Dean-Kawasaki Dynamics: Ill-posedness vs. Triviality

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Abstract

We prove that the Dean-Kawasaki SPDE admits a solution only in integer parameter regimes, in which case the solution is given in terms of a system of non-interacting particles.

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1. Introduction

In this paper we show a dichotomy of non-existence vs. triviality of solutions to a certain class of nonlinear SPDE which arise e.g. in macroscopic fluctuation theory in physics. As a prototype one might consider the model

$$\partial_t \rho = T\Delta\rho + \nabla \cdot \left(\sqrt{T\rho}\dot{W}\right), \tag{1}$$

which is a particular instance of the more general class of formal Ginzburg-Landau stochastic phase field models (cf.[30]) of the form

$$\partial_t \phi + \nabla \cdot \left(-L(\phi)\nabla \frac{\delta H}{\delta \phi}(\phi) + \sqrt{TL(\phi)}\dot{W}\right) = 0,$$

where H is a Hamiltonian, L is an Onsager coefficient and $\dot{W}$ is vector valued space-time white noise. The particular equation (1) was proposed independently in [8] and [20] as mesoscopic description for interacting particles and is referred to as Dean-Kawasaki equation today. Since then, together with several variants, it has been an active research topic in various branches of non-equilibrium statistical mechanics over the last years (c.f.[9, 12, 16, 18, 21, 24, 32]). Mathematically, interest in equations like (1) comes from the fact that it appears to describe an 'intrinsic' random perturbation of the gradient flow for the entropy functional on Wasserstein space by a noise which is locally uniformly distributed in terms of dissipated energy, e.g. by a noise that is aligned with Otto's formal Riemannian structure [28] for optimal transportation. To see this, consider the rescaled Hamiltonian $\mathcal{H}(\mu,\phi) := \lim_{\epsilon\to 0} \epsilon e^{-\frac{1}{\epsilon}\langle\mu,\phi\rangle}\mathcal{G}^\epsilon e^{\frac{1}{\epsilon}\langle\mu,\phi\rangle}$, with $\mathcal{G}^\epsilon$ being the generator associated to $\{\mu_{\epsilon t}\}_{t\geq 0}$. Following [10], one expects the short time asymptotics of μ to be governed by the large deviation rate function

$$\mathcal{A}(\nu) = \int_0^1 \sup_{\phi\in C_0^\infty(M)} \{\langle\dot{\nu},\phi\rangle - \mathcal{H}(\nu,\phi)\rangle\}dt,$$

for regular curves ν in the Wasserstein-space. But since $\mathcal{H}(\mu,\phi) = \frac{1}{2}\langle\mu, |\nabla\phi|^2\rangle$, we see that $\mathcal{A}$ is precisely the energy functional which determines the Wasserstein-metric by means of the Benamou-Brenier-formula (c.f. [3] and Appendix D in [14]).

In spite of significant continuing interest in Dean-Kawasaki type models both in physics and mathematics, rigorous results on existence and uniqueness exist so far only for appropriate regularisations of the original equation, see e.g. [7, 13, 25]. On the other hand, in [33] Sturm and the third author succeeded in constructing a process having the Wasserstein distance as its intrinsic metric and which can formally be seen as a solution to the SPDE

$$\partial_t\mu = \alpha\Delta\mu + \Xi(\mu) + \nabla\cdot\left(\sqrt{\mu}\dot{W}\right),$$

where Ξ is some nonlinear operator. Particle approximations of these dynamics as well as analytic and geometric properties of the corresponding entropic measure were investigated afterwards in [1, 31] and [5, 11], respectively. Based on Arratia flows, a new candidate for a process with Wasserstein-short-time asymptotics, but with different drift component as in [33] was recently studied by the latter two authors in [22, 23] and subsequently, in [26].

