LECTURE NOTES, Part I

Literature

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Contents

1 Jordan measure

1.1 Notation

We consider the Euclidean space $\mathbb{R}^n$. Its elements are n-tuples of real numbers denoted by $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), \ldots$

- For $x \in \mathbb{R}^n$, we denote by $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$ the Euclidean *norm* of x.
- The (open) *ball* of radius r centered at $x \in \mathbb{R}^n$ is the set of points in $\mathbb{R}^n$ whose distance to x is smaller than r:

$$B(x, r) = \{y \in \mathbb{R}^n \; : \; \|y - x\| < r\}.$$

- A set A of elements in $\mathbb{R}^n$ is *open* if for any $x \in A$ there exists $r > 0$ such that $B(x, r) \subseteq A$.
- The *boundary* of $A \subseteq \mathbb{R}^n$ is

$$\partial A = \{y \in \mathbb{R}^n \; : \; \forall r > 0, \; B(y, r) \cap A \neq \emptyset \text{ and } B(y, r) \cap A^c \neq \emptyset\},$$

  where $A^c = \mathbb{R}^n \setminus A$ is the complement of A.
- The *closure* of $A \subseteq \mathbb{R}^n$ is the set

$$\overline{A} = A \cup \partial A$$

  and the *interior* of A is

$$\text{int}A = A \setminus \partial A.$$

  Note that $\partial A = \overline{A} \setminus \text{int}A$.
- The set A is *closed* if $\overline{A} = A$. One can prove that A is closed if and only if A^c is open. Only two sets are open and closed at the same time: $\mathbb{R}^n$ and $\emptyset$.

1.2 Inner and outer Jordan measures

Our aim is to measure the size of sets in $\mathbb{R}^n$, extending the familiar notions of length, area, volume. We will do it in two steps. We first postulate the measure of *elementary sets*, these form a reach class of sets with rather simple geometry. We then define the measure of any set by approximating it from inside and from outside by elementary sets.

Definition 1.1. A *box* in $\mathbb{R}^n$ is any set of the form

$$B = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] = \{x \in \mathbb{R}^n \; : \; a_i \leq x_i \leq b_i, \, i \in \{1, \ldots, n\}\},$$

where $a_i < b_i$ are real numbers. We define the *measure* of B by

$$m(B) = \prod_{i=1}^{n} (b_i - a_i)$$

(based on the analogy with $n = 1, 2, 3$).

Exercise 1.2. Prove that for each box B,

(a) $m(B) = \lim_{k\to\infty} \frac{1}{k^n} \sharp(B \cap \frac{1}{k}\mathbb{Z}^n)$,

(b) $\lim_{k\to\infty} \frac{1}{k^n} \sharp(\partial B \cap \frac{1}{k}\mathbb{Z}^n) = 0$.

Here $\frac{1}{k}\mathbb{Z}^n = \{(\frac{a_1}{k}, \dots, \frac{a_n}{k}) : a_1, \dots, a_n \in \mathbb{Z}\}$ and $\sharp A$ is the cardinality of set A (i.e., the number of elements in A).

Definition 1.3. We say that two sets A_1, A_2 in $\mathbb{R}^n$ are *almost disjoint* if $\text{int}A_1 \cap \text{int}A_2 = \emptyset$.

Definition 1.4. A set B is *elementary* if

$$B = B_1 \cup B_2 \cup \dots B_k,$$

for some boxes $B_1, B_2, \dots, B_k$.

Exercise 1.5. Prove that for any elementary set B, there exist pairwise almost disjoint boxes $B_1', \dots, B_m'$ (i.e., for all $i \neq j$, B_i' and B_j' are almost disjoint) such that $B = B_1' \cup \dots B_m'$.

Proposition 1.6. *Let B be an elementary set in $\mathbb{R}^n$. Then there exists a number denoted by $m(B)$ such that for any pairwise almost disjoint boxes $B_1', \dots, B_m'$ such that $B = B_1' \cup \dots B_m'$,*

$$m(B_1') + \dots + m(B_m') = m(B).$$

The number $m(B)$ is called the measure *of B.*

(Note that by Exercise 1.5, at least one choice of such boxes is possible.)

Proof. Let $B_1', \dots, B_m'$ be as in the statement. By Exercise 1.2,

$$m(B_1') + \dots + m(B_m') = \sum_{i=1}^{m} \lim_{k\to\infty} \frac{1}{k^n} \sharp(B_i' \cap \frac{1}{k}\mathbb{Z}^n) = \lim_{k\to\infty} \frac{1}{k^n} \sum_{i=1}^{m} \sharp(B_i' \cap \frac{1}{k}\mathbb{Z}^n)$$
$$= \lim_{k\to\infty} \frac{1}{k^n} \sharp(\cup_{i=1}^m B_i' \cap \frac{1}{k}\mathbb{Z}^n) = \lim_{k\to\infty} \frac{1}{k^n} \sharp(B \cap \frac{1}{k}\mathbb{Z}^n).$$

We conclude that the limit on the right hand side exists. Furthermore, since the right hand side only depends on B and not on the choice of $B_1', \dots, B_m'$, the left hand side takes the same value for any choice of the partition $B_1', \dots, B_m'$ of B into pairwise almost disjoint boxes. □

We can now define the measure of any set S in $\mathbb{R}^n$ by approximating S with elementary sets. It can be done in two ways, which results in two definitions of the measure.

Definition 1.7. Let S be a set in $\mathbb{R}^n$.

- The *inner Jordan measure* of S is defined as

$$\mu_*(S) = \sup\{m(B) : B \text{ is elementary}, B \subseteq S\}.$$

- The *outer Jordan measure* of S is defined as

$$\mu^*(S) = \inf\{m(B) : B \text{ is elementary}, B \supseteq S\}.$$

Instead of optimizing over all elementary sets, one can define the inner and outer Jordan measures as limits of measures of explicit sequences of elementary sets. For that, partition $\mathbb{R}^n$ into pairwise almost disjoint boxes of side length $\frac{1}{2^k}$:

$$T_k = \left\{ \left[\frac{a_1}{2^k}, \frac{a_1+1}{2^k}\right] \times \ldots \times \left[\frac{a_n}{2^k}, \frac{a_n+1}{2^k}\right] \ : \ a_1, \ldots, a_n \in \mathbb{Z} \right\}. \tag{1.1}$$

Note that any $Q \in T_k$ is the union of 2^n boxes from T_{k+1}.

For each bounded set $S \subset \mathbb{R}^n$, let $q_k(S)$ be the union of all boxes from T_k that are contained in S, and $Q_k(S)$ the union of all boxes from T_k that intersect S. Note that

- $q_k \subseteq Q_k$,
- $q_0 \subseteq q_1 \subseteq \ldots$,
- $Q_0 \supseteq Q_1 \supseteq \ldots$.

Thus, the sequence $m(q_k(S))$ is bounded non-decreasing and the sequence $m(Q_k(S))$ is bounded non-increasing, so both have finite limits. In fact, these limits are precisely the inner and outer Jordan measures of S:

Proposition 1.8. *For any set S,*

- $\mu_*(S) = \lim_{k\to\infty} m(q_k(S))$
- $\mu^*(S) = \lim_{k\to\infty} m(Q_k(S))$

Proof. Exercise. □

Next, we summarize basic properties of the inner and outer Jordan measures. These follow easily either from the definition of the measures or from Proposition 1.8.

Proposition 1.9.

- *For any $S \subseteq \mathbb{R}^n$, $\mu_*(S) \leq \mu^*(S)$.*
- *For any elementary S, $\mu_*(S) = \mu^*(S) = m(S)$.*
- *For any sets $S_1 \subseteq S_2$, $\mu_*(S_1) \leq \mu_*(S_2)$ and $\mu^*(S_1) \leq \mu^*(S_2)$.*
- *For any sets S_1, S_2, $\mu^*(S_1 \cup S_2) \leq \mu^*(S_1) + \mu^*(S_2)$.*
- *For any disjoint sets S_1, S_2, $\mu_*(S_1 \cup S_2) \geq \mu_*(S_1) + \mu_*(S_2)$.*
- *The definitions of $\mu_*(S)$ and $\mu^*(S)$ do not depend on the choice of Cartesian coordinates. (In particular, any rotation of S preserves its inner and outer Jordan measures.)*

Proof. Exercise. □

1.3 Jordan measurable sets

The natural question is of course if inner and outer Jordan measures have the same value. In general, the answer is no. Indeed, consider the set $S = \mathbb{Q}^n \cap [0,1]^n$. It is easy to see either from Definition 1.7 or by Proposition 1.9 that $\mu_*(S) = 0$ and $\mu^*(S) = 1$.

In fact, the same example shows that the intuitive conclusion that the measure of the union of two disjoint sets is the sum of the measures is in general false! Indeed, if $S_1 = \mathbb{Q}^n \cap [0,1]^n$ and $S_2 = [0,1]^n \setminus \mathbb{Q}^n$, then $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = [0,1]^n$, $\mu_*(S_1) = \mu_*(S_2) = 0$, $\mu^*(S_1) = \mu^*(S_2) = 1$, but $\mu_*(S_1 \cup S_2) = \mu^*(S_1 \cup S_2) = 1$. Thus, $\mu_*(S_1 \cup S_2) > \mu_*(S_1) + \mu_*(S_2)$ and $\mu^*(S_1 \cup S_2) < \mu^*(S_1) + \mu^*(S_2)$.

Nevertheless, for many "natural" sets, the inner and outer Jordan measures are the same. Furthermore, if only such sets are considered, then intuitive properties of the measure — such that the measure of the union of disjoint sets is the sum of the measures — do hold, see Proposition 1.17.

Definition 1.10. A set S is called *Jordan measurable* if $\mu_*(S) = \mu^*(S)$. In this case, the common value is denoted by $\mu(S)$ and called the *Jordan measure* of S.

As we have just seen, not every set is Jordan measurable ($\mathbb{Q}^n \cap [0,1]^n$ is not). Peano curve (a curve filling $[0,1]^n$) is also an example of a non-measurable set. Curiously, there exist open sets which are not Jordan measurable. For instance, enumerate elements of $\mathbb{Q}^n \cap [0,1]^n$ as $q_1, q_2, \ldots$ and take $S = \cup_{i=1}^{\infty} B(q_i, \frac{1}{2^{i+2}})$.

A better notion of measure — Lebesgue measure — will be considered in Mathematics 4. Many sets that are not Jordan measurable are Lebesgue measurable (in particular, all open sets), but not all. Constructing a Lebesgue non-measurable set is rather tricky though.

Our next goal is to understand, which sets are Jordan measurable. For that, it will be useful first to discuss sets of measure zero.

Definition 1.11. A set S is a (Jordan) *null set* if $\mu^*(S) = 0$.

Proposition 1.12 (Properties of null sets)**.**

1. *Any null set is Jordan measurable.*
2. *If S is a null set and $S' \subseteq S$, then S' is a null set, i.e., any subset of a null set is a null set.*
3. *If S_1, S_2 are null sets, then $S_1 \cup S_2$ is a null set. By induction, this implies that any finite union of null sets is a null set.*

Proof. Exercise. □

Remark 1.13. Although any *finite* union of null sets is a null set, a *countable* union of null sets is in general not a null set, possibly even not measurable. As an example, let $S_x = \{x\}$, $x \in \mathbb{R}^n$. Then S_x is a null set for each $x \in \mathbb{R}^n$, but the union over all $x \in \mathbb{Q}^n \cap [0,1]^n$ is not measurable, thus, by Proposition 1.12 (1), not a null set.

Such issue disappears, when we slightly change the definition of the outer measure. For each $S \subseteq \mathbb{R}^n$, let

$$\lambda^*(S) = \inf\{\sum_{i=1}^{N} m(B_i) \,:\, N \in \mathbb{N} \cup \{\infty\},\, \cup_{i=1}^{N} B_i \supseteq S,\, B_i \text{ are boxes}\} \tag{1.2}$$

be the *outer Lebesgue measure* of S. The only difference to Definition 1.7 is that we now approximate S from outside by finite or countable unions of boxes. This slightly altered notion of the outer measure changes its properties dramatically. For instance, if $\lambda^*(S_i) = 0$ for all i, then $\lambda^*(\cup_{i=1}^{\infty} S_i) = 0$ (try to prove it). In particular, $\lambda^*(\mathbb{Q}^n \cap [0,1]^n) = 0$ and even $\lambda^*(\mathbb{Q}^n) = 0$!

In general, since the infimum is taken over a larger set of approximations, $\lambda^*(S) \leq \mu^*(S)$ for all sets. The equality holds for all Jordan measurable sets. The outer Lebesgue measure will be studied in details in Mathematics 4.

The main criterion of Jordan measurability is the following theorem.

Theorem 1.14. *Let S be a bounded set in $\mathbb{R}^n$. Then S is Jordan measurable if and only if its boundary ∂S is a null set.*

Proof. Omitted. □

Many of the sets that we will consider in this course have a boundary which consists of finitely many pieces, each of which is a graph of a continous function (possibly after a rotation). The next proposition implies that all such sets are Jordan measurable.

Proposition 1.15. *Let f be a continuous function on a bounded closed set A in $\mathbb{R}^n$. The graph of f is a subset of $\mathbb{R}^{n+1}$ defined as*

$$S = \{(x, f(x)) \,:\, x \in A\} = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \,:\, (x_1, \ldots, x_n) \in A\}\,.$$

Then S is a null set.

Proof. Let T_k be the set boxes in $\mathbb{R}^{n+1}$ introduced in (1.1) and Q_k those boxes from T_k that intersect S. We subdivide Q_k into sets of boxes $Q_{k,l}$ with the same projection $P_{k,l}$ onto $\mathbb{R}^n \times \{0\}$. Then

$$m(Q_k) = \sum_l m(Q_{k,l}) = \sum_l m(P_{k,l}) \frac{N_{k,l}}{2^k},$$

where $N_{k,l}$ is the number of boxes in $Q_{k,l}$.

Let $\omega(\delta) = \sup\{|f(x) - f(y)| : x, y \in A, \|x - y\| < \delta\}$ be the modulus of continuity of the function f in A. Then

$$\frac{N_{k,l}}{2^k} \leq \omega\left(\frac{\sqrt{n}}{2^k}\right) + 2\,\frac{1}{2^k}.$$

Thus,

$$m(Q_k) \leq \left(\sum_l m(P_{k,l})\right)\left(\omega\left(\frac{\sqrt{n}}{2^k}\right) + 2\,\frac{1}{2^k}\right). \tag{1.3}$$

By Proposition 1.8, $m(Q_k) \to \mu^*_{n+1}(S)$ as $k \to \infty$, where we write "$n+1$" to emphasize that this is a measure of a set in $\mathbb{R}^{n+1}$. Thus, it suffices to prove that the right hand side of (1.3) vanishes as $k \to \infty$.

Again by Proposition 1.8, $\sum_l m(P_{k,l}) \to \mu^*_n(A)$, which is finite because of boundedness of A. Furthermore, by the Heine-Cantor theorem, any continuous function on a bounded closed set in $\mathbb{R}^n$ is *uniformly continuous* on this set. In our case, this reads that the modulus of continuity of f, $\omega(\delta)$ tends to 0 as $\delta \to 0$. In particular, $\omega\left(\frac{\sqrt{n}}{2^k}\right) + 2\,\frac{1}{2^k} \to 0$ as $k \to \infty$. Putting things together, by passing to the limit $k \to \infty$ in (1.3), we obtain that S is a null set in $\mathbb{R}^{n+1}$. □

Another useful class of null sets are rectifiable curves, i.e., curves with finite length.

Proposition 1.16. *Any rectifiable curve in $\mathbb{R}^n$ is a null set.*

Proof. Exercise. □

Propositions 1.15 (in any $\mathbb{R}^n$) and 1.16 (in $\mathbb{R}^2$) give many explicit examples of Jordan measurable sets, for instance a ball, a cylinder, a cone, etc. We finish this section with some useful properties of measurable sets, which follow easily from Theorem 1.14.

Proposition 1.17. *Let S_1, S_2 be Jordan measurable sets in $\mathbb{R}^n$. Then*

1. *$S_1 \cup S_2$ and $S_1 \cap S_2$ are Jordan measurable,*
2. *if S_1, S_2 are almost disjoint, then $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$,*
3. *if $S_1 \subseteq S_2$, then $S_2 \setminus S_1$ is Jordan measurable and $\mu(S_2 \setminus S_1) = \mu(S_2) - \mu(S_1)$,*
4. *$\overline{S_1}$ is Jordan measurable and $\mu(\overline{S_1}) = \mu(S_1)$.*

2 Multiple integral

2.1 Definition and basic properties

Our aim in this section is to obtain an extension of the classical Riemann intergral to functions of several variables. This can be done in several ways. Here we will follow closely the construction on Riemann integral of a function over an interval $\int_a^b f(x)dx$. It is natural to start by recalling this construction.

A tagged partition of $[a, b]$ consists of a partition $a = x_0 < x_1 < \ldots < x_n = b$ and numbers $\xi_i \in [x_{i-1}, x_i]$, $1 \le i \le n$. The mesh of the partition is the length of the largest subinterval $[x_{i-1}, x_i]$. To each tagged partition, one writes the Riemann sum of f, $\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$ and, loosely speaking, defines the Riemann intergral of f over $[a, b]$ as the limit of Riemann sums as the mesh tends to 0. What is really meant here is that a real number I is the integral if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that for } \textit{any} \text{ tagged partition with the mesh} < \delta,$$
$$|I - \textstyle\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})| < \varepsilon.$$

In other words, all the Riemann sums of tagged partitions with sufficiently small mesh have approximately the same value, which is called the Riemann integral.

