EXERCISES, Week 14 (submit by 30.01.2017)

1. Let $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ be the unique solutions to initial-boundary value problems

$$\begin{cases} u_t^{(i)} = a^2 u_{xx}^{(i)} + f^{(i)}(x,t) & 0 < x < l, t > 0 \\ u^{(i)}(x,0) = \varphi^{(i)}(x) & 0 \le x \le l \\ u^{(i)}(0,t) = \alpha^{(i)}(t) & t \ge 0 \\ u^{(i)}(l,t) = \beta^{(i)}(t) & t \ge 0. \end{cases}$$

Assume that $\varphi^{(1)}(x) \ge \varphi^{(2)}(x)$ for all $0 \le x \le l$, $\alpha^{(1)}(t) \ge \alpha^{(2)}(t)$, $\beta^{(1)}(t) \ge \beta^{(2)}(t)$ for all $t \ge 0$.

   (a) Assume that $f^{(1)}(x,t) = f^{(2)}(x,t)$ for all $0 \le x \le l$, $t \ge 0$. Prove that $u^{(1)}(x,t) \ge u^{(2)}(x,t)$ for all $0 \le x \le l$, $t \ge 0$.

   (b) Assume that $f^{(1)}(x,t) \ge f^{(2)}(x,t)$ for all $0 \le x \le l$, $t \ge 0$. Prove that $u^{(1)}(x,t) \ge u^{(2)}(x,t)$ for all $0 \le x \le l$, $t \ge 0$.

2. Let $u(x,t)$ be a solution to the heat equation $u_t = a^2 u_{xx}$. Prove that for any $\alpha \in \mathbb{R}$, $v(x,t) = u(\alpha x, \alpha^2 t)$ is also a solution to this equation. Find the general solution to the heat equation which satisfies $u(x,t) = u(\alpha x, \alpha^2 t)$ for all $\alpha > 0$.

   [Hint: Such a solution has the form $u(x,t) = f(\frac{x}{\sqrt{t}})$ for a suitable f. Introduce the new variable $z = \frac{x}{\sqrt{t}}$ and find $f(z)$.]

3. Use the Fourier method to solve the following initial-boundary value problems.

   (a)
$$\begin{cases} u_{tt} = 4u_{xx} & 0 < x < 1, t > 0 \\ u(x,0) = \sin \pi x & 0 \le x \le 1 \\ u_t(x,0) = 0 & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & t \ge 0. \end{cases}$$

   (b)
$$\begin{cases} u_{tt} = a^2 u_{xx} + \sin \omega t & 0 < x < 1, t > 0 \\ u(x,0) = u_t(x,0) = 0 & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & t \ge 0. \end{cases}$$

   (c)
$$\begin{cases} u_t = u_{xx} & 0 < x < 1, t > 0 \\ u(x,0) = \sin \pi x & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & t \ge 0. \end{cases}$$

   (d)
$$\begin{cases} u_t = u_{xx} + t & 0 < x < 1, t > 0 \\ u(x,0) = 0 & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & t \ge 0. \end{cases}$$