The main contribution of this note asserts that some correction term Ξ is in fact necessary for the existence of (nontrivial) solutions to these DK-type models. More precisely, in Theorem 1 below we find that the (uncorrected) generalized Dean-Kawasaki equation

$$\begin{aligned}\partial_t\mu &= \frac{\alpha}{2}\Delta\mu + \nabla\cdot\left(\sqrt{\mu}\dot{W}\right),\\ \mu_{|t=0} &= \mu_0,\end{aligned} \qquad (\mathrm{DK})^\alpha_{\mu_0}$$

corresponding to a Ginzburg-Landau model with $H = \alpha\mathrm{Ent}$, $T = 1$ and $L =$ identity operator, admits solutions only for a discrete spectrum of parameters α and atomic intial measures. Moreover, for these particular choices solutions are trivial, in being just 'measure-valued lifts' of the martingale-problem for $\frac{\alpha}{2}\Delta$.

Finally, given the apparent similarity of $(\mathrm{DK})^\alpha_{\mu_0}$ to the SPDE description of the Dawson-Watanbe ('Super Brownian Motion') process

$$\partial_t\mu = \beta\Delta\mu + \sqrt{\mu}\dot{W},$$

admitting unique in law solutions for every $\beta > 0$, our result is interesting also from an independent SPDE point of view.

2. Statement and proof of the main result

As for notation, given a Polish space E, we will write $\mathcal{M}_1(E)$ for the set of all probability measures on E and for any function f on E, we write as usual $\langle \mu, f\rangle = \int_E f(x)\mu(dx)$, whenever the integral is well-defined. By $C_b(E)$ we denote space of real-valued, bounded continuous functions on E.

Let us briefly motivate, what we will refer to as a solution to the Dean-Kawasaki equation. Typically, one would call a time-continuous process $t \mapsto \mu_t$, which takes values in absolutely continuous measures on $\mathbb{R}^d$, a solution to $(\mathrm{DK})^\alpha_{\mu_0}$, provided that for all $t \in [0,T]$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz-space) it holds true that $\mathbb{P}$-a.s

$$\langle \mu_t, \phi\rangle = \langle \mu_0, \phi\rangle + \frac{\alpha}{2}\int_0^t \langle \mu_s, \Delta\phi\rangle ds - \sum_{i=1}^d \int_0^t \int_{\mathbb{R}^d} \sqrt{\mu_s(x)}\partial_i\phi(x)W^i(dsdx), \tag{2}$$

with W^i being mutually independent space-time white noises (for instance in the sense of Walsh [34]).

Of course, for such a process μ we knew that

$$[0,T] \ni t \mapsto \langle \mu_t, \phi\rangle - \langle \mu_0, \phi\rangle - \frac{\alpha}{2}\int_0^t \langle \mu_s, \Delta\phi\rangle ds$$

is martingale with quadratic variation

$$\int_0^t \langle \mu_s, |\nabla\phi|^2\rangle ds.$$

Rather then the weak formulation in (2), it is this martingale characterisation that we will study in the following, however in slightly more general setup.

Let E be a Polish state-space and (E, π, Γ) be the Markov-triple associated to some symmetric Markov Diffusion operator L in the classical sense, as e.g. in [2]. In this setting, we know L has the diffusion property, i.e.

$$L\psi(f) = \psi'(f)Lf + \psi''(f)\Gamma f$$

for every $\psi \in C^2$ and $f \in \mathcal{D}(L)$. Here, Γ is the is the carré du champs operator, defined on some algebra $\mathcal{A} \subset C^b(E)$ which is dense in $L^p(\pi)$, by

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf)$$

for $(f,g) \in \mathcal{A}\times\mathcal{A}$ and $\Gamma f = \Gamma(f,f)$. Additionally, we impose henceforth the following regularity hypothesis: the Markov-semigroup P_t belonging to L, satisfies a gradient bound, i.e. for some $\rho \in \mathbb{R}$, we have

$$\Gamma P_t f \leq e^{-2\rho t} P_t \Gamma f. \tag{3}$$

This requirement is certainly fulfilled for the Brownian semigroup on Euclidean space, but also for instance when $L = \Delta_{\mathfrak{g}}$ is the Laplace-Beltrami operator on a Riemannian manifold $(E = M, \mathfrak{g})$ with Ricci curvature bounded from below.

