RETAKE SOLUTIONS, 3 April 2017, 15:15 – 17:15

1. (4 points) Let $S = \{(x,y) \ : \ 0 < x \le 2, \ x^2 \le y \le x^3\}$. Compute
$$\iint_S \frac{4y}{x^5}\,dxdy\,.$$

Answer: $3 - 2\log 2$.

Solution. Since $x^2 > x^3$ when $x \in (0,1)$, $S = \{(x,y) : 1 \le x \le 2, \ x^2 \le y \le x^3\}$. By Fubini,
$$\iint_S \frac{4y}{x^5}\,dxdy = \int_1^2 dx \int_{x^2}^{x^3} \frac{4y}{x^5}\,dy = \int_1^2 \frac{4}{x^5}\frac{1}{2}(x^6 - x^4)dx = 3 - 2\log 2.$$

□

2. (4 points) For $a, b > 0$, let S be the subset of $\mathbb{R}^2$ bounded by the curve $x = a\sin t$, $y = b\sin 2t$, $t \in [0,\pi]$. Compute
$$\iint_S \frac{1}{x}\,dxdy\,.$$

Answer: πb.

Solution. The set S is parametrized by $x = ar\sin t$, $y = br\sin 2t$, with $0 \le t \le \pi$ and $0 \le r \le 1$. The Jacobian
$$\frac{\partial(x,y)}{\partial(r,t)} = \begin{vmatrix} a\sin t & b\sin 2t \\ ar\cos t & 2br\cos 2t \end{vmatrix} = -2abr\sin^3 t.$$

By the change of variables formula and using the fact that $\sin t \ge 0$ for all $t \in [0,\pi]$,
$$\iint_S \frac{1}{x}\,dxdy = \int_0^1 dr \int_0^\pi \frac{1}{ar\sin t} 2abr\sin^3 t\,dt = 2b\int_0^\pi \sin^2 t\,dt = \pi b.$$

□

3. (4 points) Let γ be the curve in $\mathbb{R}^3$ parametrized by $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = t$, $t \in [0,2]$. Compute the line integral
$$\int_\gamma z\,ds\,.$$

Answer: $2\sqrt{2}$.

Solution.

$$\int_\gamma z\,ds = \int_0^2 t\,\sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\,dt = \sqrt{2}\int_0^2 tdt = 2\sqrt{2}.$$

□

4. (4 points) Let $a, b, c > 0$. Let S be the boundary of the rectangular box $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, oriented outwards. Compute the surface integral

$$\iint_S x^2dydz + y^2dzdx + z^2dxdy\,.$$

Answer: $abc(a+b+c)$.

Solution. By the Gauss-Ostrogradsky formula,

$$\begin{aligned}\iint_S x^2dydz + y^2dzdx + z^2dxdy &= \iiint_V (2x+2y+2z)dxdydz \\ &= 2\int_0^a dx\int_0^b dy\int_0^c (x+y+z)dz = 2bc\int_0^a xdx + 2ac\int_0^b ydy + 2ab\int_0^c zdz \\ &= abc(a+b+c).\end{aligned}$$

Another approach is to use the definition of the surface integral. One splits S into 6 rectangles. For instance, if $S_1 = S\cap\{x=0\}$, then the unit normal to S_1 is $(-1,0,0)$, and $\iint_{S_1} x^2dydz + y^2dzdx + z^2dxdy = \iint_{S_1}(-x^2)dS = 0$, if $S_2 = S\cap\{x=a\}$, then the unit normal to S_2 is $(1,0,0)$, and $\iint_{S_2} x^2dydz + y^2dzdx + z^2dxdy = \iint_{S_2} x^2dS = a^2\text{Area}(S_2) = a^2bc$. The remaining 4 cases are similar. □

5. (4 points) Let γ be the curve on the intersection of the ellipsoid $3x^2+4y^2+20z^2 = 17$ and the plane $x+y=1$ oriented clockwise when viewed from the point $(0,0,0)$. Compute the line integral

$$\int_\gamma 2zdx + (x+y)dy + (x-y)dz\,.$$

Answer: 0.

Solution. Let $F = (2z, x+y, x-y)$. Let S be the surface $\{3x^2+4y^2+20z^2 \le 17\}\cap\{x+y=1\}$ oriented away from $(0,0,0)$. By Stokes' theorem, $\int_\gamma 2zdx + (x+y)dy + (x-y)dz = \iint_S \text{curl}(F)\cdot dS$. We compute $\text{curl}(F) = (-1,1,1)$. Furthermore, the unit normal to S is the same as the unit normal to the plane $x+y=1$, i.e., $n = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Since $\text{curl}(F)\cdot n = 0$, $\iint_S \text{curl}(F)\cdot dS = \iint_S(\text{curl}(F)\cdot n)\,dS = 0$. □

6. (4 points) For which values of $\alpha \ge 0$, the function $f(z) = |z|^\alpha$ is holomorphic on $\mathbb{C}$?

   Answer: $\alpha = 0$.

Solution. The function f only takes real values. A real valued function is holomorphic if and only if it is constant. This is the case only when $\alpha = 0$. $\square$

7. (4 points) Compute the integral

$$\oint\limits_{|z|=1} \frac{1}{z^2} e^z \, dz.$$

Answer: $2\pi i$.

Solution. We use the Cauchy formula for derivatives:

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \oint\limits_{|z-z_0|=\epsilon} \frac{f(z)}{(z-z_0)^{k+1}} \, dz.$$

In our case, $z_0 = 0$, $f(z) = e^z$, $k = 1$. Thus, $\oint\limits_{|z|=1} \frac{1}{z^2} e^z \, dz = 2\pi i \, (e^z)'|_{z=0} = 2\pi i$.