We would like to adopt this construction to functions defined on subsets S of $\mathbb{R}^n$. For this construction, we restrict to sets that are Jordan measurable and from now on we refer to such sets as measurable.

Let S be a measurable set in $\mathbb{R}^n$ and f a function defined on $\overline{S}$.

- A finite family of sets $\tau = \{S_1, \ldots, S_{\ell_\tau}\}$ is a *partition* of S if
  - all S_i's are measurable,
  - for all $i \neq j$, S_i, S_j are almost disjoint,
  - $\cup_{i=1}^{\ell_\tau} S_i = S$.
- A *tagged partition* is a partition τ and numbers $\xi_i \in \overline{S}_i$, $1 \le i \le \ell_\tau$.
- The *mesh* of a tagged partition, δ_τ is the largest of the diameters of S_i's, i.e.,
$$\delta_\tau = \max_{1 \le i \le \ell_\tau} \sup\{\|x - y\| \,:\, x, y \in S_i\}.$$
- The *Riemann sum* of f corresponding to a tagged partition is
$$\sum_{i=1}^{\ell_\tau} f(\xi_i)\mu(S_i).$$

Finally, we define the integral of f over S as the limit of Riemann sums as the mesh tends to 0:

Definition 2.1. For any measurable set S in $\mathbb{R}^n$ and a function $f : \overline{S} \to \mathbb{R}$, a real number I is the *(multiple or Riemann) integral* of f over S if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \textit{for all} \text{ tagged partitions } \tau = \{S_1, \ldots, S_{\ell_\tau}\},\ (\xi_1, \ldots, \xi_{\ell_\tau})$$
$$\text{with mesh } \delta_\tau < \delta,\ |I - \textstyle\sum_{i=1}^{\ell_\tau} f(\xi_i)\mu(S_i) - I| < \varepsilon.$$

The integral is denoted by $\int_S f(x)dx$ or $\underset{S}{\int \ldots \int} f(x_1, \ldots, x_n)dx_1 \ldots dx_n$. A function for which the integral over S is defined called *integrable on S*.

- If $n = 2$, then the coordinate variables in $\mathbb{R}^2$ are usually denoted by x and y and then the integral is denoted as $\iint_S f(x, y)dxdy$.
- If $n = 3$, then the coordinate variables in $\mathbb{R}^3$ are usually denoted by x, y, z and then the integral is denoted as $\iiint_S f(x, y, z)dxdydz$.

Remark 2.2. It follows from the definition that a function f is integrable on a (measurable) set S if and only if it is integrable on the interior of S, and

$$\int_S f(x)dx = \int_{\text{int}S} f(x)dx.$$

(In particular, if $\text{int}S = \emptyset$, then $\int_S f(x)dx = 0$ for any f.) Thus, it will be sufficient to consider the integral over open sets.

Two questions we are after:

- Which functions are integrable?
- How to compute the integral of a given function over a given set?

The first question will be answered in Section 2.2 and the second in Sections 2.4 and 2.5.

2.2 Conditions for integrability

We begin with a necessary condition for integrability.

Proposition 2.3. *Let S be an open measurable set, $f : \overline{S} \to \mathbb{R}$. If f is integrable on S, then f is bounded on $\overline{S}$.*

Proof. Let f be integrable on an open set S and define $I = \int_S f(x)dx$. By the definition of I, there exists $\delta > 0$ such that for any tagged partition $(\tau, \xi) = (S_1, \dots, S_{\ell_\tau}; \xi_1, \dots, \xi_{\ell_\tau})$ of S with the mesh smaller than δ,

$$\left| \sum_{i=1}^{\ell_\tau} f(\xi_i)\mu(S_i) - I \right| \le 1.$$

Fix one of such tagged partitions and such that each S_i has non-empty interior.

Assume that f is unbounded. Then there exists i_0 such that f is unbounded on S_{i_0}. In particular, there exists $\eta_{i_0} \in \overline{S_{i_0}}$ such that $|f(\xi_{i_0}) - f(\eta_{i_0})|\mu(S_{i_0}) > 2$. (Here we use the fact that the measure of any open measurable set is positive.) Consider the tagged partition $(\tau, \eta) = (S_1, \dots, S_{\ell_\tau}; \eta_1, \dots, \eta_{\ell_\tau})$ of S, where $\eta_i = \xi_i$ for $i \ne i_0$. Since the sets S_i are the same as before, the mesh of this partition is smaller that δ. Thus,

$$\left| \sum_{i=1}^{\ell_\tau} f(\eta_i)\mu(S_i) - I \right| \le 1.$$

By the triangle inequality,

$$\left|\sum_{i=1}^{\ell_\tau} f(\xi_i)\mu(S_i) - \sum_{i=1}^{\ell_\tau} f(\eta_i)\mu(S_i)\right| \le \left|\sum_{i=1}^{\ell_\tau} f(\xi_i)\mu(S_i) - I\right| + \left|\sum_{i=1}^{\ell_\tau} f(\eta_i)\mu(S_i) - I\right| \le 2. \tag{2.1}$$

However, by the definition of η_i's, the left hand side of (2.1) equals

$$\left|\sum_{i=1}^{\ell_\tau}(f(\xi_i) - f(\eta_i))\mu(S_i)\right| = |f(\xi_{i_0}) - f(\eta_{i_0})|\mu(S_{i_0}) > 2.$$

This is a contradiction to (2.1). Thus, our assumption that f is unbounded is false. □

The next theorem gives a criterion for integrability, which will be particularly useful, when we study properties of the multiple integral. For a set A and a function $f : A \to \mathbb{R}$, define the *fluctuation* of f on A by

$$\omega(f; A) = \sup_{x,y\in A} |f(x) - f(y)|.$$

Theorem 2.4. *Let S be an open measurable set, $f : \overline{S} \to \mathbb{R}$. Then f is integrable on S if and only if*

1. *f is bounded on $\overline{S}$ and*
2. *for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition $\tau = (S_1, \dots, S_{\ell_\tau})$ of S with mesh $< \delta$,*

$$\sum_{i=1}^{\ell_\tau} \omega(f; \overline{S_i})\mu(S_i) < \varepsilon.$$

Proof.[Sketch.] The necessity of conditions 1 and 2 follows from Proposition 2.3 and the definition of the integral. Sufficiency is more delicate as it requires to show that the integral exists even though conditions 1 and 2 do not suggest what the value of the integral should be. The proof uses upper and lower Darboux sums

$$D_\tau = \sum_{i=1}^{\ell_\tau} M_i\mu(S_i) \quad \text{and} \quad d_\tau = \sum_{i=1}^{\ell_\tau} m_i\mu(S_i),$$

where $M_i = \sup_{x\in\overline{S_i}} f(x)$ and $m_i = \inf_{x\in\overline{S_i}} f(x)$. Using special (monotonicity) properties of Darboux sums, one can show that there exist numbers $\overline{I}$ and $\underline{I}$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all partitions τ with mesh smaller that δ, $|D_\tau - \overline{I}| < \varepsilon$ and $|d_\tau - \underline{I}| < \varepsilon$. Thus, f is integrable if and only if $\overline{I} = \underline{I}$, in which case the common value is precisely the value of the integral. On the other hand, it is easy to see that $\sum_{i=1}^{\ell_\tau} \omega(f; \overline{S_i})\mu(S_i) = D_\tau - d_\tau$. Thus, Condition 2 of Theorem 2.4 holds if and only if $\overline{I} = \underline{I}$. We omit technical details of the proof of existence of $\overline{I}$ and $\underline{I}$. □

Remark 2.5. 1. Conditions 1 and 2 of Theorem 2.4 are sufficient for integrability of f over *any* measurable set.

2. Another criterion for integrability of a function is the Lebesgue theorem, which states that a function f is integrable on S if and only if $\lambda^*(D) = 0$, where λ^* is the outer Lebesgue measure, defined in (1.2), and D is the set of all points in S where f is not continuous. In particular, this implies that any bounded monotone function on $[a, b] \subset \mathbb{R}$ is integrable on $[a, b]$.

Most of the functions that we consider in this course are continuous. The next theorem states that they all are integrable.

Theorem 2.6. *Let S be an open measurable set and f a continuous function on $\overline{S}$. Then f is integrable on S.*

Proof. We will use the criterion from Theorem 2.4.

The set $\overline{S}$ is bounded and closed and f is continuous on $\overline{S}$. By the Weierstrass theorem, f is bounded on $\overline{S}$, which is Condition 1 from Theorem 2.4. Furthermore, by the Heine-Cantor theorem, f is *uniformly* continuous on $\overline{S}$, that is,

$$\begin{gathered} \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \textit{for all } x, y \in \overline{S} \text{ with } \|x - y\| < \delta, \\ |f(x) - f(y)| < \varepsilon. \end{gathered} \tag{2.2}$$

To show that Condition 2 of Theorem 2.4 holds, we take an arbitrary $\varepsilon > 0$ and choose δ as in (2.2). Then, for any partition $\tau = (S_1, \dots, S_{\ell_\tau})$ of S with the mesh $< \delta$, the fluctuation $\omega(f; \overline{S_i})$ of f on $\overline{S_i}$ is smaller than ε. Thus,

$$\sum_{i=1}^{\ell_\tau} \omega(f; \overline{S_i}) \mu(S_i) < \varepsilon \sum_{i=1}^{\ell_\tau} \mu(S_i) = \varepsilon \mu(S).$$

Since $\mu(S) < \infty$, this implies Condition 2 of Theorem 2.4 (after redefining ε and δ suitably). Finally, as the two conditions of Theorem 2.4 hold, it follows that f is integrable on S. ☐

Remark 2.7. The value of the integral does not depend on the values of the function f on any given null set. Namely, if S is an open measurable set, f is integrable on S and g is a function on $\overline{S}$ such that the set $\{x \in \overline{S} : g(x) \neq f(x)\}$ is a null set, then g is integrable on S and $\int_S g(x)dx = \int_S f(x)dx$.

2.3 Properties of the multiple integral

1. Let S be a measurable set, then

$$\int_S 1 dx = \mu(S).$$

Proof. Note that for any tagged partition, $\sum_{i=1}^{\ell_\tau} 1\mu(S_i) = \mu(S)$, thus the same equality holds after "passing to the limit as $\delta_\tau \to 0$". $\square$

2. Let $S' \subset S$ be measurable and f integrable on S. Then f is integrable on S'.

   Proof. Without loss of generality, S, S' are open. We apply Theorem 2.4. f is integrable on S, thus

   - f is bounded on S, in particular, f is bounded on S',
   - $\forall \varepsilon > 0\ \exists \delta > 0$ such that for all partitions $\tau = (S_1, \dots, S_{\ell_\tau})$ of S with the mesh $< \delta$, $\sum_{i=1}^{\ell_\tau} \omega(f; \overline{S_i})\mu(S_i) < \varepsilon$.
     Take $\varepsilon > 0$ and δ as above. Let τ' be any partition of S' and τ'' any partition of $S \setminus S'$, both with mesh $< \delta$. Then, $\tau = \tau' \cup \tau''$ is a partition of S with mesh $< \delta$. By the choice of δ, $\sum_{i=1}^{\ell_\tau} \omega(f; \overline{S_i})\mu(S_i) < \varepsilon$. But $\sum_{i=1}^{\ell_\tau} \omega(f; \overline{S_i})\mu(S_i) = \sum_{i=1}^{\ell_{\tau'}} \omega(f; \overline{S_i'})\mu(S_i') + \sum_{i=1}^{\ell_{\tau''}} \omega(f; \overline{S_i''})\mu(S_i'') \geq \sum_{i=1}^{\ell_{\tau'}} \omega(f; \overline{S_i'})\mu(S_i')$. Thus, $\forall \varepsilon > 0\ \exists \delta > 0$ such that for all partitions τ' of S' with the mesh $< \delta$, $\sum_{i=1}^{\ell_{\tau'}} \omega(f; \overline{S_i'})\mu(S_i') < \varepsilon$.

   Since conditions 1 and 2 of Theorem 2.4 are satisfied, f is integrable on S'. $\square$

3. Let S', S'' be disjoint measurable sets, $S = S' \cup S''$, and f a function integrable on S. Then
$$\int_S f(x)dx = \int_{S'} f(x)dx + \int_{S''} f(x)dx.$$

   Proof. By property 2, f is integrable on S' and S'', so all the integrals are defined. Let $I = \int_S f(x)dx$, $I' = \int_{S'} f(x)dx$ and $I'' = \inf_{S''} f(x)dx$. We need to show that $I = I' + I''$. By the definition of the integral, $\forall \varepsilon > 0\ \exists \delta > 0$ such that for all tagged partitions (τ, ξ), (τ', ξ'), (τ'', ξ'') of S, S', S'', respectively, with mesh $< \delta$,
$$\left| \sum_{i=1}^{\ell_\tau} f(\xi_i)\mu(S_i) - I \right| < \frac{\varepsilon}{3}, \left| \sum_{i=1}^{\ell_{\tau'}} f(\xi_i')\mu(S_i') - I' \right| < \frac{\varepsilon}{3}, \left| \sum_{i=1}^{\ell_{\tau''}} f(\xi_i'')\mu(S_i'') - I'' \right| < \frac{\varepsilon}{3}.$$
   Note that $(\tau', \xi') \cup (\tau'', \xi'')$ is a tagged partition of S with mesh $< \delta$. Thus,
$$\left| \left(\sum_{i=1}^{\ell_{\tau'}} f(\xi_i')\mu(S_i') + \sum_{i=1}^{\ell_{\tau''}} f(\xi_i'')\mu(S_i'') \right) - I \right| < \frac{\varepsilon}{3}.$$
   Now, by the triangle inequality,
$$\begin{aligned} \left| I - (I' + I'') \right| \leq & \left| I - \left(\sum_{i=1}^{\ell_{\tau'}} f(\xi_i')\mu(S_i') + \sum_{i=1}^{\ell_{\tau''}} f(\xi_i'')\mu(S_i'') \right) \right| \\ & + \left| \sum_{i=1}^{\ell_{\tau'}} f(\xi_i')\mu(S_i') - I' \right| + \left| \sum_{i=1}^{\ell_{\tau''}} f(\xi_i'')\mu(S_i'') - I'' \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus, for any $\varepsilon > 0$, $|I - (I' + I'')| < \varepsilon$. Since the left hand side is a non-negative number independent of ε, the inequality can only hold for all $\varepsilon > 0$ if $I = I' + I''$. □

4. If f, g are integrable functions on S, $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is an integrable function on S and

$$\int_S (\alpha f + \beta g)(x)dx = \alpha \int_S f(x)dx + \beta \int_S g(x)dx.$$

Proof. If $\alpha = \beta = 0$, the result trivially holds, thus assume that at least one of them $\neq 0$. Note that for any tagged partition (τ, ξ) of S,

$$\sum_{i=1}^{\ell_\tau} (\alpha f + \beta g)(\xi_i)\mu(S_i) = \alpha \sum_{i=1}^{\ell_\tau} f(\xi_i)\mu(S_i) + \beta \sum_{i=1}^{\ell_\tau} g(\xi_i)\mu(S_i),$$

by the linearity of finite sums. Since f and g are integrable on S, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all tagged partitions (τ, ξ) of S with mesh $< \delta$,

$$\left| \sum_{i=1}^{\ell_\tau} f(\xi_i)\mu(S_i) - \int_S f(x)dx \right| < \frac{\varepsilon}{|\alpha| + |\beta|}$$
$$\left| \sum_{i=1}^{\ell_\tau} g(\xi_i)\mu(S_i) - \int_S g(x)dx \right| < \frac{\varepsilon}{|\alpha| + |\beta|}.$$

Thus, for any tagged partition (τ, ξ) of S with mesh $< \delta$, also

$$\begin{aligned} &\left| \sum_{i=1}^{\ell_\tau} (\alpha f + \beta g)(\xi_i)\mu(S_i) - \left(\alpha \int_S f(x)dx + \beta \int_S g(x)dx \right) \right| \\ &\quad \leq |\alpha| \left| \sum_{i=1}^{\ell_\tau} f(\xi_i)\mu(S_i) - \int_S f(x)dx \right| + |\beta| \left| \sum_{i=1}^{\ell_\tau} g(\xi_i)\mu(S_i) - \int_S g(x)dx \right| \\ &\quad\quad \leq |\alpha| \frac{\varepsilon}{|\alpha| + |\beta|} + |\beta| \frac{\varepsilon}{|\alpha| + |\beta|} = \varepsilon. \end{aligned}$$

We proved that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all tagged partitions (τ, ξ) of S with mesh $< \delta$,

$$\left| \sum_{i=1}^{\ell_\tau} (\alpha f + \beta g)(\xi_i)\mu(S_i) - \left(\alpha \int_S f(x)dx + \beta \int_S g(x)dx \right) \right| < \varepsilon.$$

Thus, $\alpha \int_S f(x)dx + \beta \int_S g(x)dx = \int_S (\alpha f + \beta g)(x)dx$. □

5. Let f, g be integrable functions on S, then fg is integrable on S.

Proof. Without loss of generality, S is open. We prove that fg satisfis conditions of Theorem 2.4. Since f, g are integrable, by Theorem 2.4, f, g are bounded on $\overline{S}$, i.e., there exists M such that $|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in \overline{S}$. Thus, $|(fg)(x)| = |f(x)||g(x) \leq M^2$ for all $x \in \overline{S}$, i.e., fg is bounded on $\overline{S}$.

