RETAKE SOLUTIONS, 6 April 2018, 10:00 – 12:00

1. Find the area bounded by the curves $y = x$, $y = 2x$, $y = x^2$.

   Answer: $\frac{7}{6}$.

   Solution. Denote the given set by S. We change variables by $u = \frac{y}{x}$, $v = x$, then the set S is parametrized as $1 \le u \le 2$, $0 \le v \le u$. The inverse change of variables is $x = v$, $y = uv$. Thus, the Jacobian equals $\begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = -v$, and the area is

$$\iint_S dS = \int_1^2 du \int_0^u vdv = \int_1^2 \frac{u^2}{2} dy = \frac{7}{6}.$$

   Another way would be to note that S is normal and use Fubini's theorem. □

2. Let γ be the circle given by the equations $x^2 + y^2 + z^2 = 1$, $y = x$. Compute the line integral

$$\int_\gamma \frac{ds}{\sqrt{2y^2 + z^2}}.$$

   Answer: 2π.

   Solution. Note that on γ, $2y^2 + z^2 = x^2 + y^2 + z^2 = 1$. Thus, the integral equals to the length of γ. Since γ is a great circle of the unit sphere, its length is 2π.

   One could also solve this problem by finding an explicit parametrization of γ, for instance, $x = y = \frac{1}{\sqrt{2}} \cos\varphi$, $z = \sin\varphi$, where $0 \le \varphi \le 2\pi$. Then, $2y^2 + z^2 = 1$, and the integral equals to

$$\int_0^{2\pi} \sqrt{(x')^2 + (y')^2 + (z')^2} d\varphi = \int_0^{2\pi} \sqrt{(\frac{-\sin\varphi}{\sqrt{2}})^2 + (\frac{-\sin\varphi}{\sqrt{2}})^2 + (\cos\varphi)^2} d\varphi = 2\pi.$$

   □

3. Let γ be the curve $y = \log x$, $1 \le x \le e$. Compute the line integral $\int_\gamma \frac{y}{x} dx + dy$.

   Answer: $\frac{3}{2}$.

   Solution. By the definition of line integral,

$$\int_\gamma \frac{y}{x} dx + dy = \int_1^e \left(\frac{\ln x}{x} 1 + 1 \frac{1}{x} \right) dx = \frac{1}{2} (\ln x + 1)^2 \big|_1^e = \frac{3}{2}.$$

   □

4. Let S be the surface given by $z = \sqrt{x^2 + y^2}$, $z \le 1$. Compute the surface integral

$$\iint_S (x^2 + y^2) dS.$$

   Answer: $\frac{\pi}{\sqrt{2}}$.

Solution. Consider the parametrization of S: $x = u\cos v$, $y = u\sin v$, $z = u$, with $0 \le v \le 2\pi$, $0 \le u \le 1$. We first find the coefficients of the first fundamental form of S in this parametrization. The basis of the tangent plane is $r_u = (\cos v, \sin v, 1)$ and $r_v = (-u\sin v, u\cos v, 0)$. Thus, $E = 2$, $F = 0$, $G = u^2$, and $\sqrt{EG - F^2} = u\sqrt{2}$. By the definition of surface integral and the Fubini theorem,

$$\iint_S (x^2 + y^2) dS = \int_0^{2\pi} dv \int_0^1 u^2\, u\sqrt{2}\, du = \frac{\pi}{\sqrt{2}}.$$

□

5. Let S be the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ oriented inward. Let $F = (x + y, 0, x + z)$. Compute the surface integral $\iint_S F \cdot dS$.

   Answer: -16π.

   Solution. The problem can be solved directly by using the definition of the surface integral, however it is simpler to use the Gauss-Ostrogradsky theorem. Indeed, let V be the volume surrounded by S. Then,

$$\iint_S F \cdot dS = -\iiint_V \operatorname{div} F\, dV = -2 \iiint_V dV.$$

   (Here we put a minus sign, since S is oriented inward.) Now, the volume of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ is $\frac{4}{3}\pi abc$, and the above integral equals to $-2\,\frac{4}{3}\pi\, 2\, 3\, 1 = -16\pi$.

