system of three ordinary differential equations

$$\frac{dt}{ds} = a, \quad \frac{dx}{ds} = b, \quad \frac{du}{ds} = c$$

gives the solution $u(t(s), x(s))$ along a (characteristic) curve $(t(s), x(s))$.

- *Classification of* 2*nd order PDEs.* Elliptic, hyperbolic, parabolic. Canonical form.

Canonical form of the PDE

$$au_{xx} + 2bu_{xy} + cu_{yy} + F(x,y,u,\nabla u) = 0$$

in a neighborhood of (x_0, y_0), where a, b, c are twice continuously differentiable, not zeros all at the same time, can be found as follows:

(1) Solve characteristic equation: $a(z_x)^2 + 2bz_xz_y + c(z_y)^2 = 0$. Assuming $a \neq 0$, it reduces to two first order PDEs $z_x + \lambda_1 z_y = 0$ or $z_x + \lambda_2 z_y = 0$, where $\lambda_1 = \frac{b-\sqrt{b^2-ac}}{a}$, $\lambda_2 = \frac{b+\sqrt{b^2-ac}}{a}$.

(2) If $d = b^2 - ac > 0$, the PDE is hyperbolic. The characteristic equation has two real solutions $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$. Taking ξ and η for new variables leads to the equation $\widetilde{u}_{\xi\eta} + \widetilde{F} = 0$.

(3) If $d = 0$, the PDE is parabolic. The characteristic equation has one real solution $\xi = \xi(x,y)$. Let $\eta = \eta(x,y)$ be such that $|\frac{\partial(\xi,\eta)}{\partial(x,y)}| \neq 0$ in a neighborhood of (x_0, y_0). Taking ξ and η for new variables leads to the equation $\widetilde{u}_{\eta\eta} + \widetilde{F} = 0$.

(4) If $d < 0$, the PDE is elliptic. The characteristic equation has two solutions $\alpha = \alpha(x,y)$ and $\beta = \beta(x,y)$ such that $\beta = \overline{\alpha}$. Taking $\xi = \mathrm{Re}(\alpha)$ and $\eta = \mathrm{Im}(\alpha)$ for new variables leads to the equation $\widetilde{u}_{\xi\xi} + \widetilde{u}_{\eta\eta} + \widetilde{F} = 0$.

Remark: If coefficients in front of the highest derivatives are constants, the canonical form can be further simplified by substitution $u = e^{\lambda\xi+\mu\eta}v$, for suitably chosen λ, μ.

- *Wave equation.* Initial conditions. Boundary conditions. Correctly stated problem.

  Theorem (uniqueness): There exists at most one $u(x,t)$ twice continuously differentiable in $(0,\ell)\times(0,+\infty)$ such that u and u_t are continuous in $[0,\ell]\times[0,+\infty)$, which solves the initial-boundary value problem

$$\begin{cases} \rho u_{tt} = (ku_x)_x + f & 0<x<\ell, t>0 \\ u(x,0)=\varphi(x) & 0\le x\le \ell \\ u_t(x,0)=\psi(x) & 0\le x\le\ell \\ u(0,t)=\mu_1(t) & t\ge 0 \\ u(\ell,t)=\mu_2(t) & t\ge 0. \end{cases}$$

  Existence of solution. Method of characteristics / method of travelling waves.

$$\begin{cases} u_{tt}=a^2u_{xx} & x\in\mathbb{R}, t>0 \\ u(x,0)=\varphi(x) & x\in\mathbb{R} \\ u_t(x,0)=\psi(x) & x\in\mathbb{R}. \end{cases}$$

  D'Alembert's formula:

$$u(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(z)dz.$$

  Theorem (existence): If φ is twice continuously differentiable, ψ continuously differentiable, then u given by d'Alembert's formula solves the above initial value problem. The solution is unique and depends continuously on the initial data.

  Remark: The unique solution to the inhomogeneous problem $u_{tt}=a^2u_{xx}+f$, $u|_{t=0}=\varphi$, $u_t|_{t=0}=\psi$ is given by the d'Alembert's formula

$$u(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(z)dz+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi\,d\tau.$$

  Proposition: If in the statement of the homogeneous problem φ,ψ are odd with respect to x_0, then $u(x_0,t)=0$ for all t, if they are even, then $u_x(x_0,t)=0$ for all t.

  Solution to the wave equation on $[0,+\infty)$ and $[0,\ell]$ using reflections of initial data.

- *Fourier series.* Trigonometric series.

  Theorem: If $\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$ converges uniformly on $[-\pi,\pi]$, then its sum $f(x)$ is continuous on $[-\pi,\pi]$, $f(\pi)=f(-\pi)$, and the Euler-Fourier relations hold:

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos ktdt,\qquad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin ktdt.$$

  Fourier series of an absolutely integrable function,

$$f(x)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx).$$

Piecewise continuous functions on $[a,b]$, $\mathrm{PC}[a,b]$. Piecewise continuously differentiable functions on $[a,b]$, $\mathrm{PC}'[a,b]$.

Theorem (Dirichlet): Let $f \in \mathrm{PC}'[-\pi,\pi]$ and $f(x) \sim \frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k \cos kx+b_k \sin kx)$. Then

(1) For all $x \in (-\pi,\pi)$, $\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k \cos kx+b_k \sin kx) = f(x)$,

for $x \in \{-\pi,\pi\}$, $\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k \cos kx+b_k \sin kx) = \frac{f(\pi)+f(-\pi)}{2}$,

(2) The Fourier series converges uniformly on any $[a,b] \subset (-\pi,\pi)$,

(3) If $f(-\pi)=f(\pi)$, then the Fourier series converges uniformly to f on $[-\pi,\pi]$.