Furthermore, since f, g are integrable, by Theorem 2.4, $\forall \varepsilon > 0\ \exists \delta > 0$ such that for all partitions τ of S with mesh $< \delta$,

$$\sum_{i=1}^{\ell_\tau} \omega(f; \overline{S_i})\mu(S_i) < \frac{\varepsilon}{2M\mu(S)}, \quad \sum_{i=1}^{\ell_\tau} \omega(g; \overline{S_i})\mu(S_i) < \frac{\varepsilon}{2M\mu(S)}.$$

However, for each set A,

$$\begin{aligned} \omega(fg; A) &= \sup_{x,y \in A} |f(x)g(x) - f(y)g(y)| \\ &= \sup_{x,y \in A} |(f(x) - f(y))g(x) + f(y)(g(x) - g(y))| \\ &\leq \sup_{z \in A} |g(z)|\omega(f; A) + \sup_{z \in A} |f(z)|\omega(g; A). \end{aligned}$$

Thus,

$$\omega(fg; \overline{S_i}) \leq M\frac{\varepsilon}{2M\mu(S)} + M\frac{\varepsilon}{2M\mu(S)} = \frac{\varepsilon}{\mu(S)}$$

and

$$\sum_{i=1}^{\ell_\tau} \omega(fg; \overline{S_i})\mu(S_i) < \frac{\varepsilon}{\mu(S)} \sum_{i=1}^{\ell_\tau} \mu(S_i) = \varepsilon.$$

We proved that the function fg satisfies conditions of Theorem 2.4, which implies that fg is integrable on S. □

6. Let f, g be integrable on S and $f(x) \leq g(x)$ for all $x \in \overline{S}$. Then

$$\int_S f(x)dx \leq \int_S g(x)dx.$$

Proof. Exercise. □

7. Let f be integrable on S. Then $|f|$ is also integrable on S and

$$\left| \int_S f(x)dx \right| \leq \int_S |f(x)|dx.$$

Proof. It suffices to prove that $|f|$ is integrable. The inequality then follows from property 6, since $-|f(x)| \leq f(x) \leq |f(x)|$ for all x.

Without loss of generality, we may assume that S is open. Since f is bounded on S, $|f|$ is also bounded on S. Furthermore, for each set A, by the triangle inequality $||a| - |b|| \leq |a - b|$,

$$\omega(|f|; A) = \sup_{x,y \in A} ||f(x)| - |f(y)|| \leq \sup_{x,y \in A} |f(x) - f(y)| = \omega(f; A).$$

Now, since condition 2 of Theorem 2.4 is satisfied by f, from the above inequality it follows that it is also satisfied by $|f|$.

We conclude that $|f|$ satisfies both conditions of Theorem 2.4, thus it is integrable on S. □

8. Let $S' \subseteq S$ be measurable sets, f an integrable function on S such that $f(x) \geq 0$ for all $x \in S$. Then

$$\int_{S'} f(x)dx \leq \int_{S} f(x)dx.$$

Proof. By property 2, f is integrable on S', so the integral over S' is defined. By property 3,

$$\int_{S} f(x)dx = \int_{S'} f(x)dx + \int_{S \setminus S'} f(x)dx.$$

By property 6, since $f(x) \geq 0$ on $S \setminus S'$, $\int_{S \setminus S'} f(x)dx \geq 0$ and the result follows. □

2.4 Iterated integrals

In this section we introduce the main tool for computing the multiple integral, which relates the multiple integral and the classical Riemann integral over an interval. This will allow to use methods of classical intergral calculus, for instance, the Newton-Leibniz theorem, to compute multiple integrals.

We begin with the two dimensional case. Fix Cartesian coordinates in $\mathbb{R}^2$.

Definition 2.8. A set S in $\mathbb{R}^2$ is *normal with respect to y-axis* if there exist $a < b$ and continuous functions φ, ψ on $[a, b]$ such that

$$S = \{(x, y) \in \mathbb{R}^2 \,:\, a \leq x \leq b,\, \varphi(x) \leq y \leq \psi(x)\}.$$

Similarly, S is *normal with respect to x-axis* if there exist $c < d$ and continuous functions α, β on $[c, d]$ such that

$$S = \{(x, y) \in \mathbb{R}^2 \,:\, c \leq y \leq d,\, \alpha(y) \leq x \leq \beta(y)\}.$$

Remark 2.9. Note that any set S normal with respect to x- or y-axis is measurable. Indeed, by Proposition 1.15 the boundary of any such S is a null set, thus by Theorem 1.14 S is measurable.

The following theorem, usually called the *Fubini theorem*, gives a relation between the multiple integral in $\mathbb{R}^2$ and the classical Riemann integral.

Theorem 2.10. *Let* $S = \{(x,y) \in \mathbb{R}^2 \,:\, a \le x \le b,\, \varphi(x) \le y \le \psi(x)\}$ *and* f *a continuous function on* S*. Then*

$$\iint_S f(x,y)dxdy = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right] dx. \tag{2.3}$$

Similarly, if $S = \{(x,y) \in \mathbb{R}^2 \,:\, c \le y \le d,\, \alpha(y) \le x \le \beta(y)\}$*, then*

$$\iint_S f(x,y)dxdy = \int_c^d \left[\int_{\alpha(y)}^{\beta(y)} f(x,y)dx\right] dy. \tag{2.4}$$

Remark 2.11.

1. Note that the right and side of (2.3) is an iterated caclulation of two Riemann integrals: (1) for each *fixed* $x \in [a,b]$, $F(x) = \int_{\varphi(x)}^{\psi(x)} f(x,y)dy$ and (2) $\int_a^b F(x)dx$. These integrals can be computed using tools from the basic integral calculus.

   The existence of the first integral is immediate, since for each fixed x, $f(x,y)$ is a continuous function of y. The existence of the second integral follows from the fact that the function $F(x)$ is continuous on $[a,b]$. (Prove it!) In particular, the right hand side of (2.3) is well-defined.

   Same remark applies to the right hand side of (2.4).

2. In order to simplify the notation (this will be particulary useful in higher dimensions), one usually writes

$$\int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)g(x)h(y)dy\right] dx$$

   as

$$\int_a^b g(x)dx \int_{\varphi(x)}^{\psi(x)} f(x,y)h(y)dy \quad \text{or} \quad \int_a^b dxg(x) \int_{\varphi(x)}^{\psi(x)} dyf(x,y)h(y)dy$$

   by placing differentials directly after the corresponding integrals. These are just for notational convenience and do not change the meaning of the iterated integrals, namely, the computation starts with the rightmost integral and proceeds leftwards.

3. If S is normal with respect to x- *and* y-axes, i.e.,

$$\begin{aligned} S &= \{(x,y) \in \mathbb{R}^2 \,:\, a \le x \le b,\, \varphi(x) \le y \le \psi(x)\} \\ &= \{(x,y) \in \mathbb{R}^2 \,:\, c \le y \le d,\, \alpha(y) \le x \le \beta(y)\}. \end{aligned}$$

Then

$$\int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right] dx = \int_c^d \left[\int_{\alpha(y)}^{\beta(y)} f(x,y)dx\right] dy.$$

This is called changing order of integration in iterated integral and can be useful for computations.

4. Finally, if S is not normal, but can be written as a finite union of pairwise almost disjoint normal sets, say $S = S_1 \cup \ldots \cup S_m$, then by properties of the multiple integral,

$$\iint_S f(x,y)dxdy = \sum_{i=1}^m \iint_{S_i} f(x,y)dxdy,$$

so this integral can be computed by applying Theorem 2.10 to each of the integrals in the sum.

Proof.[Proof of Theorem 2.10] Let $S = \{(x,y) \in \mathbb{R}^2 \, : \, a \leq x \leq b, \, \varphi(x) \leq y \leq \psi(x)\}$ and f a continuous function on S.

For integer $k \geq 1$, consider

$$x_i = a + \frac{b-a}{k} i, \quad 1 \leq i \leq k$$

and

$$\varphi_j(x) = \varphi(x) + \frac{\psi(x) - \varphi(x)}{k} j, \quad 1 \leq j \leq k$$

and define the partition $\tau_k = \{S_{ij}, 1 \leq i,j \leq k\}$ of S with

$$S_{ij} = \{(x,y) \, : \, x_{i-1} \leq x \leq x_i, \, \varphi_{j-1}(x) \leq y \leq \varphi_j(x)\}, \quad 1 \leq i,j \leq k.$$

Then, using properties of the Riemann integral,

$$\begin{aligned}\int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right] dx &= \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right] dx \\ &= \sum_{i=1}^k \sum_{j=1}^k \int_{x_{i-1}}^{x_i} \left[\int_{\varphi_{j-1}(x)}^{\varphi_j(x)} f(x,y)dy\right] dx.\end{aligned}$$

Let

$$m_{ij} = \inf_{(x,y)\in\overline{S_{ij}}} f(x,y), \qquad M_{ij} = \sup_{(x,y)\in\overline{S_{ij}}} f(x,y).$$

Then, for each i,j,

$$\int_{x_{i-1}}^{x_i} \left[\int_{\varphi_{j-1}(x)}^{\varphi_j(x)} f(x,y)dy\right] dx \geq m_{ij} \int_{x_{i-1}}^{x_i} \left[\varphi_j(x) - \varphi_{j-1}(x)\right] dx = m_{ij}\mu(S_{ij})$$

and, similarly,

$$\int_{x_{i-1}}^{x_i} \left[\int_{\varphi_{j-1}(x)}^{\varphi_j(x)} f(x,y)dy\right] dx \leq M_{ij}\mu(S_{ij}).$$

Since the functions φ and ψ are continuous on $[a,b]$, the mesh of partition τ_k tends to 0 as $k \to \infty$ (prove this!). Therefore, since f is integrable on S, the Darboux sums converge to the integral:

$$\lim_{k\to\infty} \sum_{i,j=1}^{k} m_{ij}\mu(S_{ij}) = \lim_{k\to\infty} \sum_{i,j=1}^{k} M_{ij}\mu(S_{ij}) = \iint_S f(x,y)dxdy.$$

□

Now, we generalize Theorem 2.10 to functions of three variables. Its proof is essentially the same as the proof of Theorem 2.10.

Similarly to Definition 2.8, define $S \subset \mathbb{R}^3$ to be *normal with respect to z-axis*, if there exist a measurable in $\mathbb{R}^2$ set $S' \subset \mathbb{R}^2$ and continuous functions φ_2 and ψ_2 on S' such that

$$S = \{(x,y,z) \in \mathbb{R}^3 \,:\, (x,y) \in S',\, \varphi_2(x,y) \le z \le \psi_2(x,y)\}.$$

Similarly, one can define sets normal with respect to the other two coordinate axes.

Theorem 2.12. *Let S' be a measurable subset of $\mathbb{R}^2$, φ_2, ψ_2 continuous functions on S', $S = \{(x,y,z) \in \mathbb{R}^3 \,:\, (x,y) \in S',\, \varphi_2(x,y) \le z \le \psi_2(x,y)\}$, and f a continuous function on S. Then*

$$\iiint_S f(x,y,z)dxdydz = \iint_{S'} \left[\int_{\varphi_2(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right] dxdy. \tag{2.5}$$

Remark 2.13.

1. Similar statements hold for sets normal with respect to the other two axes by changing the role of coordinates in (2.5).

2. The right hand side of (2.5) is an iterated caclulation of a Riemann integral and a multiple integral: (1) for each *fixed* $(x,y) \in S'$, $F(x,y) = \int_{\varphi_2(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$ and (2) $\iint_{S'} F(x,y)dxdy$.

   The existence of the first integral is immediate, since for each fixed x, y, $f(x,y,z)$ is a continuous function of z. The existence of the second integral follows from Theorem 2.6 and the fact that the function $F(x,y)$ is continuous on S'. (Prove it!) In particular, the right hand side of (2.5) is well-defined.

3. If S' in the statement of Theorem 2.12 is normal with respect to y-axis, i.e., has the form $S' = \{(x,y) \in \mathbb{R}^2 \,:\, a \le x \le b,\, \varphi_1(x) \le y \le \psi_1(x)\}$, then by Theorem 2.10,

$$\iiint_S f(x,y,z)dxdydz = \int_a^b \left[\int_{\varphi_1(x)}^{\psi_1(x)} \left[\int_{\varphi_2(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right] dy\right] dx \tag{2.6}$$

and again we use the convention of placing differentials just after the respective integral sign, where appropriate, so that the iteraged integral on the right hand side becomes

$$\int_a^b dx \int_{\varphi_1(x)}^{\psi_1(x)} dy \int_{\varphi_2(x,y)}^{\psi_2(x,y)} f(x,y,z)dz.$$

Sometimes, it is useful to express the integral $\iiint_S f(x,y,z)dxdydz$ as iterated integrals, where the outer integral is over an interval and the inner over a subset of $\mathbb{R}^2$. We provide the result without proof.

Theorem 2.14. *Let S be a measurable subset of $\mathbb{R}^3$ such that for each z_0, the set*

$$S(z_0) = \{(x,y,z) \in S \,:\, z = z_0\}$$

is measurable in $\mathbb{R}^2$.

Let $a < b$ be such that for all $z_0 \notin [a,b]$, $S(z_0) = \emptyset$. Let f be a continuous function on S. Then

$$\iiint_S f(x,y,z)dxdydz = \int_a^b dz \iint_{S(z)} f(x,y,z)dxdy. \tag{2.7}$$

Finally, Theorems 2.10 and 2.12 can be naturally generalized to any $\mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is *normal with respect to x_n-axis* if there exist a measurable set $S' \subset \mathbb{R}^{n-1}$ and continuous functions $\varphi_{n-1}, \psi_{n-1}$ on S' such that

$$S = \{(x_1,\dots,x_n) \,:\, (x_1,\dots,x_{n-1}) \in S',\, \varphi_{n-1}(x_1,\dots,x_{n-1}) \le x_n \le \psi_{n-1}(x_1,\dots,x_{n-1})\}.$$

Theorem 2.15. *Let $n \ge 2$ be an integer. Let S' be a measurable subset of $\mathbb{R}^{n-1}$, $\varphi_{n-1}, \psi_{n-1}$ continuous functions on S', $S = \{(x_1,\dots,x_n) \,:\, (x_1,\dots,x_{n-1}) \in S',\, \varphi_{n-1}(x_1,\dots,x_{n-1}) \le x_n \le \psi_{n-1}(x_1,\dots,x_{n-1})\}$, and f a continuous function on S. Then*

$$\int_S f(x)dx = \int_{S'} dx_1 \dots dx_{n-1} \int_{\varphi_{n-1}(x_1,\dots,x_{n-1})}^{\psi_{n-1}(x_1,\dots,x_{n-1})} f(x_1,\dots,x_n)dx_n. \tag{2.8}$$

Theorem 2.15 allows to reduce the problem of computing an integral over n-dimensional set S to the computation of a Riemann integral and an integral over $(n-1)$-dimensional set S'. If S' is normal with respect to one of the coordinates, then Theorem 2.15 can be applied to S' and so on. In particular, if

$$S = \left\{ (x_1,\dots,x_n) \,:\, \begin{array}{l} a \le x_1 \le b,\, \varphi_1(x_1) \le x_2 \le \psi_1(x_1),\, \dots, \\ \varphi_{n-1}(x_1,\dots,x_{n-1}) \le x_n \le \psi_{n-1}(x_1,\dots,x_{n-1}) \end{array} \right\},$$

then

$$\int_S f(x)dx = \int_a^b dx_1 \int_{\varphi_1(x_1)}^{\psi(x_1)} dx_2 \ldots \int_{\varphi_{n-1}(x_1,\ldots,x_{n-1})}^{\psi_{n-1}(x_1,\ldots,x_{n-1})} f(x_1,\ldots,x_n)dx_n.$$

Example 2.16. 1. Compute $I = \iint_S \frac{dxdy}{(1+x+y)^2}$, where S is the triangle bounded by the lines $y = 2x$, $y = \frac{x}{2}$, $x + y = 6$.

The domain of integration S is not normal. To apply Fubini's theorem, we should partition it into normal subsets. We partition S into two triangles S_1 and S_2 by the line $x = 2$. Then both S_1 and S_2 are normal with respect to y-axis, more precisely, $S_1 = \{(x, y) : 0 \le x \le 2, \frac{x}{2} \le y \le 2x\}$, $S_2 = \{(x, y) : 2 \le x \le 4, \frac{x}{2} \le y \le 6 - x\}$. Thus, by Fubini's theorem,

$$I_1 = \iint_{S_1} \frac{dxdy}{(1+x+y)^2} = \int_0^2 dx \int_{\frac{x}{2}}^{2x} \frac{dy}{(1+x+y)^2} = \int_0^2 dx \int_{1+\frac{3x}{2}}^{1+3x} \frac{dz}{z^2}$$
$$= \int_0^2 \left(\frac{1}{1+\frac{3x}{2}} - \frac{1}{1+3x} \right) dx = \frac{2}{3} \ln 4 - \frac{1}{3} \ln 7.$$

Similarly,

$$I_2 = \iint_{S_2} \frac{dxdy}{(1+x+y)^2} = \int_2^4 dx \int_{\frac{x}{2}}^{6-x} \frac{dy}{(1+x+y)^2}$$
$$= \int_2^4 \left(\frac{1}{1+\frac{3x}{2}} - \frac{1}{7} \right) dx = \frac{2}{3} \ln \frac{7}{4} - \frac{2}{7}.$$

Thus, $I = I_1 + I_2 = \frac{1}{3} \ln 7 - \frac{2}{7}$.

