EXAM SOLUTIONS, 21 February 2017, 10:00 – 12:00

1. (4 points) Compute the iterated integral $\int_0^\pi dx \int_x^\pi \frac{\sin y}{y}\, dy$.

   Answer: 2.

   Solution. Since $\{(x,y) : 0 \le x \le \pi, x \le y \le \pi\} = \{(x,y) : 0 \le y \le \pi, 0 \le x \le y\}$,

$$\int_0^\pi dx \int_x^\pi \frac{\sin y}{y}\, dy = \int_0^\pi dy \int_0^y \frac{\sin y}{y}\, dx = \int_0^\pi \left[\frac{\sin y}{y} \cdot y\right] dy = \int_0^\pi \sin y\, dy = 2.$$

□

2. (4 points) Let γ be the curve $\{(x,y) : x^2 + y^2 = 1,\, y \ge 0\}$. Compute $\int_\gamma x^2\, ds$.

   Answer: $\frac{\pi}{2}$.

   Solution. Parametrize the curve γ by $x = \cos t$, $y = \sin t$, $t \in [0, \pi]$. Then,

$$\int_\gamma x^2\, ds = \int_0^\pi x(t)^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^\pi \cos^2 t\, dt = \int_0^\pi \frac{1 + \cos 2t}{2}\, dt = \frac{\pi}{2}.$$

□

3. (4 points) Let γ be the curve $y = \log_2(1 + x)$, $0 \le x \le 1$, connecting points $(0,0)$ and $(1,1)$ in $\mathbb{R}^2$. Compute the line integral $\int_\gamma xdx + ydy$.

   Answer: 1.

   Solution. Note that the vector field (x,y) is irrotational, thus conservative. In particular, if γ_1 is the line segment connecting $(0,0)$ and $(1,1)$, then

$$\int_\gamma xdx + ydy = \int_{\gamma_1} xdx + ydy = \int_0^1 2xdx = 1.$$

   One could also notice that $(x,y) = \nabla\varphi(x,y)$, where $\varphi(x,y) = \frac{1}{2}(x^2 + y^2)$. Thus, $\int_\gamma xdx + ydy = \varphi(1,1) - \varphi(0,0) = 1$. □

4. (4 points) Let S be the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Compute $\iint_S \frac{1}{\sqrt{x^2+y^2}}\, dS$.

   Answer: $2\pi^2$.

   Solution. Consider parametrization of the unit sphere by spherical coordinates: $x = \cos\varphi\cos\psi$, $y = \sin\varphi\cos\psi$, $z = \sin\psi$, $\varphi \in [0, 2\pi]$, $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The corresponding tangent vectors are

$$\begin{aligned} r_\varphi &= (x_\varphi,\, y_\varphi,\, z_\varphi) &= (-\sin\varphi\cos\psi,\, \cos\varphi\cos\psi,\, 0) \\ r_\psi &= (x_\psi,\, y_\psi,\, z_\psi) &= (-\cos\varphi\sin\psi,\, -\sin\varphi\sin\psi,\, \cos\psi)\,. \end{aligned}$$

Thus, the coefficients of the first fundamental form of the sphere are

$$E = r_\varphi^2 = \cos^2\psi, \quad G = r_\psi^2 = 1, \quad F = r_\varphi r_\psi = 0.$$

Therefore, $\sqrt{EG - F^2} = \cos\psi$. Since also $\sqrt{x^2+y^2} = \cos\psi$, we have

$$\iint_S \frac{1}{\sqrt{x^2+y^2}}\, dS = \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cos\psi} \cos\psi\, d\psi = 2\pi^2.$$

□

5. (4 points) Let $a, b, c > 0$. Let S be the outer side of the cone $0 \le z \le c$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le \frac{z^2}{c^2}$. Compute the surface integral $\iint_S x^2 dydz + y^2 dzdx + z^2 dxdy$.
   Answer: $\frac{1}{2}\pi abc^2$.

