EXERCISES, Week 5 (submit by 14.11.2016)

1. Use Green's formula to compute $I = \int_\gamma P\,dx + Q\,dy$.

   (a) $I = \int_\gamma (3x^2 - y)\,dx + (1-2x)\,dy$, where γ is the positively oriented boundary of the triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$.

   (b) $I = \int_\gamma \frac{x\,dy + y\,dx}{x^2+y^2}$, where γ is the circle $(x-1)^2 + (y-1)^2 = 1$ oriented counter-clockwise.

   (c) $I = \int_\gamma (e^x \sin y - y)\,dx + (e^x \cos y - 1)\,dy$, where γ is the positively oriented boundary of the set $\{(x,y) \mid x^2 + y^2 < ax,\ y > 0\}$. (Here $a > 0$.)

2. Let γ be a curve connecting A to B. Compute $I = \int_\gamma P\,dx + Q\,dy$ by identifying a scalar potential for (P,Q).

   (a) $I = \int_\gamma x\,dy + y\,dx$, $A = (-1,3)$, $B = (2,2)$.

   (b) $I = \int_\gamma x\,dx + y\,dy$, $A = (-1,0)$, $B = (-3,4)$.

   (c) $I = \int_\gamma \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$, A lies on the sphere $x^2+y^2+z^2 = r^2$, B lies on the sphere $x^2+y^2+z^2 = R^2$.

3. Find the area bounded by the curve

   (a) $x = 12\sin^3 t$, $y = 3\cos^3 t$, $t \in [0, 2\pi]$.

   (b) $x = \frac{3t}{1+t^3}$, $y = \frac{3t^2}{1+t^3}$, $t > 0$.

4. Let S be a subset of $\mathbb{R}^2$ allowed by Green's formula. Let u, v be twice continuously differentiable functions with $v(x,y) = 0$ for all $(x,y) \in \partial S$. Let $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. Prove that

$$\iint_S v\Delta u\,dxdy = -\iint_S \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\,dxdy\,.$$

   (The integral on the right hand side may also be written as $\iint_S \nabla u \cdot \nabla v\,dxdy$.)