   One can, of course, verify the volume formula directly by using the generalized spherical coordinates: $x = 2r\cos\varphi\sin\psi$, $y = 3r\sin\varphi\sin\psi$, $z = r\cos\psi$, with $0 \le \varphi \le 2\pi$, $0 \le \psi \le \pi$. Indeed, the Jacobian equals to $-6r^2\sin\psi$, so

$$-2\iiint_V dV = -2\int_0^{2\pi} d\varphi \int_0^{\pi} d\psi \int_0^1 \left(6r^2\sin\psi\right) dr = -16\pi.$$

□

6. Let γ be the boundary of the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 2)$, positively oriented with respect to the vector $(0, 1, 0)$. Let $F = (y, z, x)$. Use Stokes' theorem to compute the line integral $\int_\gamma F \cdot ds$.

   Answer: $-\frac{5}{2}$.

   Solution. Let S be the given triangle, positively oriented with respect to the vector $(0, 1, 0)$, i.e., the upper side of the triangle. By Stokes' theorem, $\int_\gamma F \cdot ds = \iint_S (\operatorname{curl} F \cdot n) dS$, where n is the unit normal to S.

   We first compute the curl, $\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(1, 1, 1)$. To find the normal n, denote the vertices of the triangle by $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 2)$

and let $a = \overrightarrow{AB} = (-1,1,0)$, $b = \overrightarrow{AC} = (-1,0,2)$. Then $n = \frac{a\times b}{\|a\times b\|}$. Since $a\times b = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix} = (2,2,1)$, the unit normal is $n = \left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$. Thus,

$$\iint_S (\mathrm{curl} F\cdot n)dS = -\frac{5}{3}\iint_S dS = -\frac{5}{3}\frac{\|a\times b\|}{2} = -\frac{5}{2}.$$

$\square$

7. Is the function $f(z) = \begin{cases} z^2\sin\frac{1}{z} & z\neq 0 \\ 0 & z=0 \end{cases}$ holomorphic at 0? Give a proof.

   Answer: no.

   Solution. One way to solve this problem is to show that f is not even continuous at 0. Indeed,

$$f\left(\frac{i}{n}\right) = \frac{-1}{n^2}\sin(-in) = \frac{-1}{n^2}\frac{1}{2i}\left(e^n - e^{-n}\right) \not\to 0.$$

   Another way is to use that $z=0$ is a zero of f. Indeed, if f is holomorphic at 0, then 0 must be an *isolated* zero of f, but it is not, since $f(\frac{1}{\pi n}) = \frac{1}{(\pi n)^2}\sin(\pi n) = 0$ and $\frac{1}{\pi n}\to 0$ as $n\to\infty$. $\square$

8. Compute the integral $\frac{1}{2\pi i}\oint\limits_{|z-i|=1}\frac{dz}{z^2-i}$.

   Answer: $\frac{1-i}{2\sqrt{2}}$.

   Solution. First write $z^2 - i = \left(z - e^{\frac{\pi i}{4}}\right)\left(z + e^{\frac{\pi i}{4}}\right)$. Note that the circle $|z-i|=1$ surrounds the point $e^{\frac{\pi i}{4}}$, but not the point $-e^{\frac{\pi i}{4}}$. Thus, by Cauchy's integral formula,

$$\frac{1}{2\pi i}\oint\limits_{|z-i|=1}\frac{dz}{z^2-i} = \frac{1}{z+e^{\frac{\pi i}{4}}}\Big|_{z=e^{\frac{\pi i}{4}}} = \frac{1}{2}e^{-\frac{\pi i}{4}} = \frac{1-i}{2\sqrt{2}}.$$

$\square$

9. Compute the integral $\frac{1}{2\pi i}\oint\limits_{|z|=1} z\cos\frac{1}{z}\,dz$.

   Answer: $-\frac{1}{2}$.

