EXERCISES, Week 8 (submit by 05.12.2016)

1. Find the real and imaginary parts of

$$\text{(a) } \cos(2+i) \qquad \text{(b) } \log i \qquad \text{(c) } 2^i \qquad \text{(d) } i^i$$

2. Let $a, b \in \mathbb{C}$. Which of the following sets are equal?

$$a^{2b}, \quad (a^b)^2, \quad (a^2)^b$$

3. Prove that for each $z \in \mathbb{C}$, $\lim\limits_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z$.

   Hint: Use the fact that a sequence of complex numbers $z_n = r_n e^{i\varphi_n}$ converges to $z = re^{i\varphi}$ as $n \to \infty$ if and only $r_n \to r$ and $\varphi_n \to \varphi$ as $n \to \infty$.

4. For which $a \in \mathbb{C}$ the following function is continuous at 0?

$$f(z) = \begin{cases} \frac{\text{Re}z}{z} & \text{if } z \neq 0 \\ a & \text{if } z = 0. \end{cases}$$

5. For $a, b, c \in \mathbb{R}$ and $z = x + iy$, let $f(z) = x + ay + i(bx + cy)$. For which values of a, b, c the function f is holomorphic?

6. Check that $f(z) = \cos z$ satisfies the Cauchy-Riemann conditions and show that $(\cos z)' = -\sin z$.

7. Prove that the funciton $f(z) = \overline{z}$ is nowhere differentiable.

8. Is the function $f(z) = z\text{Re}z$ differentiable at $z = 0$? Is it holomorphic at $z = 0$?

9. For $z = x + iy$, let $f(z) = \sqrt{|xy|}$. Prove that f satisfies the Cauchy-Riemann conditions at $z = 0$, but f is not differentiable at $z = 0$.

10. Let $f(z) = z^2$.

    (a) Determine the angle of rotation of the complex plane by f at the point $z = 1+i$.

    (b) Which part of the complex plane is stretched and which is contracted by f?