2. Compute $I = \iint_S y^2 dxdy$, where S is bounded by the rotated parabola $x = y^2$ and the line $y = x - 2$.

Note that S is normal with respect to x-axis, $S = \{(x, y) : -1 \le y \le 2, y^2 \le x \le y+2\}$. Thus, by Fubini's theorem,

$$I = \int_{-1}^2 dy \int_{y^2}^{y+2y^2} dx = \int_{-1}^2 y^2(y + 2 - y^2)dy = \frac{63}{20}.$$

[Another way to solve this example would be to partition S by the line $x = 1$ into two subsets normal with respect to y-axis and compute the two integrals using (2.3). The results will be the same, but this way the calculation is longer.]

3. Often the domain of integration naturally suggests which of the iterated integrals to use, (2.3) or (2.4). So, in the previous example, it was more natural to use (2.4), even though any of the two would lead to the same result. The present example demonstrates that sometimes the nature of the integrated function may seriously obstruct the use of one of the iterated integrals (2.3) or (2.4), but not both of them.

Compute the iterated integral $I = \int_0^1 dx \int_x^1 \sqrt[4]{1 - y^2} dy$.

The problem with this example is that the antiderivative of $\sqrt[4]{1-y^2}$ cannot be expressed in elementary functions, so the integral $\int_x^1 \sqrt[4]{1-y^2}dy$ cannot be computed with the Newton-Leibniz formula. However, things simplify a lot, if we change the order of integration. Note that $\{(x,y) : 0 \le x \le 1, x \le y \le 1\}$ is a triangle, which can be also written as $\{(x,y) : 0 \le y \le 1, 0 \le x \le y\}$. So,

$$I = \int_0^1 dy \int_0^y \sqrt[4]{1-y^2}dx = \int_0^1 y\sqrt[4]{1-y^2}dy = \frac{1}{2}\int_0^1 \sqrt[4]{z}dz = \frac{2}{5}.$$

4. The following example demonstrates that sometimes it is more convenient not to partition the domain of integration into normal subsets, but rather express it as a set difference of normal sets.

   Compute $I = \iint_S xdxdy$, where S is the set of points that lie in the disc $\{(x,y) : x^2 + y^2 \le R\}$ but not in the disc $\{(x,y) : (x-r)^2 + y^2 \le r^2\}$. Here $0 < 2r \le R$.

   One could of course compute I by partitioning S into normal subsets, but one can do better. Let B_1 be the big disc and B_2 the smaller one. Note that $f(x,y) = x$ is continuous on B_1, so it is integrable on B_1 and

$$I = I_1 - I_2 = \iint_{B_1} xdxdy - \iint_{B_2} xdxdy.$$

   Both B_1 and B_2 are normal with respect to y-axis. Thus, by Fubini's theorem (2.3),

$$I_1 = \iint_{B_1} xdxdy = \int_{-R}^{R} dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} xdx = 0$$

   and

$$I_2 = \iint_{B_2} xdxdy = \int_{-r}^{r} dy \int_{r-\sqrt{r^2-y^2}}^{r+\sqrt{r^2-y^2}} xdx = 2r\int_{-r}^{r} \sqrt{r^2-y^2}dy = \pi r^3,$$

   since 2
   $int_{-r}^{r}\sqrt{r^2-y^2}dy$ is the area of a disc of radius r, which equals πr^2.
   Finally, $I = I_1 - I_2 = -\pi r^3$.

5. Compute $I = \iiint_V (x+y+z)dxdydz$, where V is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

   Note that $V = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x-y\}$. Thus, by Fubini's theorem (2.6),

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x+y+z)dz = \frac{1}{2}\int_0^1 dx \int_0^{1-x} (1-(x+y)^2)dy$$
$$= \frac{1}{2}\int_0^1 (1 - x - \frac{1}{3}(1-x^3))dx = \frac{1}{8}.$$

6. Compute $I = \iiint_V ydxdydz$, where $V = \{(x, y, z) : |x| \leq z, 0 \leq z \leq 1, z \leq y, x^2 + y^2 + z^2 \leq 4\}$.

   Note that V is normal with respect to y-axis, $V = \{(x, y, z) : |x| \leq z, 0 \leq z \leq 1, z \leq y \leq \sqrt{4 - x^2 - z^2}\}$. Furthermore, the set $\{(x, z) : |x| \leq z, 0 \leq z \leq 1\} = \{(x, z) : 0 \leq z \leq 1, -z \leq x \leq z\}$ is normal with respect to x-axis. Thus, by (2.6),

$$I = \int_0^1 dz \int_{-z}^{z} dx \int_{z}^{\sqrt{4-x^2-z^2}} ydy = \ldots = \frac{17}{12}.$$

7. Finally, we give an application of Theorem 2.14.

   Compute $I = \iiint_V \frac{dxdydz}{(x+y+z)^3}$, where V is the pyramid bounded by the planes

$$4x + 3z = 12, \quad 4x + z = 4, \quad 4y + 3z = 12, \quad 4y + z = 4, \quad z = 0.$$

   Note that the section of V by the plane $z = c$ $(0 \leq c \leq 4)$ is the rectangle $S(c) = \{(x, y) : \frac{4-c}{4} \leq x \leq \frac{12-3c}{4}, \frac{4-c}{4} \leq y \leq \frac{12-3c}{4}\}$. Thus, by (2.7),

$$I = \int_0^4 dz \iint_{S(z)} \frac{dxdydz}{(x+y+z)^3} = \int_0^4 dz \int_{\frac{4-c}{4}}^{\frac{12-3c}{4}} dx \int_{\frac{4-c}{4}}^{\frac{12-3c}{4}} \frac{dz}{(x+y+z)^3} = \ldots = \ln 3 - 1.$$

2.5 Changes of variables

Fubini theorem gives us a tool to compute multiple integral $\int_S f(x)dx$ over sets normal with respect to one of the coordinates. However, any set S in $\mathbb{R}^n$ with sufficiently smooth boundary can be expressed as a finite union of sets normal with respect to one of the coordinates, thus an integral over it can be reduced to iterated Riemann integrals. Still, the function f or the functions φ_j, ψ_j that define the boundaries of the normal sets can be quite complicated. It turns out that it may be possible to simplify the domain of integration considerably by parametrizing it with new variables.

Let S be a measurable subset of $\mathbb{R}^n$. Assume that every point $(x_1, \ldots, x_n)$ in S is in a one-to-one correspondence with a point $(u_1, \ldots, u_n)$ in a set D, i.e., there is a bijection $\varphi : D \to S$. (Here one could think of a set D that has a much simpler geometry in comparison to S.) If φ is sufficiently smooth, it is possible to replace integration over S of a function with the integration over D of some other function. More precisely, the following theorem holds.

Theorem 2.17. *Let S, D be measurable subsets of $\mathbb{R}^n$, $\varphi : D \to S$ a continuously differentible bijection such that its Jacobian is non-zero everywhere in D. Then*

$$\int_S f(x)dx = \int_D f(\varphi(u)) \left| \frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)} \right| du. \tag{2.9}$$

Remark 2.18. Change of variables formula (2.9) can be used to simplify the function as well as the domain of integraltion.

Example 2.19. 1. (polar coordinates)

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad (r > 0, \varphi \in [0, 2\pi)) \qquad \frac{\partial(x, y)}{\partial(r, \varphi)} = r$$

2. (cylindrical coordinates)

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \qquad (r > 0, \varphi \in [0, 2\pi), z \in \mathbb{R}) \qquad \frac{\partial(x, y, z)}{\partial(r, \varphi, z)} = r$$

3. (spherical coordinates)

$$\begin{cases} x = r\cos\varphi\sin\psi \\ y = r\sin\varphi\sin\psi \\ z = r\cos\psi \end{cases} \qquad (r > 0, \varphi \in [0, 2\pi), \psi \in [0, \pi]) \qquad \frac{\partial(x, y, z)}{\partial(r, \varphi, \psi)} = r^2\sin\psi$$

Remark 2.20. (Geometrical interpretation of Jacobian) The aim of this remark is to explain why the modulus of Jacobian appears in (2.9). For notational convenience, we only consider the case of two variables, which we denote by (u, v) instead of (u_1, u_2) and (x, y) instead of (x_1, x_2). The general case is similar. First, assume that φ is an affine map:

$$\begin{array}{rcl} x(u, v) & = & a_1 + a_{11}u + a_{12}v \\ y(u, v) & = & a_2 + a_{21}u + a_{22}v \end{array} \qquad (u, v) \in D \subset \mathbb{R}^2.$$

Then, a square $\Gamma' = [u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v]$ is mapped by φ to the parallelogramm Γ spanned by vectors $r_u = (a_{11}\Delta u, a_{21}\Delta u)$ and $r_v = (a_{12}\Delta v, a_{22}\Delta v)$ applied at $(x_0, y_0) = \varphi(u_0, v_0)$. In particular, the area of Γ equals the length of the cross product of the above two vectors:

$$\mu(\Gamma) = \|r_u \times r_v\| = \left| \begin{array}{cc} a_{11}\Delta u & a_{21}\Delta u \\ a_{12}\Delta v & a_{22}\Delta v \end{array} \right| = \left| \begin{array}{cc} a_{11} & a_{21} \\ a_{12} & a_{22} \end{array} \right| \Delta u \Delta v = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mu(\Gamma'), \qquad (2.10)$$

where the Jacobian is evaluated at (u_0, v_0).

Now, consider dyadic squares from T_k (recall (1.1)) contained in D. Denote them by Γ_i' and their bottom left corner by ξ_i'. Then, the image of each $Gamma_i'$ is a parallelogramm Γ_i contained in S, and the collection of Γ_i's is almost a partition of S (it may leave out part of S close to its boundary). Still, one can show by approximating integrals with Riemann sums (we omit details here) that

$$\iint_S f(x, y)dxdy \approx \sum_i f(\varphi(\xi_i'))\, \mu(\Gamma_i) \overset{(2.10)}{=} \sum_i f(\varphi(\xi_i')) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| (\xi_i')\, \mu(\Gamma_i')$$
$$\approx \iint_{S'} f(\varphi(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv.$$

In the general case, if φ is continuously differentiable, then by Tailor's formula, for all $(u,v) \in \Gamma'$ (Γ' is a rectangle as above),

$$x(u,v) = x(u_0,v_0) + \tfrac{\partial x}{\partial u}(u_0,v_0)(u-u_0) + \tfrac{\partial x}{\partial v}(u_0,v_0)(v-v_0) + o\left(\sqrt{(\Delta u)^2 + (\Delta v)^2}\right)$$
$$y(u,v) = y(u_0,v_0) + \tfrac{\partial y}{\partial u}(u_0,v_0)(u-u_0) + \tfrac{\partial y}{\partial v}(u_0,v_0)(v-v_0) + o\left(\sqrt{(\Delta u)^2 + (\Delta v)^2}\right).$$

Consider $\varphi_0(u,v) = (a_1 + a_{11}u + a_{12}v, a_2 + a_{21}u + a_{22}v)$, where $a_1 = x(u_0,v_0) - \frac{\partial x}{\partial u}(u_0,v_0)u_0 - \frac{\partial x}{\partial v}(u_0,v_0)v_0$, $a_{11} = \frac{\partial x}{\partial u}(u_0,v_0)$, $a_{12} = \frac{\partial x}{\partial v}(u_0,v_0)$, $a_2 = \dots$ Then φ_0 is an affine map and

$$\varphi(u,v) \approx \varphi_0(u,v), \qquad (u,v) \in \Gamma', \text{ when } \Delta u, \Delta v \to 0.$$

Furthermore, one can show (we omit the details) that

$$\mu(\varphi(\Gamma')) \approx \mu(\varphi_0(\Gamma')) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| (u_0,v_0)\, \mu(\Gamma') \qquad \text{when } \Delta u, \Delta v \to 0.$$

In other words, although φ is generally not an affine transformation, in particular, the image of a small rectangle Γ' is generally not a parallelogramm, φ can be well approximated locally by an affine transformation and so sufficiently small rectangles are mapped to sets close to parallelogramms spanned by the vectors $(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u)$ and $(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v)$ with their areas close to $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$.

Example 2.21. 1. Compute $I = \iint_S x dxdy$, where $S = \{(x,y) \in \mathbb{R}^2 : 2x \le x^2 + y^2 \le 6x, y \le x\}$.

Note that S is normal neither with respect to x-axis nor y-axis, so it needs to be partitioned into normal subsets before Fubini's theorem can be applied. However, S is normal in polar coordinates. Indeed, $S = \{(x,y) : x = r\cos\varphi, y = r\sin\varphi, (r,\varphi) \in S'\}$, where $S' = \{(r,\varphi) : -\frac{\pi}{2} \le \varphi \le \frac{\pi}{4}, 2\cos\varphi \le r \le 6\cos\varphi\}$. Thus, by Theorem 2.17 (recall that Jacobian for polar coordinates is r),

$$I = \iint_{S'} r\cos\varphi\, r dr d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos\varphi d\varphi \int_{2\cos\varphi}^{6\cos\varphi} r^2 dr = \frac{13}{6}(9\pi + 8).$$

2. Compute $I = \iint_S \frac{dxdy}{y}$, where S is bounded by the lines $y = x$, $y = 2x$, $y = 1 - \frac{x}{2}$, $y = 4 - 2x$.

Note that it is more suggestive to write the equations for the lines as $\frac{y}{x} = 1$, $\frac{y}{x} = 2$, $\frac{y}{2-x} = \frac{1}{2}$, $\frac{y}{2-x} = 2$. From this, it is natural to change variables to $u = \frac{y}{x}$, $v = \frac{y}{2-x}$. In the new variables, the domain of integration is $S' = \{(u,v) : 1 \le u \le 2, \frac{1}{2} \le v \le 2\}$.

We now write $x = \frac{2v}{u+v}$ and $y = \frac{2uv}{u+v}$, then differentiation with respect to u and v gives the Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \frac{4uv}{(u+v)^3}$ (which is positive for $(u,v) \in S'$). Thus,

$$I = \iint_{S'} \frac{u+v}{2uv} \frac{4uv}{(u+v)^3} dudv = 2\int_1^2 du \int_{\frac{1}{2}}^2 \frac{dv}{(u+v)^2} = 2\ln\frac{5}{4}.$$

3. Compute $I = \iiint_V \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} dxdydz$, where $V = \{(x,y,z) : \sqrt{x^2+y^2} \le z \le a\}$ is a cone.

   Consider cylindrical coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$. In the new coordinates, the domain of integration is $V' = \{(r,\varphi,z) : 0 \le \varphi \le 2\pi, 0 \le z \le a, 0 \le r \le z\}$. Furthermore, the Jacobian of cylindrical coordinates equals r. Thus,

$$I = \iiint_{V'} \frac{r^2}{\sqrt{r^2+z^2}} r d\varphi dr dz = \int_0^{2\pi} d\varphi \int_0^a dz \int_0^z \frac{r^3 dr}{\sqrt{r^2+z^2}} = \frac{\pi}{6}(2-\sqrt{2})a^4.$$

4. Sometimes it is difficult to draw the domain of integration. In such cases, it may still be possible to argue purely algebraically, as in this example.

   Compute $I = \iiint_V dxdydz$, where $V = \{(x,y,z) : (x^2+y^2+z^2)^2 \le 4xyz, x \ge 0, y \ge 0\}$.

   Consider spherical coordinates $x = r\cos\varphi\sin\psi$, $y = r\sin\varphi\sin\psi$, $z = r\cos\psi$, for $r \ge 0$, $0 \le \varphi \le 2\pi$, $0 \le \psi \le \pi$. Rewrite conditions that define V in terms of the new variables:

$$\begin{cases} r^4 \le 4r^3 \cos\varphi \sin\varphi \sin^2\psi \cos\psi \\ r\cos\varphi\sin\psi \ge 0 \\ r\sin\varphi\sin\psi \ge 0 \end{cases}$$

   Since $r \ge 0$ and $\sin\psi \ge 0$ (for $0 \le \psi \le \pi$), these inequalities are equivalent to

$$\begin{cases} r \le 4\cos\varphi\sin\varphi\sin^2\psi\cos\psi \\ \cos\varphi \ge 0 \\ \sin\varphi \ge 0 \end{cases}$$

   The first ineqality implies that $\psi \in [0, \frac{\pi}{2}]$ and the last two that $\varphi \in [0, \frac{\pi}{2}]$, thus the domain of integration in the new variables is

$$V' = \{(r,\varphi,\psi) : 0 \le \varphi \le \frac{\pi}{2}, 0 \le \psi \le \frac{\pi}{2}, 0 \le r \le 4\cos\varphi\sin\varphi\sin^2\psi\cos\psi\}.$$

   Finally, recall that the Jacobian of spherical coordinates is $r^2\sin\psi$. Thus,

$$I = \iiint_{V'} r^2 \sin\psi \, dr d\varphi d\psi = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{4\cos\varphi\sin\varphi\sin^2\psi\cos\psi} r^2 \sin\psi \, dr = \frac{2}{45}.$$

2.6 Improper integrals

Sometimes multiple integral can be made sense of in cases when either f or S is unbounded. This is done by taking an additional limit of a sequence of integrals computed over family of finite growing sets, called exhaustions.