   Solution. Let V be the volume surrounded by the surface S and $F = (x^2, y^2, z^2)$. By the Gauss-Ostrogradsky formula,

$$\iint_S x^2 dydz + y^2 dzdx + z^2 dxdy = \iiint_V \operatorname{div} F\, dxdydz = 2 \iiint_V (x+y+z)\, dxdydz.$$

   Consider the change of variables $(x,y,z) \mapsto (r, \varphi, t)$, $x = ar\cos\varphi$, $y = br\sin\varphi$, $z = ct$. The Jacobian is $\begin{vmatrix} a\cos\varphi & -ar\sin\varphi & 0 \\ b\sin\varphi & br\cos\varphi & 0 \\ 0 & 0 & c \end{vmatrix} = abcr$, and in the new variables, $V = \{(r,\varphi,t) \ :\ 0 \le t \le 1,\ 0 \le \varphi \le 2\pi,\ 0 \le r \le t\}$. Thus,

$$\begin{aligned} 2\iiint_V (x+y+z)\, dxdydz &= 2\int_0^{2\pi} d\varphi \int_0^1 dt \int_0^t dr\, (ar\cos\varphi + br\sin\varphi + ct)\ abcr \\ &= 2\int_0^{2\pi} d\varphi \int_0^1 dt \int_0^t (ct\, abcr)\, dr = \frac{1}{2}\pi abc^2. \end{aligned}$$

   The second equality follows from the observation $\int_0^{2\pi} \cos\varphi d\varphi = \int_0^{2\pi} \sin\varphi d\varphi = 0$. □

6. (4 points) Let γ be the unit circle $x^2 + y^2 = 1$, $z = 2$ in $\mathbb{R}^3$, oriented clockwise with respect to the point $(0,0,0)$. Compute the line integral $\int_\gamma yzdx + 2zxdy + 21xydz$.
   Answer: 2π.

   Proof. Although the integral can be calculated by the definition, it is simpler to use Stokes' formula. Let S be the upper side of the unit disc $x^2 + y^2 \le 1$, $z = 2$, and $F = (P, Q, R) = (yz, 2zx, 21xy)$. Note that S is oriented by the unit normal $n = (0,0,1)$, and the orientations of S and γ agree. Thus, by Stokes' formula

$$\begin{aligned} \int_\gamma yzdx + 2zxdy + 21xydz = \iint_S (\operatorname{curl} F \cdot n)\, dS &= \iint_S (Q_x - P_y)\, dS \\ &= \iint_S z dS = 2\operatorname{Area}(S) = 2\pi. \end{aligned}$$

□

7. (4 points) Let $f(z) = |z|^2$, $z \in \mathbb{C}$. Is f differentiable at 0? Is f holomorphic at 0?

Answer: Differentiable, but not holomorphic.

Solution. We first note that f is differentiable at $z = 0$ and $f'(0) = \lim_{z\to 0} \frac{|z|^2}{z} = 0$. Next, we show that f is not differentiable for any $z \neq 0$, in particular, f is not holomorphic at $z = 0$. We do this by showing that f satisfies the Cauchy-Riemann conditions if and only if $z = 0$. Let $z = x + iy$ and $f = u + iv$, where $u = x^2 + y^2$, $v = 0$. Note that $u_x = 2x$, $u_y = 2y$, $v_x = v_y = 0$. Thus, the Cauchy-Riemann conditions ($u_x = v_y$, $u_y = -v_x$) are satisfied if and only if $x = y = 0$. □

8. (4 points) Let γ be the contour consisting of the upper half of the circle $|z| = 1$ and the line segment $-1 \leq x \leq 1$, $y = 0$. Compute the integral $\oint_\gamma |z|^2\overline{z}\, dz$.

Answer: πi.

Solution. Let γ_1 be the line segment from -1 to 1, and γ_2 the upper half of the circle $|z| = 1$ oriented from 1 to -1. Consider parametrizations $\gamma_1 = \{z(x) = x : -1 \leq x \leq 1\}$ and $\gamma_2 = \{z(t) = e^{it} : 0 \leq t \leq \pi\}$. Then,

$$\int_{\gamma_1} |z|^2\overline{z}\, dz = \int_{-1}^{1} x^3 dx = 0, \qquad \int_{\gamma_2} |z|^2\overline{z}\, dz = \int_0^\pi e^{-it}\, e^{it}\, idt = \pi i.$$

Finally, $\oint_\gamma |z|^2\overline{z}\, dz$, as the sum of the above two integrals, equals πi. □

9. (4 points) Compute the integral $\oint\limits_{|z|=2} (1+z)\left(e^{\frac{1}{z}} + e^{\frac{1}{z-1}}\right) dz$.

Answer: $8\pi i$.

