EXAM SOLUTIONS, 19 February 2018, 10:00 – 12:00

1. Let $S = \{(x,y) : 4x^2 + y^2 \le 4, x \ge 0\}$. Compute

$$I = \iint_S x\,dxdy.$$

Answer: $\frac{4}{3}$.

Solution. S is normal with respect to x-axis, $S = \{(x,y) : -2 \le y \le 2, 0 \le x \le \sqrt{1-\frac{y^2}{4}}\}$. Thus, by Fubini's theorem,

$$I = \int_{-2}^{2} dy \int_0^{\sqrt{1-\frac{y^2}{4}}} xdx = \int_{-2}^{2} \frac{1}{2}\left(1 - \frac{y^2}{4}\right) dy = \frac{4}{3}.$$

[One could instead use that S is normal with respect to y-axis or pass to generalized polar coordinates $x = r\cos\varphi$, $y = 2r\sin\varphi$, where $0 \le r \le 1$ and $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$.] □

2. Let $a > 0$. Let γ be the circle $x^2 + y^2 = 2ax$. Compute

$$I = \int_\gamma (y - x)ds.$$

Answer: $-2\pi a^2$.

Solution. Parametrize γ by $\begin{cases} x = a + a\cos t \\ y = a\sin t \end{cases}$, where $0 \le t \le 2\pi$. Then,

$$I = \int_0^{2\pi} (a\sin t - a - a\cos t) \underbrace{\sqrt{(-a\cos t)^2 + (a\sin t)^2}}_{=a} dt = -2\pi a^2.$$

□

3. Let γ be the curve $y = x^3$ for $0 \le x \le 2$. Compute

$$I = \int_\gamma ydx - xdy.$$

Answer: -8.

Solution. By the definition of the line integral, $I = \int_0^2 (x^3\,1 - x\,3x^2)\,dx = -8$. □

4. Let S be the surface given by $x = u\cos v$, $y = u\sin v$, $z = v$, where $0 \le u \le 1$, $0 \le v \le 2\pi$. Compute

$$I = \iint_S \frac{1}{\sqrt{1 + x^2 + y^2}} dS.$$

Answer: 2π.

Solution. We begin by computing the coefficients of the first fundamental form for S. For that, we first compute the basis vectors of the tangent plane:

$$r_u = (\cos v, \sin v, 0)\,, \qquad r_v = (-u\sin v, u\cos v, 1)\,.$$

Thus, $E = r_u^2 = 1$, $F = r_u \cdot r_v = 0$, $G = r_v^2 = 1 + u^2$, and $\sqrt{EG - F^2} = \sqrt{1+u^2}$. We now use the definition of the surface integral:

$$I = \int_0^1 du \int_0^{2\pi} dv \,\frac{1}{\sqrt{1 + (u\cos v)^2 + (u\sin v)^2}}\,\sqrt{1+u^2} = \int_0^1 du \int_0^{2\pi} dv \ = 2\pi.$$

□

5. Let S be the outer surface of the cube $\{(x,y,z) : 1 \le x \le 2, 1 \le y \le 2, 1 \le z \le 2\}$. Let $F = (\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$. Compute $I = \iint_S F \cdot dS$.

   Answer: $-\frac{3}{2}$.

   Solution. Partition S into 6 rectangles (faces of the cube): $S_{x=1} = \{(x,y,z) : x = 1, 1 \le y \le 2, 1 \le z \le 2\}$, $S_{x=2} = \{(x,y,z) : x = 2, 1 \le y \le 2, 1 \le z \le 2\}$, $S_{y=1}$, $S_{y=2}$, $S_{z=1}$ and $S_{z=2}$.

   First consider $S_{x=1}$. The outgoing unit normal to $S_{x=1}$ is $(-1,0,0)$. Thus,

$$I_{x=1} = \iint_{S_{x=1}} F \cdot dS = \iint_{S_{x=1}} (-1)\, dS = -\text{Area}(S_{x=1}) = -1.$$

   Next, consider $S_{x=2}$. The outgoing unit normal to $S_{x=2}$ is $(1,0,0)$. Thus,

$$I_{x=2} = \iint_{S_{x=2}} F \cdot dS = \iint_{S_{x=2}} \frac{1}{2}\, dS = \frac{1}{2}\,\text{Area}(S_{x=2}) = \frac{1}{2}.$$

   The remaining 4 cases are symmetrical and give, respectively, -1, $\frac{1}{2}$, -1, $\frac{1}{2}$. Finally, $I = 3(-1 + \frac{1}{2}) = -\frac{3}{2}$.

   [Alternatively one could use the Gauss-Ostrogradsky theorem.] □

6. Let γ be the curve on the intersection of the cone $x^2 + y^2 = z^2$ and the plane $z = x + 1$ positively oriented with respect to vector $(0,0,1)$. Let $F = (z^{19}, y^2, x^{2018})$. Compute $I = \int_\gamma F \cdot ds$.

   Answer: 0.

   Solution. We apply Stokes' theorem to the surface S on the plane $z = x + 1$ surrounded by γ. The unit normal to this surface is everywhere the same as the unit normal to the plane, namely, $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ or $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$. The orientation of the surface positive with respect to vector $(0,0,1)$ corresponds to the unit normal $n = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. (The inner product of this normal and $(0,0,1)$ is positive.)