Definition 2.22. Let S be an open set in $\mathbb{R}^n$. A family S_k of sets is an *exhaustion* of S if (a) all S_k's are open and measurable, (b) $\overline{S_k} \subseteq S_{k+1}$ for all k, (c) $\cup_{k=1}^{\infty} S_k = S$.

Now we can define an integral of f over S, which admits exhaustions.

Definition 2.23. Let S be an open set, f integrable on any measurable S' such that $\overline{S'} \subset S$. We say that f is integrable on S if for any exhaustion S_k of S there exists a limit $\lim_{k\to\infty} \int_{S_k} f(x)dx$ and its value is the same for all exhaustions. This value is called the integral of f over S and denoted by $\int_S f(x)dx$ or $\int \cdots \int_S f(x_1, \dots, x_n)dx_1 \dots dx_n$. One also says that the integral converges.

Remark 2.24.

1. Improper integral can arise if S is unbounded or if f is unbounded near the boundary of S.
2. Various properties of multiple integral can be suitably extended to improper integrals.
3. If $f \geq 0$ then either for all exhaustions S_k, $\lim_{k\to\infty} \int_{S_k} f(x)dx$ is finite or for all exhaustions it is infinite. In the second case, one usually writes $\lim_{k\to\infty} \int_{S_k} f(x)dx = +\infty$. (Note, this only applies when $f \geq 0$ everywhere.)

Sometimes, it is more important to know if the integral converges or not rather than computing it exact value. In such cases, one generally dominates the original function by a simpler one and applies a comparison criterion.

Theorem 2.25. *Let S be an open set, $0 \leq f(x) \leq g(x)$, $x \in S$. If $\int_S g(x)dx$ converges, then $\int_S f(x)dx$ also converges.*

Usually, it is helpful to compare a function at infinity or near a singularity to a polynomial. The following proposition describes integrability properties of basic polynomials.

Proposition 2.26.

1. *The integral* $\int \cdots \int\limits_{x_1^2+\dots x_n^2 \leq 1} \frac{dx_1 \dots dx_n}{(\sqrt{x_1^2+\dots x_n^2})^\alpha}$ *converges if and only if $\alpha < n$.*
2. *The integral* $\int \cdots \int\limits_{x_1^2+\dots x_n^2 \geq 1} \frac{dx_1 \dots dx_n}{(\sqrt{x_1^2+\dots x_n^2})^\alpha}$ *converges if and only if $\alpha > n$.*

Proof. Left as an exercise. (Change variables to generalized spherical coordinates.) □

3 Line integrals

In this chapter we introduce integration of scalar and vector fields along curves in $\mathbb{R}^3$ — line integrals. All the theory extends naturally to any $\mathbb{R}^n$, but for applications that we have in mind, we will need line integrals over curves in $\mathbb{R}^2$ and $\mathbb{R}^3$.

3.1 Line integral of scalar field

Let γ be a rectifiable curve with length L and $r(s) = (x(s), y(s), z(s))$, $0 \leq s \leq L$, the parametrization of γ by the length. Let f be a real-valued function defined on a subset of $\mathbb{R}^3$ that contains γ.

Definition 3.1. The *line integral* of f along γ is denoted by and defined as

$$\int_\gamma f ds = \int_0^L f(x(s), y(s), z(s)) ds, \tag{3.1}$$

where the right hand side is a usual Riemann integral over an interval.

Remark 3.2. By (3.1), the line integral $\int_\gamma f ds$ is the limit of Riemann sums

$$\sum_{i=1}^{k} f(r(\xi_i))(s_i - s_{i-1}),$$

where $s_0 = 0 < s_1 < \ldots < s_k = L$ and $\xi_i \in [s_{i-1}, s_i]$ is a tagged partition of $[0, L]$ such that the mesh $\max_i(s_i - s_{i-1})$ tends to 0 as $k \to \infty$.

Various properties of Riemann integral, for example, linearity, extend immediately to the line integral. Below we list further main properties of the line integral:

1. $\int_\gamma ds = L$. (L is the length of γ.)

   Proof. This follows from (3.1) appiled to $f = 1$. □

2. If f is continuous, then $\int_\gamma f ds$ exists.

   Proof. This follows, since the Riemann integral on the right hand side of (3.1) exists for any continous function f. □

3. Let γ^R be the time reversal of γ. Then $\int_\gamma f ds = \int_{\gamma^R} f ds$, i.e., the line integral does not depend on the orientation of the curve.

   Proof. Let $r(s)$, $0 \le s \le L$, be the parametrization of γ by its length. Then, the parametrization of γ^R by its length is $r(L - s)$, $0 \le s \le L$. Thus,

   $$\int_{\gamma^R} f ds \overset{(3.1)}{=} \int_0^L f(r(L-t)) dt \overset{(s=L-t)}{=} \int_0^L f(r(s)) ds \overset{(3.1)}{=} \int_\gamma f ds.$$

   □

4. $\widetilde{r}(t) = (\widetilde{x}(t), \widetilde{y}(t), \widetilde{z}(t))$, $a \le t \le b$, be some parametrization of γ, where $\widetilde{x}, \widetilde{y}, \widetilde{z}$ are continuously differentiable functions such that $\widetilde{x}'(t)^2 + \widetilde{y}'(t)^2 + \widetilde{z}'(t)^2 > 0$ for all $t \in [a, b]$. Then, for any continuous function f,

   $$\int_\gamma f ds = \int_a^b f(\widetilde{r}(t)) \sqrt{\widetilde{x}'(t)^2 + \widetilde{y}'(t)^2 + \widetilde{z}'(t)^2}\, dt. \tag{3.2}$$

   In particular, the right hand side is the same for *any* parametrization of γ with above properties.

Proof. First of all, under the given assumptions, γ is rectifiable and $\int_\gamma f ds$ exists. Let $r(s)$, $0 \le s \le L$, be the parametrization of γ by the length. By definition, $\int_\gamma f ds = \int_0^L f(r(s))ds$.

For each $t \in [a,b]$, denote the length of the subcurve $\{\widetilde{r}(t'),\ a \le t' \le t\}$ by $s(t)$. Then, $s(t)$ is a continuous bijection between $[a,b]$ and $[0,L]$. Furthermore, by the Pythagoras theorem, $\left|\frac{ds}{dt}\right| = \sqrt{\widetilde{x}'(t)^2 + \widetilde{y}'(t)^2 + \widetilde{z}'(t)^2}$. Thus, by changing variable s to t in the Riemann integral, we get

$$\int_\gamma f ds = \int_0^L f(r(s))ds \overset{s=s(t)}{=} \int_a^b f(\widetilde{r}(t)) \sqrt{\widetilde{x}'(t)^2 + \widetilde{y}'(t)^2 + \widetilde{z}'(t)^2}\, dt.$$

□

Example 3.3. Compute $I = \int_\gamma (x+y)ds$, where γ is the boundary of triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(1,1)$.

Let γ_1 be the line segment connecting $(0,0)$ and $(1,0)$. We parametrize it by $\{(x,0);\ 0 \le x \le 1\}$. Then,

$$I_1 = \int_{\gamma_1} (x+y)ds = \int_0^1 x dx = \frac{1}{2}.$$

Let γ_2 be the line segment connecting $(1,0)$ and $(1,1)$. We parametrize it by $\{(1,y);\ 0 \le y \le 1\}$. Then,

$$I_2 = \int_{\gamma_2} (x+y)ds = \int_0^1 (1+y)dy = \frac{3}{2}.$$

Let γ_3 be the line segment connecting $(0,0)$ and $(1,1)$. We parametrize it by $\{(x,x);\ 0 \le x \le 1\}$. Then,

$$I_3 = \int_{\gamma_3} (x+y)ds = \int_0^1 (x+x)\sqrt{1^2+1^2}dx = \sqrt{2}.$$

Finally, $I = I_1 + I_2 + I_3 = 2 + \sqrt{2}$.

[Note that we chose γ_3 to be the line segment from $(0,0)$ to $(1,1)$ and not from $(1,1)$ to $(0,0)$. The reason is that the parametrization is slightly simpler in this case. The result, of course, does not depend on the choice of orientation, cf. Property 3 above.]

3.2 Line integral of vector field

Let, $F = (P,Q,R)$ be a vector field on $\mathbb{R}^3$ and γ a curve parametrized by the vector function $r(t) = (x(t), y(t), z(t))$, $a \le t \le b$, where x, y, z are continuously differentiable functions and $\|r'(t)\|^2 = x'(t)^2 + y'(t)^2 + z'(t)^2 > 0$ for all t.

Definition 3.4. The *line integral* of F along γ is denoted by and defined as

$$\int_\gamma F \cdot ds = \int_a^b (F(r(t)) \cdot r'(t))dt = \int_a^b \left(P(r(t))x'(t) + Q(r(t))y'(t) + R(r(t))z'(t)\right) dt, \tag{3.3}$$

where the right hand side is a usual Riemann integral over the interval $[a,b]$.

Definition 3.4 seems to rely on the choice of parametrization of the curve γ. Thus, to complete the definition, we need to show that the right hand side of (3.3) is *independent of the parametrization of* γ:

Proof. Let $\widetilde{r}(\widetilde{t}) = (\widetilde{x}(\widetilde{t}), \widetilde{y}(\widetilde{t}), \widetilde{z}(\widetilde{t}))$, $\widetilde{a} \le \widetilde{t} \le \widetilde{b}$, be another parametrization of γ. As in the proof of Property 4 of the line integral of scalar field, consider the (continuously differentiable, strictly increasing) map

$$\varphi : [a,b] \to [\widetilde{a}, \widetilde{b}] \quad \text{such that } r(t) = \widetilde{r}(\widetilde{t}) \text{ for } \widetilde{t} = \varphi(t).$$

Then

$$\int_{\widetilde{a}}^{\widetilde{b}} \big(P(\widetilde{r}(\widetilde{t}))\widetilde{x}'(\widetilde{t}) + Q(\widetilde{r}(\widetilde{t}))\widetilde{y}'(\widetilde{t}) + R(\widetilde{r}(\widetilde{t}))\widetilde{z}'(\widetilde{t})\big)\, d\widetilde{t}$$
$$\overset{\widetilde{t}=\varphi(t)}{=} \int_a^b \big(P(\widetilde{r}(\varphi(t)))\widetilde{x}'(\varphi(t)) + Q(\widetilde{r}(\varphi(t)))\widetilde{y}'(\varphi(t)) + R(\widetilde{r}(\varphi(t)))\widetilde{z}'(\varphi(t))\big)\, \varphi'(t) dt$$
$$= \int_a^b \big(P(r(t))x'(t) + Q(r(t))y'(t) + R(r(t))z'(t)\big)\, dt,$$

where the last equality follows from the chain rule identities $\widetilde{x}'(\varphi(t))\varphi'(t) = \frac{d}{dt}\widetilde{x}(\varphi(t)) = x'(t)$, $\widetilde{y}'(\varphi(t))\varphi'(t) = \ldots$. □

Remark 3.5. Often the line integral in (3.3) is denoted by

$$\int_\gamma P dx + Q dy + R dz.$$

While the line integral of a scalar field over a curve does not depend on the orientation of the curve—basically, $\int_\gamma f ds$ is the *mass* of the curve γ with (inhomogeneous) density f—the line integral of a vector field $\int_\gamma F \cdot ds$ is the *work of the force* F, so it usually does depend on the orientation of the curve in the following way:

- Let γ^R be the reversal of γ. Then

$$\int_{\gamma^R} F \cdot ds = -\int_\gamma F \cdot ds. \tag{3.4}$$

Proof. Let $r(s) = (x(s), y(s), z(s))$, $0 \le s \le L$ be the parametrization of γ by its length. Then $\widetilde{r}(s) = r(L-s)$, $0 \le s \le L$, is the parametrization of γ^R by the length. Hence,

$$\int_{\gamma^R} F \cdot ds = \int_0^L \big(P(\widetilde{r}(s))\widetilde{x}'(s) + Q(\widetilde{r}(s))\widetilde{y}'(s) + R(\widetilde{r}(s))\widetilde{z}'(s)\big)\, ds$$
$$= -\int_0^L \big(P(r(L-s))x'(L-s) + Q(r(L-s))y'(L-s) + R(r(L-s))z'(L-s)\big)\, ds$$
$$= -\int_0^L \big(P(r(s))x'(s) + Q(r(s))y'(s) + R(r(s))z'(s)\big)\, ds = \int_\gamma F \cdot ds,$$

where the second equality follows after substituting $\widetilde{r}(s) = r(L-s)$ and using $\widetilde{r}'(s) = -r'(L-s)$ and the third equality follows by changing variables $L - s \mapsto s$. □

Example 3.6. 1. Compute $I = \int_\gamma ydx + xdy$, where $\gamma = \{(x, x^2);\ 0 \le x \le 1\}$ is a piece of parabola in $\mathbb{R}^2$ connecting $(0,0)$ and $(1,1)$.

By Definition 3.4,

$$I = \int_0^1 (x^2\,1 + x\,2x)dx = 1.$$

2. Compute $I = \int_\gamma ydx + xdy$, where γ is the curve given by

$$\begin{cases} x = 1 - \cos t \\ y = \sin t \end{cases} \qquad t \in [0, \frac{\pi}{2}].$$

By Definition 3.4,

$$I = \int_0^{\frac{\pi}{2}} (y\,x' + x\,y')dt = \int_0^{\frac{\pi}{2}} (\sin t\ \sin t + (1 - \cos t)\ \cos t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \cos tdt - \int_0^{\frac{\pi}{2}} \cos 2tdt = 1.$$

Remark 3.7. It is not a coincidence that we get the same result in both examples. In fact, no matter what the curve γ is in this case, as long as it is connects $(0,0)$ and $(1,1)$, the value of $\int_\gamma ydx + xdy$ is always 1. This follows from the fact that the vector field (y, x) is conservative, cf. Propositions 3.13 and 3.16.

Knowing that the vector field under integration is conservative may help to simplify calculations enormously, if one replaces a given contour γ with a simples possible, for instance, with a line segment connecting two points.

3.3 Green's formula

The line integral of a vector field over a closed contour in $\mathbb{R}^2$—a circulation of the vector field along the contour—has an important connection to the multiple integral defined in (2.1). The following theorem is a two dimensional analogue of both the Gauss-Ostrogradsky theorem and the Stokes' theorem considered lated in these notes.

We begin with a definition. Let S be a subset of $\mathbb{R}^2$ such that its boundary ∂S is a simple curve, from now on we call it γ. The (simple, closed) curve γ is *positively oriented* if the curve interior (the set S) stays to the left when traveling along γ, and *negatively oriented* if the curve interor stays to the right. Basically, positively oriented curves globally rotate counterclockwise and negatively oriented curves clockwise.

Theorem 3.8. *[Green's formula] Let $F = (P, Q)$, where P and Q are continuously differentiable functions on $\overline{S}$. Then*

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_\gamma F \cdot ds, \tag{3.5}$$

where γ is positively oriented *boundary of S. The right hand side is the line integral of vector field F in $\mathbb{R}^2$.*

Proof. It suffices to prove that the theorem holds for the vector fields $F_1 = (P, 0)$ and $F_2 = (0, Q)$. If so, then adding up the results for F_1 and F_2 gives (3.5). The proofs for F_1 and F_2 are essentially the same, so we only consider F_1.