Definition Let $\alpha > 0$, $\mu_0 \in \mathcal{M}^1(E)$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be some probability base. We say that a $\mathbb{P}$-almost surely continuous $\mathcal{M}^1(E)$-valued process μ is a solution to the martingale problem $(\mathrm{MP})^{\alpha}_{\mu_0}$ iff

1. for all $\phi \in \mathcal{D}(L)$

$$M_t(\phi) := \langle \mu_t, \phi\rangle - \langle \mu_0, \phi\rangle - \frac{\alpha}{2}\int_0^t \langle \mu_s, L\phi\rangle ds, \quad t \in [0, T], \tag{4}$$

   is an $\mathcal{F}_t$-adapted martingale,

2. whose quadratic variation is given by

$$\langle M(\phi)\rangle_t = \int_0^t \langle \mu_s, \Gamma\phi\rangle ds.$$

With this notation, our main results reads as follows.

Theorem 1. *Solutions to* $(\mathrm{MP})^{\alpha}_{\mu_0}$ *exist, if and only if* $\alpha = n \in \mathbb{N}$ *and*

$$\mu_0 = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}, \tag{5}$$

with $x_1, \dots, x_n \in E$. *In case of existence, the solution is, uniquely in law, given by the empirical measure*

$$\mu_t = \frac{1}{n}\sum_{i=1}^n \delta_{X^i_{nt}}, \tag{6}$$

where the X^i *are* n *independent diffusion processes, each with generator* $\frac{1}{2}L$ *and starting point* x_i.

Certainly, the Dean-Kawasaki dynamic fits in as the special case of taking $L = \Delta$ on $E = \mathbb{R}$.

Theorem 1 will be proven in several steps. We start by the almost trivial observation that the empirical measures (6) provide solutions to $(\mathrm{MP})^{n}_{\mu_0}$. Consider first the case $n = 1$. Plugging $\mu_t = \delta_{X_t}$ into the martingale problem, immediately yields that

$$M_t(\phi) = \phi(X_t) - \phi(X_0) - \frac{1}{2}\int_0^t L\phi(X_s)ds$$

is a martingale, plainly since X is the solution of the martingale problem for $\frac{1}{2}L$. Also necessarily, the quadratic variation of M satisfies

$$\langle M(\phi)\rangle_t = \int_0^t \Gamma\phi(X_s)ds = \int_0^t \langle \mu_s, \Gamma\phi\rangle ds.$$

For the general case, denote by M^i the martingale associated to X^i. Then

$$M_t(\phi) = \frac{1}{n}\sum_{i=1}^n \left(\phi(X^i_{nt}) - \phi(X^i_0) - \frac{n}{2}\int_0^t L\phi(X^i_{ns})ds\right) = \frac{1}{n}\sum_{i=1}^n M^i_{nt}(\phi)$$

is a $\mathcal{F}_{nt}$-martingale with quadratic variation

$$\langle M(\phi)\rangle_t = \frac{1}{n^2}\sum_{i=1}^n \langle M^i(\phi)\rangle_{nt} = \frac{1}{n}\sum_{i=1}^n \int_0^t \Gamma\phi(X_{ns})ds = \int_0^t \langle \mu_s, \Gamma\phi\rangle ds,$$

as needed.

Our next aim is to show that these solutions are unique in law. In fact, the statement that we prove is much stronger, namely that any solution, provided its existence, must be unique in law. The proof adopts the argument which is used in order proof weak uniqueness for super-Brownian motion, by Laplace-duality to some reaction-diffusion equation (c.f.[15, 27, 29]).