One could also use the residue theorem: $\oint\limits_{|z|=1} \frac{1}{z^2} e^z \, dz = 2\pi i \operatorname{Res}_{z=0} \left(\frac{1}{z^2} e^z \right) = 2\pi i$. $\square$

8. (4 points) Compute the integral

$$I = \int_{-\infty}^{+\infty} \frac{x}{1+x^2} \, e^{-ix} \, dx.$$

Answer: $-\pi i e^{-1}$.

Solution. Consider the complex conjugate of I. Since the polynomial $x^2 + 1$ does not have real roots, has degree strictly bigger than 1, and its only root with positive imaginary part is i, we have

$$\overline{I} = \int_{-\infty}^{+\infty} \frac{x}{1+x^2} \, e^{ix} \, dx = 2\pi i \operatorname{Res}_{z=i} \left(\frac{z}{1+z^2} \, e^{iz} \right) = 2\pi i \, \frac{1}{2} e^{-1} = \pi i e^{-1}.$$

Thus, $I = -\pi i e^{-1}$. $\square$

9. (4 points) Solve the initial value problem

$$u_t + t x u_x = 1, \quad u(0, x) = x.$$

Answer: $u(t, x) = t + x e^{-\frac{1}{2} t^2}$.

Solution. We use the method of characteristics. Each characteristic is a curve $(t(s), x(s))$ that satisfies the system of ODEs $\frac{dt}{ds} = 1$, $\frac{dx}{ds} = tx$. The general solution to this system is $t = s$, $x = Ce^{\frac{1}{2}s^2}$. This characteristic curve intersects the line $t = 0$ at the point $(0, C)$. In particular, the unique characteristic curve that passes through the point (t, x) intersects the line $t = 0$ at the point $(0, xe^{-\frac{1}{2}t^2})$.

On characteristics, the function $u(t(s), x(s))$ satisfies the ODE $\frac{du}{ds} = 1$, i.e., $u(t(s), x(s)) = s + u(0, C) = s + C$. By choosing the characteristic that passes through (t, x), we obtain $u(t, x) = t + xe^{-\frac{1}{2}t^2}$. $\square$

10. (4 points) Determine the type of the second order PDE:

$$x^3 u_{xx} + 2xyu_{xy} + y^3 u_{yy} + x^2 u_x + y^2 u_y + 3u = 0.$$

Answer: Parabolic if $x = 0$ or $y = 0$ or $xy = 1$, elliptic if $xy > 1$, hyperbolic if $xy < 1$.

Solution. If the main part of the second order PDE is $a(x, y, u)u_{xx} + 2b(x, y, u)u_{xy} + c(x, y, u)u_{yy}$ and $\Delta(x, y) = \begin{vmatrix} a & b \\ b & c \end{vmatrix}$, then the PDE is elliptic at (x, y) if $\Delta(x, y) > 0$, parabolic if $\Delta(x, y) = 0$ and hyperbolic if $\Delta(x, y) < 0$. In our case $\Delta(x, y) = \begin{vmatrix} x^3 & xy \\ xy & y^3 \end{vmatrix} = (xy)^2(xy - 1)$. Thus, on the lines $x = 0$ or $y = 0$ and on the hyperbola $xy = 1$ the PDE is parabolic, if $xy > 1$ it is elliptic and if $xy < 1$ hyperbolic. $\square$

11. (4 points) Solve the initial value problem

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < 1, t > 0 \\ u(0, t) = u(1, t) = 0 & t \geq 0 \\ u(x, 0) = \sin \pi x & 0 \leq x \leq 1 \\ u_t(x, 0) = \sin \pi x \cos \pi x & 0 \leq x \leq 1. \end{cases}$$

Answer: $u(t, x) = \cos \pi t \sin \pi x + \frac{1}{4\pi} \sin 2\pi t \sin 2\pi x$.

Solution. We use the separation of variables method. We first find all non-trivial solutions to the PDE which satisfy the boundary condition and have the form $u(t, x) = T(t)X(x)$. From the PDE, the functions T and X must satisfy $\frac{T''}{T} = \frac{X''}{X}$. Since we are interested in non-trivial solutions, there must exist $\lambda \in \mathbb{R}$, such that $\frac{T''}{T} = \frac{X''}{X} = \lambda$. Boundary condition further gives that $X(0) = X(1) = 0$. The Sturm-Liouville problem $X'' - \lambda X = 0$, $X(0) = X(1) = 0$, has non-trivial solutions only if $\lambda = -(\pi n)^2$, $n \in \mathbb{N}$, giving $X_n(x) = \sin \pi n x$. The general solution to $T'' + (\pi n)^2 T = 0$ is $T_n(t) = A_n \sin \pi n t + B_n \cos \pi n t$.

Next we identify such A_n and B_n for which the sum of all the found solutions satisfies the initial conditions. We write $u(t, x) = \sum_{n=1}^{\infty} (A_n \sin \pi n t + B_n \cos \pi n t) \sin \pi n x$. Since $u(0, x) = \sum_{n=1}^{\infty} B_n \sin \pi n x = \sin \pi x$, we get $B_1 = 1$ and $B_n = 0$ for all $n \geq 2$. Since $u_t(0, x) = \sum_{n=1}^{\infty} A_n \pi n \sin \pi n x = \sin \pi x \cos \pi x = \frac{1}{2} \sin 2\pi x$, we get $A_2 = \frac{1}{4\pi}$ and $A_n = 0$ for all $n \neq 2$. Thus, $u(t, x) = \cos \pi t \sin \pi x + \frac{1}{4\pi} \sin 2\pi t \sin 2\pi x$. $\square$