To apply Stokes' theorem, we also need to compute the curl of F:

$$\mathrm{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^{19} & y^2 & x^{2018} \end{vmatrix} = \left(0, 19z^{18} - 2018x^{2017}, 0\right).$$

In fact, we are only interested in the projection of the curl onto n: $\mathrm{curl} F \cdot n = 0$. Thus, by Stokes' theorem, $I = \iint_S \mathrm{curl} F \cdot n \, dS = 0$. $\square$

7. Let $f(z) = e^{\overline{z}}$, $z \in \mathbb{C}$. Is f differentiable at 0? Give a proof.

   Answer: no.

   Solution. Note that $f(0) = 1$ and for $x, y \in \mathbb{R}$,

$$\lim_{x\to 0} \frac{f(x) - 1}{x} = \lim_{x \to 0} \frac{e^x - 1}{x} = 1, \quad \lim_{y \to 0} \frac{f(iy) - 1}{iy} = \lim_{y \to 0} \frac{e^{-iy} - 1}{iy} = -1.$$

   Thus, f is not differentiable at 0.

   [One could instead show that f does not satisfy Cauchy-Riemann relations at 0.] $\square$

8. Let γ be the circle $|z - 1| = 3$. Compute

$$I = \frac{1}{2\pi i} \oint_{\gamma} \frac{z dz}{\sin z}.$$

   Answer: $-\pi$.

   Solution. We use the residue theorem. Note that γ surrounds two singularities of $\frac{z}{\sin z}$, 0 and π. 0 is removable and π is a simple pole. Thus,

$$I = \mathrm{Res}_{\pi} \frac{z}{\sin z} = \left.\frac{z}{\cos z}\right|_{z=\pi} = -\pi.$$

   $\square$

9. Find the Laurent series for the function $f(z) = \frac{1}{z^2+1}$ in $0 < |z - i| < 2$.

   Answer: $\sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{(2i)^{n+2}} (z - i)^n$.

   Solution. Note that $\frac{1}{z^2+1} = \frac{1}{z-i} \frac{1}{z+i}$. The first fraction already has the correct form, we only need to expand the second one:

$$\frac{1}{z+i} = \frac{1}{2i + (z-i)} = \frac{1}{2i} \frac{1}{1 + \frac{z-i}{2i}} = \frac{1}{2i} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2i)^n} (z - i)^n,$$

   where in the last step we used $\left|\frac{z-i}{2i}\right| < 1$. Now substitute into the first expression:

$$\frac{1}{z^2+1} = \frac{1}{z-i} \frac{1}{2i} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2i)^n} (z-i)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2i)^{n+1}} (z-i)^{n-1} = \sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{(2i)^{n+2}} (z-i)^n.$$

   [Note: $z = i$ is a simple pole, which agrees with the form of the found Laurent series.] $\square$

10. For which values of $a \in \mathbb{R}$ is the following PDE elliptic in the positive quadrant $\{x > 0, y > 0\}$ of $\mathbb{R}^2$?

$$y^2\, u_{xx} + axy\, u_{xy} + x^2\, u_{yy} + a^2\, u_x - 12ax^3\, u_y + 21 = 0.$$

Answer: $-2 < a < 2$.

Solution. The type of second order PDE is determined only by the coefficients of the second derivatives. A PDE is elliptic, if the determinant of the symmetric matrix of these coefficients is positive:

$$\begin{vmatrix} y^2 & \frac{1}{2}axy \\ \frac{1}{2}axy & x^2 \end{vmatrix} = x^2y^2\left(1 - \frac{1}{4}a^2\right) > 0 \quad \overset{(x,y>0)}{\Longleftrightarrow} \quad |a| < 2.$$

□

11. Solve the initial-boundary value problem

$$\begin{cases} u_t = u_{xx} + \sin \pi x & 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0 & t \geq 0 \\ u(x,0) = \sin 2\pi x & 0 \leq x \leq 1. \end{cases}$$

Answer: $u(x,t) = \frac{1}{\pi^2}\left(1 - e^{-\pi^2 t}\right) \sin \pi x + e^{-(2\pi)^2 t} \sin 2\pi x$.

Solution. We know that a solution to this problem exists and unique.

Taking into account the boundary conditions, we search for the solution in the form of the infinite series $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\, \sin \pi n x$.

From the initial condition, the unknown functions $u_n(t)$ should satisfy

$$\sin 2\pi x = u(x,0) = \sum_{n=1}^{\infty} u_n(0)\, \sin \pi n x \quad \Longleftrightarrow \quad u_2(0) = 1,\, u_n(0) = 0\, (n \neq 2).$$

We formally substitute the expression for u in the PDE:

$$\sum_{n=1}^{\infty} \left(u_n'(t) + (\pi n)^2\, u_n(t)\right) \sin \pi n x = \sin \pi x,$$

which corresponds to

$$\begin{cases} u_1'(t) + \pi^2 u_1(t) = 1 \\ u_n'(t) + (\pi n)^2 u_n(t) = 0 \ (n \geq 2) \end{cases} \Longleftrightarrow \begin{cases} u_1(t) = u_1(0)\, e^{-\pi^2 t} + \int_0^t e^{\pi^2(s-t)} ds \\ u_n(t) = u_n(0)\, e^{-(\pi n)^2 t} \ (n \geq 2) \end{cases}$$

$$\Longleftrightarrow \begin{cases} u_1(t) = \frac{1}{\pi^2}\left(1 - e^{-\pi^2 t}\right) \\ u_2(t) = e^{-(2\pi)^2 t} \\ u_n(t) = 0 \ (n \geq 3). \end{cases}$$

Thus, $u(x,t) = \frac{1}{\pi^2}\left(1 - e^{-\pi^2 t}\right) \sin \pi x + e^{-(2\pi)^2 t} \sin 2\pi x$. A direct check gives that u is indeed a solution. □