EXERCISES, Week 1 (submit by 18.10.2017)

1. A set B is *elementary* if it can be expressed as a union of boxes $B_1, \dots, B_k$ with disjoint interia. Its measure $m(B)$ is then defined as the sum of the measures of the boxes, i.e., $m(B) = \sum_{i=1}^{k} m(B_i)$. The *inner Jordan measure* of $S \subset \mathbb{R}^n$ is $\mu_*(S) = \sup\{m(B) : B \subseteq S, B \text{ is elementary}\}$ and the *outer Jordan measure* of S is $\mu^*(S) = \inf\{m(B) : B \supseteq S, B \text{ is elementary}\}$.

   For $k \geq 0$, let $T_k = \{[\frac{a_1}{2^k}, \frac{a_1+1}{2^k}] \times \dots \times [\frac{a_n}{2^k}, \frac{a_n+1}{2^k}] : a_1, \dots, a_n \in \mathbb{Z}\}$ and define $q_k(S)$ as the union of all boxes from T_k that are contained in S and $Q_k(S)$ as the union of all boxes from T_k that intersect S.

   Prove that $\mu_*(S) = \lim_{k\to\infty} m(q_k(S))$ and $\mu^*(S) = \lim_{k\to\infty} m(Q_k(S))$.

   [Hint: First prove that for any box B, $m(B) = \lim_{k\to\infty} m(q_k(B)) = \lim_{k\to\infty} m(Q_k(B))$. Then, prove that the same equalities hold for the measure $m(B)$ of any elementary set B. Finally, prove the main statements.]

2. Let S_1, S_2 be any bounded sets in $\mathbb{R}^n$. Prove that

   (a) $\mu^*(S_1 \cup S_2) \leq \mu^*(S_1) + \mu^*(S_2)$,

   (b) if $S_1 \cap S_2 = \emptyset$, then $\mu_*(S_1 \cup S_2) \geq \mu_*(S_1) + \mu_*(S_2)$.

3. Let S_1, S_2 be Jordan measurable sets. Prove that

   (a) $S_1 \cup S_2$ is Jordan measurable,

   (b) if $S_1 \cap S_2 = \emptyset$, then $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$.

4. Prove that any rectifiable curve in $\mathbb{R}^2$ (i.e., a curve with finite length) has Jordan measure 0.

   [Hint: Let L be the length of the curve. First prove that for any $\varepsilon > 0$, the curve can be covered by at most $\frac{L}{\varepsilon}$ squares, each of side length 2ε. Then use subadditivity of the outer Jordan measure, see Problem 2(a).]