EXERCISES, Week 13 (submit by 23.01.2017)

1. Prove that there exist at most one twice continuously differentiable solution $u(x,t)$, $x \in [0,l]$, $t \geq 0$, to

$$\begin{cases} \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) + f & 0 < x < l, t > 0 \\ u(x,0) = \varphi(x) & 0 \leq x \leq l \\ \frac{\partial u}{\partial t}(x,0) = \psi(x) & 0 \leq x \leq l \\ \frac{\partial u}{\partial x}(0,t) = \mu_1(t) & t \geq 0 \\ \frac{\partial u}{\partial x}(l,t) = \mu_2(t) & t \geq 0, \end{cases}$$

where ρ and k are continuous functions of x.

2. Let φ be twice continuously differentialbe on $\mathbb{R}$, ψ continuously differentialbe on $\mathbb{R}$, and f continuous on $\mathbb{R} \times [0,\infty)$. Prove that the unique twice continuously differentiable solution $u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, to

$$\begin{cases} u_{tt} = a^2 u_{xx} + f \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

is given by the (*d'Alembert*'s) formula

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \psi(y)dy + \frac{1}{2a} \int\limits_0^t \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau) d\xi\, d\tau\,.$$

3. Consider the problem for $x \in \mathbb{R}$, $t \geq 0$, with φ and ψ as in problem 2,

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Prove that its unique solution $u(x,t)$ satisfies the following:

   (a) if for some $x_0 \in \mathbb{R}$, $\varphi(x_0 + x) = -\varphi(x_0 - x)$ and $\psi(x_0 + x) = -\psi(x_0 - x)$ for all x (i.e., φ and ψ are *odd* with respect to x_0), then $u(x_0, t) = 0$ for all $t > 0$,

   (b) if for some $x_0 \in \mathbb{R}$, $\varphi(x_0 + x) = \varphi(x_0 - x)$ and $\psi(x_0 + x) = \psi(x_0 - x)$ for all x (i.e., φ and ψ are *even* with respect to x_0), then $u_x(x_0, t) = 0$ for all $t > 0$.

4. Find solutions to the following problems:

   (a) $u_{tt} = 9u_{xx} + \sin x$, $x \in \mathbb{R}$, $t > 0$, $u(x,0) = 1$, $u_t(x,0) = 1$

   (b) $u_{tt} = u_{xx} + \cos x$, $x \geq 0$, $t > 0$, $u(x,0) = \cos x$, $u_t(x,0) = 0$, $u_x(0,t) = 0$