Furthermore, the set S can be partitioned into subsets $S_1, \dots, S_k$ *normal* with respect to x-axis, and if formula (3.5) holds for each S_i, then adding them all up gives (3.5) for S. Indeed, it is immediate that $\sum_{i=1}^k \iint_{S_i} \left(-\frac{\partial P}{\partial y}\right) dxdy = \iint_S \left(-\frac{\partial P}{\partial y}\right) dxdy$. But the fact that $\sum_{i=1}^k \int_{\gamma_i} F \cdot ds = \int_\gamma F \cdot ds$ follows from (3.4). Thus, we can also assume that S is normal with respect to x-axis, i.e., S has the form

$$S = \{(x, y) \,:\, a \le x \le b,\, \varphi(x) \le y \le \psi(x)\}.$$

By Theorem 2.10,

$$\iint_S \left(-\frac{\partial P}{\partial y}\right) dxdy = -\int_a^b dx \int_{\varphi(x)}^{\psi(x)} \frac{\partial P}{\partial y} dy = \int_a^b P(x, \varphi(x))dx - \int_a^b P(x, \psi(x))dx,$$

where the last equality follows from the Newton-Leibniz formula. Let γ_1 be the part of γ parametrized by $(x, \varphi(x))$, $a \le x \le b$ (the "bottom" part of γ). By Definition 3.4 and using the fact that $\frac{dx}{dx} = 1$,

$$\int_a^b P(x, \varphi(x))dx = \int_{\gamma_1} F_1 \cdot ds.$$

Similarly, let γ_3 be the part of γ such that the reverse of γ_3 is parametrized by $(x, \psi(x))$, $a \le x \le b$ (the "top" part of γ), then

$$\int_a^b P(x, \psi(x))dx = \int_{\gamma_3^R} F_1 \cdot ds = -\int_{\gamma_3} F_1 \cdot ds.$$

Finally, let the remaining pieces of the curve γ be γ_2, the vertical line segment connecting the end of γ_1 with the beginning of γ_3, and γ_4, the vertical line segment connecting the end of γ_3 with the beginning of γ_1. Since the vector field F_1 is everywhere parallel to the x-axis, it is perpendicular to both γ_2 and γ_4. Thus,

$$\int_{\gamma_2} F_1 \cdot ds = \int_{\gamma_4} F_1 \cdot ds = 0.$$

We conclude that

$$\iint_S \left(-\frac{\partial P}{\partial y}\right) dxdy = \int_{\gamma_1} F_1 \cdot ds + \int_{\gamma_2} F_1 \cdot ds + \int_{\gamma_3} F_1 \cdot ds + \int_{\gamma_4} F_1 \cdot ds = \int_\gamma F_1 \cdot ds.$$

□

Corollary 3.9. *Let S be a subset of $\mathbb{R}^2$ whose boundary consists of smooth curves γ (outer boundary) oriented counterclockwise and $\gamma_1, \dots, \gamma_k$ (boundaries of "holes") oriented clockwise. Let $F = (P, Q)$, where P and Q are continuously differentiable functions on $\overline{S}$. Then*

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\gamma F \cdot ds + \sum_{i=1}^k \int_{\gamma_i} F \cdot ds. \tag{3.6}$$

Proof. Cut S into simply connected domains, apply Theorem 3.8 to each subdomain, and add up. □

Example 3.10. Compute $I = \int_\gamma x^2 y dx - xy^2 dy$, where γ is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise.

We apply Green's formula to $P = x^2 y$, $Q = -xy^2$, γ and $S = \{(x, y);\, x^2 + y^2 \le 1\}$. (Note that γ has correct orientation!) First, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -y^2 - x^2$, thus,

$$I = -\iint_S (x^2 + y^2) dS = -\int_0^{2\pi} d\varphi \int_0^1 r^3 dr = -\frac{\pi}{2}.$$

3.3.1 Area of a set bounded by a curve

Green's formula connects circulation of a planar vector field to its curl. However, it can also be considered as a mathematical tool to compute line inegrals over closed contour by reduction to area integrals and vice versa. One such example is computation of areas of sets bounded by curves. Indeed, let S be a set in $\mathbb{R}^2$ with a smooth boundary γ oriented counterclockwise. By definition, its area $\mu(S)$ equals $\iint_S dxdy$. By Green's formula, if $F = (P, Q)$ is any vector field such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ everywhere in S, then

$$\mu(S) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_\gamma F \cdot ds.$$

For instance, $P = 0$, $Q = x$ gives $\mu(S) = \int_\gamma (0, x) \cdot ds = \int_\gamma xdy$, and $P = -y$, $Q = 0$ gives $\mu(S) = \int_\gamma (-y, 0) \cdot ds = -\int_\gamma ydx$. Furthermore, the average of this two expressions gives

$$\mu(S) = \frac{1}{2} \int_\gamma xdy - ydx.$$

Example 3.11. Astroid is the curve in $\mathbb{R}^2$ given by $x(t) = a \cos^3 t, y(t) = a \sin^3 t$, for $t \in [0, 2\pi)$. Using the above formula for the area, one computes that the area of the set bounded by astroid equals

$$\frac{1}{2} \int_0^{2\pi} (x(t)y'(t) - y(t)x'(t))\, dt = \frac{3a^2}{2} \int_0^{2\pi} \left(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t\right) dt = \frac{3\pi a^2}{8}.$$

3.3.2 Sign of Jacobian

Let subsets D, S of $\mathbb{R}^2$ and $\varphi : D \to S$. satisfy conditions of Theorem 2.17. If $\varphi(u, v) = (x(u, v), y(u, v))$, then $\mu(S) = \iint_D \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$, that is, the modulus of the Jacobian $J = \frac{\partial(x,y)}{\partial(u,v)}$ is the infinitesimal change of the area under the map φ. The sign of the Jacobian also has an interpretation. Let γ be positively oriented boundary of D and $\Gamma = \varphi(\gamma)$ the boundary of S. Then, Γ is positively oriented if $J > 0$ and negatively oriented if $J < 0$.

Proof. Using formula for the area of S from previous section, $\mu(S) = \varepsilon \int_\Gamma x dy$, where $\varepsilon = +1$ if Γ is positively oriented and $\varepsilon = -1$ if Γ is negatively oriented. Let $(u(t), v(t)), t \in [a, b]$, be a parametrization of γ and $(x(t), y(t)) = \varphi(u(t), v(t))$, $t \in [a, b]$, parametrization of Γ. Then,

$$\begin{aligned}
\mu(S) &= \varepsilon \int_a^b x(t)y'(t)dt = \varepsilon \int_a^b x(t) \left(\frac{\partial y}{\partial u} u'(t) + \frac{\partial y}{\partial v} v'(t) \right) dt \\
&= \varepsilon \int_\gamma x \frac{\partial y}{\partial u} du + x \frac{\partial y}{\partial v} dv = \varepsilon \iint_D \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) dudv = \varepsilon \iint_D \frac{\partial(x, y)}{\partial(u, v)} dudv \\
&= \varepsilon \, \mathrm{sign}(J) \iint_D \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv = \varepsilon \, \mathrm{sign}(J) \mu(S).
\end{aligned}$$

Here, the first and third equalities follow from the definition of line integral, the second from the chain rule, the fourth from Green's formula applied to $P(u, v) = x \frac{\partial y}{\partial u}$ and $Q(u, v) = x \frac{\partial y}{\partial v}$. Thus, $\varepsilon \, \mathrm{sign}(J) = 1$, which gives the claim. $\square$

3.4 Conservative vector fields

Definition 3.12. A vector field F on $S \subset \mathbb{R}^n$ is called *conservative* if there exists a scalar field φ such that $F = \nabla\varphi$ in S. The field φ is called *scalar potential for F*.

Conservative vector fields are common in physics. They represent forces in systems with conserved energy. As follows from the next proposition, the work of such forces along curves only depends on the value of the scalar field at the start and the end of the curve, in particular, allowing to define potential energy.

Proposition 3.13. *Let F be a continuous vector field on $S \subset \mathbb{R}^n$. The following statements are equivalent:*

1. *F is conservative in S,*
2. *for any closed contour γ in S, $\int_\gamma F \cdot ds = 0$,*
3. *for any given points $A, B \in S$ the integral $\int_\gamma F \cdot ds$ is the same for all curves γ in S connecting A and B.*

Proof. We first prove that (1) implies (2). Let $F = \nabla\varphi$ in S. Take a closed contour γ in S parametrized as $\{r(t), t \in [a, b]\}$. Then,

$$\begin{aligned}
\int_\gamma F \cdot ds = \int_a^b F(r(t)) \cdot r'(t)dt = \int_a^b \nabla\varphi(r(t)) \cdot r'(t)dt &= \int_a^b \frac{d\varphi(r(t))}{dt} dt \\
&= \varphi(r(b)) - \varphi(r(a)) = 0.
\end{aligned}$$

Next, we prove that (2) implies (3). Let $A, B \in S$ and γ_1, γ_2 two curves in S connecting A and B. Consider the curve $\gamma = \gamma_1 \cup \gamma_2^R$, the concatenation of γ_1 and the reversal of γ_2. Note that γ is a closed contour in S. Thus, by assumption, $\int_\gamma F \cdot ds = 0$. However,

$$\int_\gamma F \cdot ds = \int_{\gamma_1} F \cdot ds + \int_{\gamma_2^R} F \cdot ds = \int_{\gamma_1} F \cdot ds - \int_{\gamma_2} F \cdot ds,$$

which implies that $\int_{\gamma_1} F \cdot ds = \int_{\gamma_2} F \cdot ds$.

It remains to prove that (3) implies (1). Assume that (3) holds. Fix $A \in S$ and define the scalar field

$$\varphi(B) = \int_\gamma F \cdot ds, \quad B \in S, \gamma \text{ connects } A \text{ to } B.$$

Importantly, by assumption, the definition of $\varphi(B)$ does not depend on the choice of γ connecting A to B. We leave it as an exercise to prove that $F = \nabla\varphi$ in S. □

Proposition 3.13 gives a number of equivalent definitions of conservative vector field. Still, using them it can be tricky to verify if a given field is conservative or not. The next proposition gives (under some additional assumptions) yet another equivalent definition of conservative vector fields in $\mathbb{R}^3$ expressed only in terms of derivatives of the field and thus most practically applicable. We begin with some definitions.

Definition 3.14. The *curl* of a vector field $F = (P, Q, R)$ in $S \subset \mathbb{R}^3$ is the vector field defined by

$$\text{curl}F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

Definition 3.15. A vector field F in $S \subset \mathbb{R}^3$ is called *irrotational* or *curl-free* if

$$\text{curl}F = 0 \quad \text{in } S.$$

Proposition 3.16. *Let F be a continuously differentiable vector field in $S \subset \mathbb{R}^3$.*

1. *If F is conservative in S, then F is irrotational in S.*
2. *If F is irrotational in S and S is* simply connected, *then F is conservative in S.*

Proof. Let F be conservative in S. Then there exists $\varphi : S \to \mathbb{R}$ such that $F = \nabla\varphi$ in S. Then

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = \frac{\partial}{\partial y}\frac{\partial \varphi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \varphi}{\partial y} = 0.$$

Similarly one shows that $\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

We only prove (2) for $F = (P, Q, 0)$ and $S \subset \mathbb{R}^2 \times \{0\}$. Since $\text{curl}F = 0$, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ in S. By Green's formula, for any closed contour γ in S surrounding the set $D \subset S$,

$$\int_\gamma F \cdot ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

Thus, by Proposition 3.13, F is conservative in S. □

Remark 3.17. The assumption of S being simply connected is essential in (2). Indeed, the vector field

$$F = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right),$$

is irrotational in $\mathbb{R}^3 \setminus \{x = y = 0\}$, but not conservative. In fact, for any (simple) closed contour γ in $\mathbb{R}^2 \times \{0\} \setminus \{x = y = 0\}$, by Green's formula, $\int_\gamma F \cdot ds = 2\pi \neq 0$. However, by Proposition 3.16(2), F is conservative in any simply connected $S \subset \mathbb{R}^3 \setminus \{x = y = 0\}$. (In fact, in any S that does not contain a contour around the line $\{x = y = 0\}$, since there $F = \nabla \arctan(\frac{y}{x})$. Note that $\arctan(\frac{y}{x})$ can only be continuously defined in sets that do not contain a contour around $\{x = y = 0\}$.)

4 Surface integrals

In this section we introduce the notion of integral of scalar and vector fields over surfaces in $\mathbb{R}^3$ and study their properties. Most importantly, these integrals are connected to multiple integrals and line integrals via Gauss-Ostrogradsky, resp., Stokes theorems. We begin with some basic differential geometry for surfaces.

4.1 Surfaces

Rouphly speaking, a surface in $\mathbb{R}^3$ is a geometrical shape that locally looks like a deformed plane. For instance, the boundary of a solid object. The exact definition often depends on the context. For this notes, we adapt the following definition.

Definition 4.1. A *surface* in $\mathbb{R}^3$ is a subset S of $\mathbb{R}^3$ that can be parametrized by a continuous vector function r:

$$S = \{r(u,v) = (x(u,v), y(u,v), z(u,v)), \ (u,v) \in \overline{D}\},$$

where D is a bounded open connected subset of $\mathbb{R}^2$ and $r(u,v) \neq r(u',v')$ for all $(u,v) \neq (u',v')$ in D. (We allow that the injectivity property of r may fail on the boundary of D.)

Example 4.2.

1. Sphere, $x^2 + y^2 + z^2 = r^2$. A natural parametrization of a sphere is by spherical coordinates:

$$\begin{cases} x = r\cos\varphi \sin\psi \\ y = r\sin\varphi\sin\psi \\ z = r\cos\psi \end{cases} \qquad \varphi \in [0, 2\pi), \psi \in [0, \pi].$$

2. We will mainly consider surfaces that are graphs of continuous functions:

$$S = \{(x, y, f(x,y)), \ (x,y) \in \overline{D}\}, \quad f \text{ is continuous on } \overline{D}.$$

   Typically a surface can be partitioned into a finite collection of subsurfaces each of which is a graph of a continuous function (modulo relabelling of coordinates) and the study of the surface is reduced to the study of the subsurfaces.

Definition 4.3. If a surface is parametrized by a continuously differentiable vector function, then it is called a *continuously differentiable surface.*

In these notes we will consider continuously differentiable surfaces often without mentioning it.

Definition 4.4. For a surface

$$S = \{r(u,v) = (x(u,v), y(u,v), z(u,v)),\ (u,v) \in \overline{D}\}$$

and $(u_0, v_0) \in D$, the lines

$$\{r(u,v_0),\ (u,v_0) \in \overline{D}\} \quad \text{and} \quad \{r(u_0,v),\ (u_0,v) \in \overline{D}\}$$

are called (u- and v-) *curvilinear coordinates* on S at $r(u_0, v_0)$. The tangent vectors to these lines are denoted respectively by

$$r_u = r_u(u_0,v_0) = \frac{\partial r}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \quad r_v = r_v(u_0,v_0) = \frac{\partial r}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right). \tag{4.1}$$

In these notes we only consider surfaces such that

$$r_u \text{ and } r_v \text{ are non-collinear at every } (u_0, v_0) \in D.$$

This condition is equivalent to $r_u \times r_v \neq 0$ and to rank $\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix} = 2$. Thus, r_u and r_v span a plane in $\mathbb{R}^3$, which is called the *tangent plane* to S at $r(u_0, v_0)$. (If S is the graph of a continuous function, then the definition of the tangent plane coincides with the one considered in Mathematics 2.) The vectors coplanar with the tangent plane are called tangent vectors to S at $r(u_0, v_0)$.

Remark 4.5. In differential geometry, vectors of the tangent plane are often denoted by dr and their coordinates with respect to the basis r_u, r_v by du, dv, i.e., $dr = r_u du + r_v dv$. This notation is motivated by the following calculation. Let $(u(t), v(t)), t \in [a,b]$, be a curve in D. Then $r(u(t), v(t))$ is a curve on S. The tangent to this curve is $\frac{dr}{dt} = r_u \frac{du}{dt} + r_v \frac{dv}{dt}$. In particular, the vector $\frac{dr}{dt}$ lies in the tangent plane and has coordinates $\frac{du}{dt}, \frac{dv}{dt}$ with respect to the basis r_u, r_v. The differentials of functions $u(t), v(t)$ are $du = \frac{du}{dt} dt$ and $dv = \frac{dv}{dt} dt$, so if we define $dr = \frac{dr}{dt} dt$, then we get an expression similar to the above.

Remark 4.6. To derive an equation for the tangent plane to S through $r_0 = r(u_0, v_0)$, note that it consists of those points $r \in \mathbb{R}^3$ such that the vector $r - r_0$ is coplanar with r_u and r_v, i.e.,

$$(r - r_0) \cdot (r_u \times r_v) = 0 \iff \begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = 0,$$

in particular, if S is the graph of a function f: $z = f(x,y), (x,y) \in \overline{D}$, then the above identity rewrites to $z = z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$, which is the familiar definition of the tangent plane to f.

Definition 4.7. The line perpendicular to the tangent plane at $r_0 \in S$ is called the *normal line* to S at r_0. Every non-zero vector parallel to the normal line at r_0 is called the *normal vector* to S at r_0 or, simply, *normal* to S at r_0.

A basic example of a normal is $r_u \times r_v$.

4.2 First fundamental form

Metric properties of a surface, such as lengths of curves on the surface, angles between curves, areas of subsurfaces, etc., are conveniently described by the first fundamental form or the metric tensor of the surface.

Definition 4.8. All tangent vectors to a surface at any fixed point form a two dimensional inner product space with the inner product induced by the canonical inner product of vectors in $\mathbb{R}^3$. This inner product is a bilinear form on tangent vectors, called the *first fundamental form* of the surface at the given point. The vectors r_u, r_v form a basis of the space of tangent vectors. The coefficients of the first fundamental form in this basis are

$$E = r_u^2, \quad F = r_u r_v, \quad G = r_v^2. \tag{4.2}$$

A surface can generally be parametrized in many different ways. Although tangent planes do not change under reparametrization of the surface, the basis r_u, r_v does change and so do the coefficients of the first fundamental form. The next proposition gives a relation between the coefficients of the first fundamental form for different parametrizations of the same surface.

