EXERCISES, Week 7 (submit by 28.11.2016)

1. Use the Gauss-Ostrogradsky theorem to compute the following surface integrals.

   (a) $\iint\limits_S (1+2x)dydz + (2x+3y)dzdx + (3y+4z)dxdy$, where S is the outer side of the boundary of the pyramid $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \le 1$, $x \ge 0$, $y \ge 0$, $z \ge 0$.
   (Here a, b, c are arbitrary positive numbers.)

   (b) $\iint\limits_S zdxdy + (5x+y)dydz$, where S is the inner side of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$.

   (c) $\iint\limits_S x^4 dydz + y^4 dzdx + z^4 dxdy$, where S is the outer side of the unit sphere $x^2 + y^2 + z^2 = 1$.

   (d) $\iint\limits_S x^2 dydz + y^2 dzdx + z^2 dxdy$, where S is the lower side of the semisphere $x^2 + y^2 + z^2 = R^2$, $z \ge 0$.

2. Let S be the smooth boundary of a solid object in $\mathbb{R}^3$ and ν its outward unit normal. Let ℓ be a fixed vector of $\mathbb{R}^3$. Prove that $\iint\limits_S \cos(\widehat{\ell, \nu})\, dS = 0$.

   (Here $\widehat{\ell, \nu}$ denotes the angle between ℓ and ν.)

3. Use the Stokes theorem to compute the following line integrals.

   (a) $\int\limits_\gamma (x+z)dx + (x-y)dy + xdz$, where γ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 1$ oriented counter-clockwise with respect to the point $(0,0,0)$.

   (b) $\int\limits_\gamma y^2 dx + z^2 dy + x^2 dz$, where γ is the triangle with vertices $(a,0,0)$, $(0,a,0)$, $(0,0,a)$ oriented clockwise with respect to the point $(0,0,0)$.

   (c) $\int\limits_\gamma ydx + zdy + xdz$, where γ is the circle $x^2 + y^2 + z^2 = R^2$, $x + y + z = 0$ oriented counter-clockwise with respect to the point $(0,0,1)$.

   (d) $\int\limits_\gamma z^2 dy + x^2 dz$, where γ is the curve $y^2 + z^2 = 9$, $4x + 3z = 5$ oriented clockwise with respect to the point $(0,0,0)$.

4. (a) Let u and v be scalar fields on $\mathbb{R}^3$. Let $F = u\nabla v$. Prove that $\nabla \times F$ is orthogonal to F.

   (b) Let F be an arbitrary vector field on $\mathbb{R}^3$. Is $\nabla \times F$ orthogonal to F?