Solution. We will use the residue theorem. Consider separately $\oint\limits_{|z|=2} (1+z)\, e^{\frac{1}{z}}\, dz$ and $\oint\limits_{|z|=2} (1+z)\, e^{\frac{1}{z-1}}\, dz$. By the residue theorem,

$$\oint\limits_{|z|=2} (1+z)\, e^{\frac{1}{z}}\, dz = 2\pi i \operatorname{Res}_0 (1+z)\, e^{\frac{1}{z}}, \qquad \oint\limits_{|z|=2} (1+z)\, e^{\frac{1}{z-1}}\, dz = 2\pi i \operatorname{Res}_1 (1+z)\, e^{\frac{1}{z-1}}.$$

The point $z = 0$ is an essential isolated singularity for the function $(1+z)e^{\frac{1}{z}}$. We find its residue at $z = 0$ by expanding $(1+z)e^{\frac{1}{z}}$ in the Laurent series at $z = 0$:

$$(1+z)e^{\frac{1}{z}} = (1+z)\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \dots\right) = z + 2 + \frac{3}{2}\frac{1}{z} + \dots$$

We conclude that $\operatorname{Res}_0 (1+z)\, e^{\frac{1}{z}} = \frac{3}{2}$.

Similarly, $z = 1$ is an essential isolated singularity for the function $(1+z)\, e^{\frac{1}{z-1}}$. We find its residue at $z = 1$ by expanding $(1+z)e^{\frac{1}{z-1}}$ in the Laurent series at $z = 1$:

$$(1+z)e^{\frac{1}{z-1}} = (2+(z-1))\left(1 + \frac{1}{z-1} + \frac{1}{2!}\frac{1}{(z-1)^2} + \dots\right) = (z-1) + 3 + \frac{5}{2}\frac{1}{z-1} + \dots$$

We conclude that $\operatorname{Res}_1 (1+z)\, e^{\frac{1}{z-1}} = \frac{5}{2}$.

Summing up, $\oint\limits_{|z|=2} (1+z)\left(e^{\frac{1}{z}} + e^{\frac{1}{z-1}}\right) dz = 2\pi i \left(\frac{3}{2} + \frac{5}{2}\right) = 8\pi i.$ $\square$

10. (4 points) Solve the initial value problem

$$\begin{cases} u_{tt} = u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = x & x \in \mathbb{R}. \end{cases}$$

Answer: xt.

Solution. Apply d'Alembert's formula $u(x,t) = \frac{\varphi(x+t)+\varphi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)dy$ to $\varphi(y) = 0$, $\psi(y) = y$. Or, guess $u(x,t) = xt$ as a solution and use uniqueness. $\square$

11. (4 points) Solve the initial-boundary value problem

$$\begin{cases} u_t = u_{xx} - 4u & 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0 & t \geq 0 \\ u(x,0) = \sin \pi x - \sin 2\pi x & 0 \leq x \leq 1. \end{cases}$$

Answer: $e^{-(4+\pi^2)t} \sin \pi x - e^{-4(1+\pi^2)t} \sin 2\pi x$.

Solution. We use separation of variables. Let $u(x,t) = X(x)T(t)$. Then $XT' = X''T - 4XT$, which implies that for some $\lambda = \text{const}$, $\frac{X''}{X} = \frac{T'+4T}{T} = \lambda$. Taking into account the boundary condition, X solves the Sturm-Liouville problem $\begin{cases} X'' - \lambda X = 0 \\ X(0) = X(1) = 0. \end{cases}$ Non-trivial solutions to this problem exist if and only if $\lambda = -(\pi n)^2$, in which case $X(x) = \sin \pi n x$. For such λ, the general T that satisfies $T' + 4T = -(\pi n)^2 T$ is $T(t) = A\, e^{-(4+(\pi n)^2)t}$. We obtain the candidate for solution in the form of an infinite series

$$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{-(4+(\pi n)^2)t} \sin \pi n x,$$

where A_n are still to be determined so that u fulfills the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \pi n x = \sin \pi n x - \sin 2\pi n x.$$

Thus, $A_1 = 1$, $A_2 = -1$, and $A_n = 0$ for all $n \geq 3$. We conclude that the solution is $u(x,t) = e^{-(4+\pi^2)t} \sin \pi x - e^{-4(1+\pi^2)t} \sin 2\pi x$.

$\square$