Exponential form: $f(x) \sim \sum_{n=-\infty}^{+\infty} c_k e^{ikx}$, $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt$.

Arbitrary interval: $f : [-\ell,\ell] \to \mathbb{R}$, $f(x) \sim \frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k \cos \frac{\pi k x}{\ell}+b_k \sin \frac{\pi k x}{\ell})$

Corollary: Every $f \in \mathrm{PC}'[0,L]$ with $f(0)=f(L)$ can be expanded in the uniformly convergent series of sines and cosines:

(1) $f(x) = \sum_{k=1}^{\infty} b_k \sin \frac{\pi k x}{L}$, $b_k = \frac{2}{L}\int_0^L f(t)\sin \frac{\pi k t}{L}dt$.

(2) $f(x) = \frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos \frac{\pi k x}{L}$, $a_k = \frac{2}{L}\int_0^L f(t)\cos \frac{\pi k t}{L}dt$.

Theorem (Bessel's inequality): If $\int_{-\infty}^{+\infty} f(x)^2dx < \infty$ and $f(x) \sim \sum_n c_n e^{inx}$, then

$$\sum_n |c_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2dx.$$

Theorem (decay of Fourier coefficients):

(1) If $f \in \mathrm{PC}[-\pi,\pi]$, then $\sum_k |c_k|^2 < \infty$.

(2) If $f \in \mathrm{PC}'[-\pi,\pi]$, $f(-\pi)=f(\pi)$, then $\sum_k |kc_k|^2 < \infty$, $\sum_k |c_k| < \infty$.

(3) If f is r times continuously differentiable, $f^{(r)} \in \mathrm{PC}'[-\pi,\pi]$, $f^{(m)}(-\pi)=f^{(m)}(\pi)$, $0 \le m \le r$, then $\sum_k |k^{r+1}c_k|^2 < \infty$, $\sum_k |k^r c_k| < \infty$.

- *Fourier method for wave equation.* Homogeneous equation.

  Theorem: If $\varphi'' \in \mathrm{PC}'[0,\ell]$, $\varphi''(0) = \varphi''(\ell) = 0$ and $\psi' \in \mathrm{PC}'[0,\ell]$. Then the initial-boundary value problem

$$\begin{cases} u_{tt} = a^2u_{xx} & x \in (0,\ell), t > 0 \\ u(x,0) = \varphi(x) & x \in [0,\ell] \\ u_t(x,0) = \psi(x) & x \in [0,\ell] \\ u(0,t) = u(\ell,t) = 0 & t \ge 0 \end{cases}$$

  has a solution that can be represented as the sum of standing waves

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{\pi n a t}{\ell} + B_n \sin\frac{\pi n a t}{\ell}\right)\sin\frac{\pi n x}{\ell},$$

  where $A_n = \frac{2}{\ell}\int_0^{\ell}\varphi(x)\sin\frac{\pi n x}{\ell}dx$, $B_n = \frac{2}{\pi n a}\int_0^{\ell}\psi(x)\sin\frac{\pi n x}{\ell}dx$.

  Solution of the inhomogeneous equation.

- *Heat equation.* Initial and boundary conditions. Uniqueness of the solution.

  Theorem (maximum principle): If $u(x,t)$ is a continuous function in $[0,\ell]\times[0,T]$ that satisfies $u_t = a^2 u_{xx}$ in $(0,\ell)\times(0,T]$, then the maximum (minimum) of u on $[0,\ell]\times[0,T]$ is attained at $t=0$ or at $x\in\{0,\ell\}$.

  Solution of the homogeneous and inhomogeneous heat equations using Fourier method.

  Heat equation in three dimensions.

  Heat equation on $\mathbb{R}$. Solution using Fourier transform.

- *Laplace and Poisson equations.* Dirichlet's and Neumann's boundary value problems. Harmonic functions. Properties of harmonic functions:

  (1) If u is harmonic in Ω, then $\iint_{\partial\Omega} \frac{\partial u}{\partial n} dS = 0$.

  (2) (mean value property) If u is harmonic in Ω then for all $M_0\in\Omega$ and all $r>0$ such that $\overline{B(M_0,r)}\subset\Omega$,

  $$u(M_0) = \frac{1}{4\pi r^2}\iint_{S(M_0,r)} u dS.$$

  (3) (maximum principle) If u is continuous in $\overline{\Omega}$ and harmonic in Ω, then the maximum and minimum of u are attained at $\partial\Omega$.

  Theorem (uniqueness): Dirichlet's boundary value problem has at most one solution.

  Existence of solution to the Laplace equation for sufficiently symmetric domains:

  (1) In the disk $\Omega = B(0,R)$, $\Delta u = 0$, $u(R,\varphi)=\mu(\varphi)$. Fourier method. Poisson formula:

  $$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \mu(\theta)\,\frac{R^2-r^2}{R^2-2rR\cos(\varphi-\theta)+r^2}\,d\theta.$$

  Remark: The Poisson formula gives solution to the Dirichlet's problem in a disk for any continuous μ.

  (2) In the rectangle $[0,a]\times[0,b]$.

  Remark: For certain Ωs, the unique solution to the Laplace equation can be found using conformal maps.

7 Literature

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