EXERCISES, Week 13 (submit by 24.01.2018)

1. Compute the following integrals using residues.

$$\text{(a)} \int_0^{2\pi} \frac{d\varphi}{a+\cos\varphi} \ (a>1) \qquad \text{(b)} \int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)^2} \ (a>0)$$

$$\text{(c)} \int_{-\infty}^{\infty} \frac{x\cos x dx}{x^2-2x+10} \qquad \text{(d)} \int_0^{\infty} \frac{x\sin ax dx}{x^2+b^2} \ (a,b>0)$$

2. Identify the type (elliptic, hyperbolic, parabolic) of the following PDEs.

   (a) $\sin^2 y\, u_{xx} - e^{2x}\, u_{yy} + u_x = 0$

   (b) $(x-y)\, u_{xx} + (xy - y^2 - x + y)\, u_{xy} = 0$

3. Let φ be twice continuously differentialbe on $\mathbb{R}$, ψ continuously differentialbe on $\mathbb{R}$, and f continuous on $\mathbb{R}\times[0,\infty)$. Prove that the unique twice continuously differentiable solution $u(x,t)$, $x\in\mathbb{R}$, $t\geq 0$, to

$$\begin{cases} u_{tt} = a^2 u_{xx} + f \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

   is given by the (*d'Alembert*'s) formula

$$u(x,t) = \frac{\varphi(x+at)+\varphi(x-at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(y)dy + \frac{1}{2a}\int\limits_0^t \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi\, d\tau\,.$$

4. Find solutions to the following problems.

   (a) $u_{tt} = 9u_{xx} + \sin x$, $x\in\mathbb{R}$, $t>0$, $u(x,0)=1$, $u_t(x,0)=1$

   (b) $u_{tt} = u_{xx} + \cos x$, $x\geq 0$, $t>0$, $u(x,0)=\cos x$, $u_t(x,0)=0$, $u_x(0,t)=0$