Proposition 4.9. *Let* $\{r(u,v),\ (u,v) \in \overline{D}\}$ *and* $\{\widetilde{r}(\widetilde{u},\widetilde{v}),\ (\widetilde{u},\widetilde{v}) \in \overline{\widetilde{D}}\}$ *be two parametrizations of a same surface related by*

$$\widetilde{r}(\varphi(u,v), \psi(u,v)) = r(u,v),$$

for some continuously differentiable vector functions $\varphi, \psi : \overline{D} \to \overline{\widetilde{D}}$. *Let* E, F, G *and* $\widetilde{E}, \widetilde{F}, \widetilde{G}$ *be the respective coefficients of the first fundamental form. Then*

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{pmatrix}^t \begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} \begin{pmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{pmatrix}.$$

Proof. Recall the general fact from linear algebra. Let V be a Euclidean vector space with a symmetric bilinear form B. Let A_B be the matrix of B in a basis $e_1, \ldots, e_n$ of V, i.e., $(A_B)_{ij} = B(e_i, e_j)$, and $\widetilde{A}_B$ be the matrix of B in a basis $\widetilde{e}_1, \ldots, \widetilde{e}_n$ of V, i.e., $(\widetilde{A}_B)_{ij} = B(\widetilde{e}_i, \widetilde{e}_j)$. If C is the transition matrix from $e_1, \ldots, e_n$ to $\widetilde{e}_1, \ldots, \widetilde{e}_n$, i.e., $(e_1, \ldots, e_n) = (\widetilde{e}_1, \ldots, \widetilde{e}_n)\, C$, then $A_B = C^t \widetilde{A}_B C$.

We differentiate relation $\widetilde{r}(\varphi(u,v), \psi(u,v)) = r(u,v)$, with respect to u and v:

$$\begin{aligned} r_u &= \widetilde{r}_{\widetilde{u}} \varphi_u + \widetilde{r}_{\widetilde{v}} \psi_u, \\ r_v &= \widetilde{r}_{\widetilde{u}} \varphi_v + \widetilde{r}_{\widetilde{v}} \psi_v. \end{aligned}$$

Thus, the matrix $C = \begin{pmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{pmatrix}$ is the transition matrix from the basis r_u, r_v to the basis $\widetilde{r}_{\widetilde{u}}, \widetilde{r}_{\widetilde{v}}$ and the claim follows from the general fact stated above. $\square$

4.2.1 Length of curve

Consider a surface S parametrized by $\{r(u,v),\ (u,v) \in \overline{D}\}$. Let $\{r(u(t), v(t)),\ t \in [a,b]\}$ be a curve on S. Then the length of this curve is given by the formula

$$L = \int_a^b \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\,dt.$$

Proof. Let $s(t)$ be the evolution of the length, i.e., the length of the curve $\{r(u(t'), v(t')),\ t' \in [a,t]\}$. Then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \left\|\frac{dr}{dt}\right\| = \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}.$$

The result follows, since $L = s(b) = s(b) - s(a) = \int_a^b s'(t)dt$. $\square$

4.2.2 Surface area

Consider a surface S parametrized by $\{r(u,v),\ (u,v) \in \overline{D}\}$. Let Γ_i be a rectangle $[u_i, u_i+\Delta u] \times [v_i, v_i+\Delta v]$ in D and S_i its image in S. Then, the area of S_i can be well approximated by the area of the parallelogramm in $\mathbb{R}^3$ spanned by the vectors $r_u\Delta u, r_v\Delta v$, as $\Delta u, \Delta v \to 0$, which equals to $\|r_u \times r_v\|\,|\Delta u\Delta v| = \|r_u \times r_v\|\,\mathrm{Area}(\Gamma_i)$. Using Riemann sum approximations, one computes the area of S:

$$\mathrm{Area}(S) = \iint_D \|r_u \times r_v\| du dv.$$

Note that

$$\begin{aligned}\|r_u \times r_v\|^2 = \|r_u\|^2\,\|r_v\|^2\,\sin^2(r_u, r_v) &= \|r_u\|^2\,\|r_v\|^2 - \|r_u\|^2\,\|r_v\|^2\,\cos^2(r_u, r_v) \\ &= \|r_u\|^2\,\|r_v\|^2 - (r_u r_v)^2 = EG - F^2.\end{aligned}$$

Thus, we have shown that the area of S can be computed by

$$\mathrm{Area}(S) = \iint_D \sqrt{EG - F^2} du dv. \tag{4.3}$$

Remark 4.10. If $\{\widetilde{r}(\widetilde{u}, \widetilde{v}),\ (\widetilde{u}, \widetilde{v}) \in \overline{\widetilde{D}}\}$ is another parametrization of S, related to the first one by $\widetilde{u} = \varphi(u,v), \widetilde{v} = \psi(u,v)$. Then the area of S can be computed by (4.3) using the two parametrizations as

$$\mathrm{Area}(S) = \iint_D \sqrt{EG - F^2} du dv = \iint_{\widetilde{D}} \sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2} d\widetilde{u} d\widetilde{v}.$$

On the other hand, we know from Theorem 2.17 about change of variables in multiple integral, that

$$\iint_{\widetilde{D}} \sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2} d\widetilde{u} d\widetilde{v} = \iint_D \sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2} \left| \frac{\partial(\widetilde{u}, \widetilde{v})}{\partial(u, v)} \right| dudv,$$

but $\frac{\partial(\widetilde{u},\widetilde{v})}{\partial(u,v)} = \begin{vmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{vmatrix}$. Since the above reasoning can be applied to any subsurface of S, particularly, to a sequence of subsurfaces shrinking to an arbitrary point on S, we conclude that at any point of S,

$$\sqrt{EG - F^2} = \sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2} \left\| \begin{matrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{matrix} \right\|. \tag{4.4}$$

The same formula also follows from the more general Proposition 4.9.

4.2.3 Examples

1. *Sphere, spherical coordinates:*

$$\begin{cases} x &= R\cos\varphi\sin\psi \\ y &= R\sin\varphi\sin\psi \\ z &= R\cos\psi \end{cases} \qquad \varphi \in [0, \pi], \psi \in [0, \pi],$$

where φ is an azimuthal coordinate or longitude, ψ is a polar coordinate or colatitude and R is the radius. The normal vector $r_\varphi \times r_\psi$ to the sphere is pointing inwards, $E = R^2 \sin\psi$, $F = 0$, $G = R^2$, and $\sqrt{EG - F^2} = \|r_\varphi \times r_\psi\| = R^2 \sin\psi$.

2. *Sphere, geographical coordinates:*

$$\begin{cases} x &= R\cos\varphi\cos\theta \\ y &= R\sin\varphi\cos\theta \\ z &= R\sin\theta \end{cases} \qquad \varphi \in [0, \pi], \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

where φ is a longitude, θ is a latitude and R is the radius. The latitude θ and colatitute ψ are related by $\theta + \psi = \frac{\pi}{2}$. For such parametrization of the sphere the normal vector $r_\varphi \times r_\theta$ to the sphere is pointing outwards, $E = R^2 \cos\theta$, $F = 0$, $G = R^2$, and $\sqrt{EG - F^2} = \|r_\varphi \times r_\theta\| = R^2 \cos\theta$.

3. *Lateral surface of cone, I:*

$$\begin{cases} x &= x \\ y &= y \\ z &= \sqrt{x^2 + y^2} \end{cases} \qquad z \geq 0$$

The normal vector $r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix}$ is pointing inwards, $E = 1 + \frac{x^2}{x^2+y^2}$, $F = \frac{xy}{x^2+y^2}$, $G = 1 + \frac{y^2}{x^2+y^2}$, and $\sqrt{EG - F^2} = \sqrt{2}$. [This is a special case of Example 6.]

4. *Lateral surface of cone, II:*

$$\begin{cases} x &= z\cos\varphi \\ y &= z\sin\varphi \\ z &= z \end{cases} \qquad z \geq 0, \varphi \in [0, 2\pi]$$

The normal vector $r_\varphi \times r_z = \begin{vmatrix} i & j & k \\ -z\sin\varphi & z\cos\varphi & 0 \\ \cos\varphi & \sin\varphi & 1 \end{vmatrix}$ is pointing outwards, $E = z^2$, $F = 0$, $G = 2$, and $\sqrt{EG - F^2} = z\sqrt{2}$.

5. *Cylinder:*

$$\begin{cases} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{cases}$$

The normal vector $r_\varphi \times r_z = \begin{vmatrix} i & j & k \\ -r\sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$ is pointing outwards, $E = r^2$, $F = 0$, $G = 1$, and $\sqrt{EG - F^2} = r$.

6. *Graph of a function:*

$$\begin{cases} x &= x \\ y &= y \\ z &= f(x, y) \end{cases}$$

The normal vector $r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$ is pointing upwards, $E = 1 + f_x^2$, $F = f_x f_y$, $G = 1 + f_y^2$, and $\sqrt{EG - F^2} = \sqrt{1 + f_x^2 + f_y^2}$.

[Note that in the special case $f(x, y) = \sqrt{x^2 + y^2}$ we recover Example 3.]

4.3 Surface integral of a scalar field

Let $S = \{r(u, v), (u, v) \in \overline{D}\}$ be a smooth surface in $\mathbb{R}^3$ and f a real-valued function defined on a subset of $\mathbb{R}^3$ that contains S (a scalar field).

Definition 4.11. The *integral of f over S* is denoted by and defined as

$$\iint_S f dS = \iint_D f(x(u, v), y(u, v), z(u, v)) \sqrt{EG - F^2}\, dudv, \tag{4.5}$$

where the right hand side is a multple integral over D (see Definition 2.1).

Remark 4.12. A physical interpretation of the integral of f over S for non-negative f is the *mass of surface S with density f*. In particular, if f is identically equal to 1, then $\iint_S 1\, dS = \iint_D \sqrt{EG-F^2}\, dudv = \text{Area}(S)$.

Lemma 4.13. *The definition of $\iint_S f dS$ is independent of parametrization of S.*

Proof. Let $\{\widetilde{r}(\widetilde{u},\widetilde{v}), (\widetilde{u},\widetilde{v}) \in \overline{\widetilde{D}}\}$ be another parametrization of S. Then $\widetilde{u} = \varphi(u,v), \widetilde{v} = \psi(u,v)$ for bijective maps φ, ψ such that $\widetilde{r}(\varphi(u,v),\psi(u,v)) = r(u,v)$. Then,

$$\iint_{\widetilde{D}} f(\widetilde{r}(\widetilde{u},\widetilde{v})\sqrt{\widetilde{E}\widetilde{G}-\widetilde{F}^2}\, d\widetilde{u}d\widetilde{v}$$

$$\overset{\text{Thm.2.17}}{=} \iint_D f(\widetilde{r}(\varphi(u,v),\psi(u,v))\sqrt{\widetilde{E}\widetilde{G}-\widetilde{F}^2}\left|\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right| dudv$$

$$\overset{(4.4)}{=} \iint_D f(r(u,v))\sqrt{EG-F^2}\, dudv.$$

□

Example 4.14. 1. Compute $I = \iint_S \frac{dS}{\sqrt{x^2+y^2+z^2}}$, where S is the lateral surface of the cylinder

$$\begin{cases} x &= r\cos u \\ y &= r\sin u \qquad 0 \le u \le 2\pi, 0 \le v \le H. \\ z &= v \end{cases}$$

By Example 5 of Section 4.2.3, the coefficients of the first fundamental form of S satisfy $\sqrt{EG-F^2} = r$. Thus,

$$I = \int_0^{2\pi}\int_0^H \frac{1}{\sqrt{r^2+v^2}} rdudv = 2\pi r \ln\frac{H+\sqrt{r^2+H^2}}{r}.$$

2. Compute $I = \iint_S z^2 dS$, where S is the surface of the cone $\sqrt{x^2+y^2} \le z \le 2$.

Let S_1 be the lateral surface of the cone and S_2 the base.

Since $z = 2$ everywhere on S_2,

$$I_2 = \iint_{S_2} z^2 dS = 4\,\text{Area}(S_2) = 16\pi.$$

To integrate over S_1, we parametrize it as in Example 3 of Section 4.2.3. As computed there, the coefficients of the first fundamental form of S_2 with such parametrization satisfy $\sqrt{EG-F^2} = \sqrt{2}$. Thus,

$$I_1 = \iint_{S_1} z^2 dS = \iint_{x^2+y^2\le 4}(x^2+y^2)\sqrt{EG-F^2}dxdy$$
$$= \sqrt{2}\iint_{x^2+y^2\le 4}(x^2+y^2)dxdy = \sqrt{2}\int_0^{2\pi}d\varphi\int_0^2 r^3 dr = 8\pi\sqrt{2}.$$

Finally, $I = I_1 + I_2 = 8(2+\sqrt{2})\pi$.

4.4 Surface integral of a vector field

Let $S = \{r(u,v), (u,v) \in \overline{D}\}$ be a smooth surface in $\mathbb{R}^3$.

- S is *orientable* if the unit normal $\frac{r_u \times r_v}{\|r_u \times r_v\|}$ is continuous in D.
- If n is a fixed continuous unit normal to S on D, then we say that S is *oriented by the normal* n.

  [Note that any orientable surface admits two orientations, namely by n and by $-n$.]

Let S be a smooth surface oriented by a unit normal n and $F = (P, Q, R)$ a vector field defined on a subset of $\mathbb{R}^3$ that contains S.

Definition 4.15. The *integral of F over S* is denoted by and defined as

$$\iint_S F \cdot dS = \iint_S (F \cdot n)\, dS, \tag{4.6}$$

where the right hand side is the surface integral of scalar field $F \cdot n$ over S (see Definition 4.11).

Remark 4.16.

1. A physical interpretation of the integral of F over S oriented by n is the *flux of F through S in the direction of n.*

2. If S is oriented by the normal $n = \frac{r_u \times r_v}{\|r_u \times r_v\|}$, then by Definition 4.11,

$$\iint_S (F \cdot n)\, dS = \iint_D F \cdot \frac{r_u \times r_v}{\|r_u \times r_v\|} \sqrt{EG - F^2}\, dudv$$
$$= \iint_D F \cdot (r_u \times r_v)\, dudv = \iint_D \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ z_v & y_v & z_v \end{vmatrix} dudv.$$

(The last equality holds since Cartesian coordinates of $\mathbb{R}^3$ have right-hand orientation.) Furthermore, the identity

$$\begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ z_v & y_v & z_v \end{vmatrix} = P\frac{\partial(y,z)}{\partial(u,v)} + Q\frac{\partial(z,x)}{\partial(u,v)} + R\frac{\partial(x,y)}{\partial(u,v)}$$

motivates the following alternative notation for the integral of F over S *when S is oriented by the normal* $\frac{r_u \times r_v}{\|r_u \times r_v\|}$, cf. Remark 3.5:

$$\iint_S F \cdot dS = \iint_S Pdydz + Qdzdx + Rdxdy.$$

[Let us emphasize that this notation applies only when $n = \frac{r_u \times r_v}{\|r_u \times r_v\|}$: if S is oriented by $-\frac{r_u \times r_v}{\|r_u \times r_v\|}$, then $\iint_S F \cdot dS = -\iint_S Pdydz + Qdzdx + Rdxdy$.]

Example 4.17. 1. Compute $I = \iint_S z dxdy$, where S is the lower part of the lateral surface of the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le H$ (i.e., oriented outwards).

We use parametrization of S as in Example 3 of Section 4.2.3, $S = \{(x,y,z) : (x,y) \in \overline{D}, z = \sqrt{x^2+y^2}\}$, where $D = \{(x, y : x^2+y^2 \le H^2\}$. By Remark 4.16(2), but noting that *vector* $r_x \times r_y$ *points inwards, which is opposite to the orientation of* S,

$$I = -\iint_D F \cdot (r_x \times r_y) dxdy = -\iint_D \sqrt{x^2+y^2}(r_x \times r_y)_z \, dxdy,$$

where $(r_x \times r_y)_z$ denotes the z-coordinate of vector $r_x \times r_y$, which equals 1. Finally, by passing to polar coordinates, we compute

$$I = -\int_0^{2\pi} d\varphi \int_0^H r^2 dr = -\frac{2}{3}\pi H^3.$$

[A sanity check: The vector field $(0,0,z)$ is pointing upwards for $z > 0$, thus its flux through S (outwards) is non-positive.]

2. Compute $I = \iint_S z^2 dxdy$, where S is a semisphere $x^2+y^2+z^2 = 1$, $y \ge 0$ oriented outwards.

Let S_1 be the part of S corresponding to points with $z \ge 0$ and S_2 the part of S corresponding to points with $z \le 0$. Then both S_1 and S_2 are graphs of functions $z = \sqrt{1-x^2-y^2}$ and $z = -\sqrt{1-x^2-y^2}$, respectively, for $(x,y) \in D = \{x^2+y^2 \le 1, y \ge 0\}$.

We parametrize S_1 and S_2 as in Example 6 of Section 4.2.3. Then $r_x \times r_y$ points outwards on S_1 and inwards on S_2 (its z-coordinate, which we denote by $(r_x \times r_y)_z$ in both cases equals to 1). Thus, by Remark 4.16(2),

$$\iint_{S_1} z^2 dxdy = \iint_D (1-x^2-y^2)(r_x \times r_y)_z dxdy = -\iint_{S_2} z^2 dxdy,$$

and $I = 0$.

Let us modify the example slightly and compute $I = \iint_S z dxdy$. In this case,

$$\iint_{S_1} z dxdy = \iint_{S_2} z dxdy = \iint_D \sqrt{1-x^2-y^2}(r_x \times r_y)_z dxdy$$

and

$$I = 2\iint_D \sqrt{1-x^2-y^2}(r_x \times r_y)_z dxdy = 2\iint_D \sqrt{1-x^2-y^2} dxdy = \frac{1}{2}\frac{4\pi}{3} = \frac{2\pi}{3},$$

since the last integral is the volume of the half of a unit ball.