   Solution. By the residue theorem, $\frac{1}{2\pi i}\oint\limits_{|z|=1} z\cos\frac{1}{z}dz = \mathrm{Res}_{z=0}\left(z\cos\frac{1}{z}\right)$. Note that $z=0$ is an essential isolated singularity for $f(z) = z\cos\frac{1}{z}$. We use the fact that $\mathrm{Res}_{z=0} f(z) = c_{-1}$, where c_{-1} is the coefficient in front of z^{-1} in the Laurent expansion of f centered at $z = 0$. We compute

$$z \cos \frac{1}{z} = z \left(1 - \frac{1}{2!\, z^2} + \frac{1}{4!\, z^4} - \frac{1}{6!\, z^6} + \dots \right) = z - \frac{1}{2z} + \frac{1}{4!\, z^3} - \dots .$$

Thus, $c_{-1} = -\frac{1}{2}$. □

10. Find the largest $r > 0$ such that the following PDE has the same type (elliptic, parabolic, or hyperbolic) at all points of the ball $(x-2)^2 + (y+1)^2 < r^2$ and determine its type in this ball.

$$2xy\, u_{xx} + y\, u_{xy} + \frac{1}{8} u_{yy} + 4u_x = 6.$$

Answer: 1, *hyperbolic.*

Solution. A second order PDE $au_{xx} + 2bu_{xy} + cu_{yy} + \ldots = 0$ is elliptic if $ac - b^2 > 0$, parabolic if $= 0$ and hyperbolic if < 0. In our case, $a = 2xy$, $b = \frac{y}{2}$, $c = \frac{1}{8}$, and $ac - b^2 = \frac{1}{4} y(x-y)$, so the PDE is elliptic if $y(x-y) > 0$, parabolic if $y(x-y) = 0$ and hyperbolic if $y(x-y) < 0$. At the center of the ball, $(2, -1)$, $y(x-y) = -3$, so the PDE is hyperbolic. Furthermore, the point $(2, -1)$ is at distance 1 from the line $y = 0$ and at distance > 1 from the line $y = x$. Thus, $r = 1$ is the largest radius, such that the PDE is hyperbolic in the ball of radius r centered at $(2, -1)$. □

11. Solve the initial value problem

$$\begin{cases} u_{tt} = u_{xx} + e^x & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_t(x,0) = 0 & x \in \mathbb{R}. \end{cases}$$

Answer: $\frac{1}{2}(e^{x-t} + e^{x+t}) - e^x$.

Solution. This immeidately follows from d'Alembert's formula for the solution to inhomogeneous wave equation.

It can also be solved directly (basically repeating the proof of d'Alembert's formula in this special case). Define the variables $\xi = x - t$, $\eta = x + t$. By the chain rule,

$$u_t = -u_\xi + u_\eta, \quad u_{tt} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}, \quad u_x = u_\xi + u_\eta, \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Thus, the equation in the new variables is $-4u_{\xi\eta} = e^x = e^{\frac{\xi+\eta}{2}}$. By integrating with respect to ξ and η we find the general form of the solution to the PDE $u_{tt} = u_{xx} + e^x$:

$$u(x,t) = -e^{\frac{\xi+\eta}{2}} + f(\xi) + g(\eta) = -e^x + f(x-t) + g(x+t),$$

where f, g are arbitrary functions. It remains to find f and g such that u satisfies the initial conditions. We have $u(x,0) = -e^x + f(x) + g(x) = 0$ and $u_t(x,0) = -f'(x) + g'(x) = 0$. Thus, $f(x) = g(x) = \frac{1}{2} e^x$ and $u(x,t) = -e^x + \frac{1}{2} e^{x-t} + \frac{1}{2} e^{x+t}$.

One other way to solve this problem is to observe that its solution can be written as $u = v - e^x$, where v solves the homogeneous equation $v_{tt} = v_{xx}$ with initial conditions $v(x,0) = e^x$, $v_t(x,0) = 0$, which, by d'Alembert's formula, now for the homogeneous wave equation, equals to $\frac{1}{2}(e^{x-t} + e^{x+t})$. □