EXERCISES, Week 1 (submit by 18.10.2016)

1. Prove that for any bounded $S_1 \subseteq S_2$,

$$\text{(a)}\quad \mu^*(S_2 \setminus S_1) \le \mu^*(S_2) - \mu_*(S_1),$$
$$\text{(b)}\quad \mu_*(S_2 \setminus S_1) \ge \mu_*(S_2) - \mu^*(S_1).$$

2. Prove that any rectifiable curve in $\mathbb{R}^2$ has Jordan measure 0.

3. Let S_1, S_2 be Jordan measurable sets. Prove that $S_1 \cup S_2$ and $S_1 \cap S_2$ are Jordan measurable.

4. Let $S_1 \subseteq S_2$ be Jordan measurable. Prove that $S_2 \setminus S_1$ is Jordan measurable and $\mu(S_2 \setminus S_1) = \mu(S_2) - \mu(S_1)$.

5. Prove that for any S in $\mathbb{R}^n$,

$$\text{(a)}\quad \mu_*(S) = \sup_{S' \subseteq S,\,\text{meas.}} \mu(S'),$$
$$\text{(b)}\quad \mu^*(S) = \inf_{S' \supseteq S,\,\text{meas.}} \mu(S'),$$

where the sup is taken over all measurable S' contained in S, and the inf over all measurable S' that contain S.

6. Let g be an integrable function on $[a, b]$ and h an integrable function on $[c, d]$ so that $\int_a^b g(z)dz = \int_c^d h(z)dz = 1$. Let $S = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$. Use the definition of Riemann integral to compute $\int_S f(x)dx$ if

$$\text{(a)}\quad f(x) = g(x_1) + h(x_2), \qquad \text{(b)}\quad f(x) = g(x_1)\, h(x_2).$$