3. Compute $I = \iint_S \frac{dydz}{x} + \frac{dzdx}{y} + \frac{dxdy}{z}$, where S is part of ellipsoid

$$\begin{cases} x = a\cos u \cos v \\ y = b \sin u \cos v \\ z = c \sin v \end{cases} \qquad \frac{\pi}{4} \le u \le \frac{\pi}{3}, \ \frac{\pi}{6} \le v \le \frac{\pi}{4},$$

oriented outwards.

Note that the vector $r_u \times r_v$ is also oriented outwards, that is S is positively oriented with respect to $r_u \times r_v$. We compute I using Remark 4.16(2). For this, we compute

$$\begin{vmatrix} \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \begin{vmatrix} \frac{1}{a\cos u\cos v} & \frac{1}{b\sin u\cos v} & \frac{1}{c\sin v} \\ -a\sin u\cos v & b\cos u\cos v & 0 \\ -a\cos u\sin v & -b\sin u\sin v & c\cos v \end{vmatrix} = p\cos v,$$

where $p = \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}$. Finally,

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} du \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} p\cos v dv = p\,\frac{\pi}{12}\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right).$$

[A sanity check: In the given range of u, v, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} > 0$, that is the vector field is pointing in the direction of the outgoing normal. Thus, $I \geq 0$.]

4.5 Divergence (Gauss-Ostrogradsky) theorem

Let S be a smooth surface surrounding a solid V in R^3 and oriented by the *outgoing* normal (such orientation will be called *positive*). Let F be a continuously differentiable vector field in $\overline{V}$. The following theorem provides a relation between the flux of F through S and the behavior of F inside of V.

Theorem 4.18.

$$\iint_S F \cdot dS = \iiint_V \text{div}F\, dV, \tag{4.7}$$

where

$$\text{div}F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{4.8}$$

is the divergence *of F.*

Remark 4.19.

1. If $\text{div}F(p)$ is positive for $p \in \mathbb{R}^3$, then p is a *source*, if negative, then a *sink*. The Gauss-Ostrogradsky theorem states that the flux of F through S equals to the "sum" of all flows from sources in V minus the "sum" of all flows to sinks in V.

2. An application of Theorem 4.18 gives that $\text{div}F$ is *independent of coordinate system* (although it is defined as sum of partial derivatives with respect to a fixed Cartesian coordinate system). Indeed, the following equality holds:

$$\text{div}F(p) = \lim_{\varepsilon \to 0} \frac{1}{\text{Vol}(B(p,\varepsilon))} \iint_{S(p,\varepsilon)} F \cdot dS,$$

which is independent of coordinates.

Proof.[Proof of Theorem 4.18] By the linearity of the right and left hand sides of (4.7) with respect to the vector field F, it suffices to prove the theorem for vector fields $(P,0,0),(0,Q,0),(0,0,R)$ and then add up the obtained results. We only consider $F=(0,0,R)$, the other are similar.

Furthermore, since V can be expressed as a union of pairwise almost-disjoint sets $V_1,\ldots,V_k$, all of which are normal with respect to z-axis (recall the definition from above Theorem 2.12), by argueing as in the proof of Green's formula (Theorem 3.8), we may assume without loss of generality that V is normal with respect to z-axis, that is $V=\{(x,y,z):(x,y)\in D,\varphi(x,y)\le z\le\psi(x,y)\}$.

Then, by the Newton-Leibniz formula,

$$\iiint_V \frac{\partial R}{\partial z}dV \overset{\text{Thm.2.12}}{=} \iint_D dxdy\int_{\varphi(x,y)}^{\psi(x,y)}\frac{\partial R}{\partial z}\,dz$$
$$= \iint_D R(x,y,\psi(x,y))dxdy - \iint_D R(x,y,\varphi(x,y))dxdy.$$

Let $S_b=\{(x,y,z):(x,y)\in D,z=\varphi(x,y)\}$ and $S_t=\{(x,y,z):(x,y)\in D,z=\psi(x,y)\}$ be the bottom and the top sides of the boundary of V, respectively. Note that on both surfaces the z-coordinate of the vector $r_x\times r_y$ equals 1, $r_x\times r_y$ is oriented outwards on S_t and inwards on S_b. Thus, we can continue the above chain of identities as

$$= \iint_{S_t} F\cdot dS + \iint_{S_b} F\cdot dS.$$

Furthermore, since V is normal with respect to z-axis, any normal to the side boundary of V is orthogonal to the z-axis, in particular, since $F=(0,0,R)$ is parallel to z-axis, any normal to the side boundary of V is orthogonal to F. Thus, the flux of F through the side boundary of V is 0 and we conclude that the last expression equals to $\iint_S F\cdot dS$. □

Corollary 4.20. *If V is a connected set with boundary consisting of smooth surfaces $S,S_1,\ldots,S_k$ (here S is the outer boundary and S_i's are holes in V)* all oriented by outgoing normals, *then*

$$\iiint_V \operatorname{div}F\,dV = \iint_S F\cdot dS + \sum_{i=1}^k \iint_{S_i} F\cdot dS.$$

Proof. Similarly to the proof of Corollary 3.9 to Green's theorem, cut V into finitely many simply connected pieces V_i, apply the Gauss-Ostrogradsky theorem to each piece and add up. The flux of F through the "cuts" is counted twice with opposite signs, so that after cancelations only the flux through the total boundary of V remains. □

Example 4.21. 1. The Gauss-Ostrogradsky formula can be used to compute volumes of solids V surrounded by smooth surfaces S. Indeed, the volume of V is $\iiint_V 1\,dV$. Note that 1 is the divergence of the vector field $\frac{1}{3}(x,y,z)$. Thus, we have the formula,

$$\text{Vol}(V)=\frac{1}{3}\iint_S (x,y,z)\cdot n\,dS, \tag{4.9}$$

where n is the outgoing unit normal to S.

As an application of (4.9), we compute the volume of an arbitrary cone with the apex at 0, the base H and the height h. Let S' be the lateral surface of the cone and let n be the outgoint unit normal. Note that the radius vector (x, y, z) is orthogonal to n everywhere on S' and $(x, y, z) \cdot n$ equals to the height of the cone everywhere on H. Therefore,

$$\mathrm{Vol(Cone)} = \frac{1}{3}\, h \,\mathrm{Area}(H).$$

2. Let S_1 be the lateral surface of the cone $x^2 + y^2 \leq z^2 \leq 1$ oriented outwards. Let $F = (x^3, y^3, z^3)$. To compute the flux of F through S_1, we first use the Gauss-Ostrogradsky theorem to compute the flux of F through the *whole* surface S of the cone V:

$$\iint_S F \cdot dS = \iiint_V \mathrm{div} F \, dV = 3 \iiint_V (x^2 + y^2 + z^2) dx dy dz.$$

We apply the change of coordinates $x = r \cos \varphi, y = r \sin \varphi, z = z$ (the Jacobian of this change equals r), so that the above integral equals

$$3 \int_0^1 dz \int_0^{2\pi} d\varphi \int_0^z (r^2 + z^2) r dr = \frac{9\pi}{10}.$$

To obtain from it the flux of F through S_1, we need to subtract the flux through the base, let us call it S_2. Note that S_2 is parallel to the xy-coordinate plane, in particular, the outgoing unit normal to S_2 is just $n = (0, 0, 1)$. Therefore,

$$\iint_{S_2} F \cdot dS = \iint_{S_2} z^3 \, dS.$$

Since $z = 1$ everywhere on S_2, the above integral equals to the area of S_2, namely (S_2 is a unit disc) to π.

We conclude that $\iint_{S_1} F \cdot dS = \frac{9\pi}{10} - \pi = -\frac{\pi}{10}$.

4.6 Stokes' theorem

Let S be a smooth surface in $\mathbb{R}^3$ oriented by a unit normal n and γ the boundary of S *positively oriented* with respect to the normal n (that is, if the thumb of the right hand points in the direction of n, then the other fingers in the direction of γ).

Let F be a continuously differentiable vector field on S and recall the definition of curl$F$ from Definition 3.14. The following theorem provides a relation between the circulation of F along γ and the behavior of curl$F$ on S.

Theorem 4.22.

$$\int_\gamma F \cdot ds = \iint_S \mathrm{curl} F \cdot dS. \tag{4.10}$$

Remark 4.23. 1. Let S be parametrized by $\{r(u,v),\ (u,v)\in\overline{D}\}$ and let Γ be the positively oriented boundary of $D\subseteq\mathbb{R}^2$ (recall the definition of positively orineted from above Theorem 3.8). Then the image of Γ, $r(\Gamma)$ is the boundary of S which is positively oriented with respect to the normal $n=\frac{r_u\times r_v}{\|r_u\times r_v\|}$. Thus, using the notation convention from Remarks 3.5 (line integral) and 4.16(2) (surface integral), if $F=(P,Q,R)$, then Stokes' theorem can be equivalently stated as

$$\int_\gamma Pdx+Qdy+Rdz=\iint_S\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dzdx+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy \tag{4.11}$$

2. Recall from Definition 3.14 that $\operatorname{curl}F=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$. (Because of a similarity of this determinant with the one in the definition of the cross product of two vectors, one often writes $\nabla\times F$ to denote curl$F$.) Thus, if S is oriented by a unit normal $n=(n_x,n_y,n_z)$, then

$$\iint_S \operatorname{curl}F\cdot dS=\iint_S(\operatorname{curl}F\cdot n)\,dS=\iint_S\begin{vmatrix} n_x & n_y & n_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}dS.$$

3. An application of Theorem 4.22 gives that curl$F$ is *independent of coordinate system* (although it is defined in terms of partial derivatives with respect to a fixed Cartesian coordinate system). Indeed, let n be a unit vector and γ_ϵ be a circle of radius ϵ centered at $p\in\mathbb{R}^3$, lying in the plane perpendicular to n and positively oriented with respect to n. Then, the projection of curl$F$ on n can be computed using Theorem 4.22 as

$$\operatorname{curl}F(p)\cdot n=\lim_{\epsilon\to 0}\frac{1}{\pi\epsilon^2}\int_{\gamma_\epsilon}F\cdot ds,$$

which is independent of coordinates.

Proof.[Proof of Theorem 4.22] As in the proof of the Gauss-Ostrogradsky theorem, it suffices to prove the theorem for the fields $(P,0,0)$, $(0,Q,0)$ and $(0,0,R)$ and then add up the results. By symmetry, all the three cases are proved similarly and we can assume without loss of generality that $F=(P,0,0)$.

We will prove Stokes' theorem in the form of (4.11). For that, let S be parametrized by $\{r(u,v),\ (u,v)\in\overline{D}\}$ and let Γ be the positively oriented boundary of $D\subseteq\mathbb{R}^2$ parametrized by $\{(u(t),v(t)),\ 0\le t\le T\}$. Then, $\gamma=r(\Gamma)$ is the boundary of S parametrized by $\{r(u(t),v(t)),\ 0\le t\le T\}$ and positively oriented with respect to the normal $n = \frac{r_u \times r_v}{\|r_u \times r_v\|}$. We compute

$$
\begin{aligned}
\int_\gamma F \cdot ds &\overset{\text{Rem.3.5}}{=} \int_\gamma P dx \\
&\overset{\text{Def.3.4}}{=} \int_0^T P(r(u(t), v(t))) \frac{\partial x(u(t), v(t))}{\partial t}\, dt \\
&= \int_0^T P(r(u(t), v(t))) \left(\frac{\partial x}{\partial u} u'(t) + \frac{\partial x}{\partial v} v'(t) \right) dt \\
&\overset{\text{Def.3.4}}{=} \int_\Gamma \left(P \frac{\partial x}{\partial u} \right) du + \left(P \frac{\partial x}{\partial v} \right) dv \\
&\overset{\text{Thm.3.8}}{=} \iint_D \left(\frac{\partial}{\partial u} \left(P \frac{\partial x}{\partial v} \right) - \frac{\partial}{\partial v} \left(P \frac{\partial x}{\partial u} \right) \right) dudv \\
&= \iint_D \left(\frac{\partial P}{\partial z} \frac{\partial(z, x)}{\partial(u, v)} - \frac{\partial P}{\partial y} \frac{\partial(x, y)}{\partial(u, v)} \right) dudv \\
&\overset{\text{Rem.4.16(2)}}{=} \iint_S \left(0, \frac{\partial P}{\partial z}, \frac{\partial P}{\partial y} \right) \cdot dS \\
&= \iint_S \text{curl} F \cdot dS.
\end{aligned}
$$

□

Example 4.24. Let γ be the curve of intersection of paraboloid $x^2 + y^2 + z = 3$ and the plane $x + y + z = 2$ oriented positively with respect to the vector $(1, 1, 1)$. (Namely, if the thumb of the right hand points in the direction of $(1, 1, 1)$, then the other fingers in the direction of γ.)

Let S be the surface in the plane spanned by γ oriented by the unit normal $n = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Note that γ is positively oriented with respect to n.

We aim to compute the integral

$$
I = \int_\gamma (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz,
$$

that is the circulation of the vector field $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$ along the contour γ. (Alternatively, $I = \int_\gamma F \cdot ds$.) We use the Stokes' theorem. We compute for $P = y^2 - z^2$, $Q = z^2 - x^2$, $R = x^2 - y^2$,

$$
\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = -2(y + z), \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = -2(x + z), \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2(x + y).
$$

By Stokes' theorem, *since γ is positively oriented with respect to n*,

$$
I = \iint_S (\text{curl} F \cdot n)\, dS = -\frac{4}{\sqrt{3}} \iint_S (x + y + z) dS.
$$

However, S is a subset of the plane $x + y + z = 2$, thus, the sum under integral equals 2 everywhere on S and we get

$$I = -\frac{8}{\sqrt{3}} \iint_S dS.$$

Furthermore, S is parametrized as $\{(x, y, z); z = 2 - x - y, (x, y) \in \overline{D}\}$ for some D, which we determine in a moment. Thus, by Definition 4.11 of the surface integral of scalar field,

$$I = -\frac{8}{\sqrt{3}} \iint_D \sqrt{EG - F^2} dxdy,$$

where E, G, F are the the coefficients of the first fundamental form for S. By Example 6 in Section 4.2.3, $\sqrt{EG - F^2} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3}$. Therefore,

$$I = -8\text{Area}(D).$$

It remains to find D and its area. The boundary of D is the projection of γ on xy-coordinate plane. To find its equation, we eliminate the dependence on z from the system of equations

$$\begin{cases} x^2 + y^2 + z = 3 \\ x + y + z = 2 \end{cases}$$

to get $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{3}{2}$, which is a circle of radius $\sqrt{\frac{3}{2}}$. Thus, the area of D is $\frac{3\pi}{2}$ and $I = -12\pi$.

4.7 Solenoidal vector fields

In Section 3.4 we introduced conservative vector fields in $\mathbb{R}^n$ and studied their properties. In this section we consider another important class of vector fields, solenoidal vector fields. We restrict to the three dimensional Euclidean space $\mathbb{R}^3$, although the definintion and properties can be extended to any $\mathbb{R}^n$.

Definition 4.25. A vector field F in $\mathbb{R}^3$ is *solenoidal*, or divergence free, in $V \subseteq \mathbb{R}^3$ if $\text{div}F = 0$ in V.

Example 4.26. 1. Coulomb's force $F = -\frac{r}{\|r\|^3} = -\left(\frac{x}{\|r\|^3}, \frac{y}{\|r\|^3}, \frac{z}{\|r\|^3}\right)$,

2. velocity field of incompressible fluid,

3. magnetic field.

The following characterisation of solenoidal fields is immediate from the Gauss-Ostrogradsky theorem.

Proposition 4.27. *Let V be an open simply connected subset of $\mathbb{R}^3$ and F a vector fields on $\overline{V}$. Then, F is solenoidal in V if and only if for any solid $V' \subset V$ with smooth boundary S', the flux of F through S' is zero.*

Proof. If F is solenoidal, then $\text{div}F = 0$ everywhere in V, thus by the Gauss-Ostrogradsky theorem, the flux through any such closed surface S' in V is zero.

If the second condition holds, then by the coordinate free definition of the divergence, see Remark 4.19(2), for any $p \in V$, $\text{div}F(p) = 0$. □

Example 4.28. Any vector field F which is a curl of some other vector field A is solenoidal. Namely, if $F = \text{curl}A$, then $\text{div}F = 0$. Combining this with the result of Proposition 3.16, we have for any scalar field φ and vector field A,

$$\text{curl}\nabla\varphi = 0, \quad \text{divcurl}A = 0,$$

where the first 0 is the zero vector field and the second 0 is the scalar.

Remark 4.29. Interestingly, in certain sense, the last example characterises all solenoidal vector fields. Indeed, the following result holds. (We omit the proof.) If in addition V is simply connected, then F is solenoidal in V if and only if $F = \text{curl}A$ for some A. Such A is called *vector potential* of F.

Helmhotz theorem (the fundamental theorem of vector calculus) states that any (smooth) vector field F is the sum of a conservative and a solenoidal vector fields, i.e., $F = \nabla\varphi + \text{curl}A$.