Before we present the duality statement in our context, we provide some preliminary considerations on viscous Hamilton-Jacobi equations. That is, we regard for some initial datum $f \in C_b(E)$, the PDE

$$\begin{aligned} \partial_t v &= \frac{\alpha}{2}Lv - \frac{1}{2}\Gamma v, \\ v_{|t=0} &= f, \end{aligned} \tag{vHJ$_f$}$$

where L and Γ are as before generator and carré du champs operator of some symmetric Markov diffusion. Of course, the unique solution is just the classical Cole-Hopf-solution given by

$$V_t f(x) = v(t,x) = -\alpha \ln(P_t e^{-\frac{1}{\alpha}f})(x),$$

where P_t is now the heat semi-group associated to $\frac{\alpha}{2}L$. Indeed, plugging in the definition of $V_t f$ into the PDE, we see

$$\partial_t V_t f = -\alpha(P_t e^{-\frac{1}{\alpha}f})^{-1} L P_t e^{-\alpha f} = -\frac{\alpha^2}{2} e^{\frac{1}{\alpha}V_t f} L e^{-\frac{1}{\alpha}V_t f} = \frac{\alpha}{2} L V_t f - \frac{1}{2}\Gamma V_t f.$$

Moreover, one can easily check that this solution satisfies maximum/minimum-principles, i.e.

$$\inf_E f \le \inf_E V_t f \qquad \text{and} \qquad \sup_E V_t f \le \sup_E f$$

for $f \in C_b(E)$. Also, by the gradient bound (3), there is $\rho \in \mathbb{R}$ such that for all $f \in \mathcal{A}$, it follows that

$$\Gamma V_t f \le \alpha^2 (P_t e^{-\frac{1}{\alpha}f})^{-2} e^{-2\rho t} P_t \Gamma e^{-\frac{1}{\alpha}f} \le 1 \vee e^{2|\rho|T} e^{\frac{2}{\alpha}\operatorname{diam} f(E)} \|\Gamma f\|_\infty. \tag{7}$$

We can now formulate our duality result, which will be crucial not only for the uniqueness, but also for the discussion of non-existence later on.

Theorem 2. *Take $\alpha > 0$. For $\mu_0 \in \mathcal{M}_1(E)$ and $f \in C_b(E)$, let μ be a solution to* $(\mathrm{MP})^\alpha_{\mu_0}$ *and $V_t f$ a solution to* $(\mathrm{vHJ})_f$. *Then*

$$\mathbb{E}^{\mu_0} e^{-\langle \mu_t, f\rangle} = e^{-\langle \mu_0, V_t f\rangle}. \tag{8}$$

Proof. Assume for now $f \in \mathcal{A}$ and let $v(t,x) = V_t f(x)$ be the Cole-Hopf solution to $(\mathrm{vHJ})_f$. For $0 \le s \le t \le T$, by Itō's formula

$$\begin{aligned} de^{-\langle \mu_s, v(t-s)\rangle} &= -e^{-\langle \mu_s, v(t-s)\rangle}\left(\langle \mu_s, (\partial_s + \frac{\alpha}{2}L)v(t-s)\rangle ds + dM_s(v(t-s)) - \frac{1}{2}d\langle M(v(t-\cdot))\rangle_s\right) \\ &= e^{-\langle \mu_s, v(t-s)\rangle}\left(\langle \mu_s, (\partial_s v)(t-s) - \frac{\alpha}{2}Lv(t-s) + \frac{1}{2}\Gamma v(t-s)\rangle ds + dM_s(v(t-s))\right) \\ &= e^{-\langle \mu_s, v(t-s)\rangle} dM_s(v(t-s)). \end{aligned}$$

Hence for $t \in [0,T]$ and $s \leq t$ the map $s \mapsto e^{-\langle \mu_s, v(t-s)\rangle}$ is a local martingale. Since by (7)

$$\mathbb{E}\int_0^T e^{-2\langle \mu_s, v(t-s)\rangle}\langle \mu_s, \Gamma v(t-s)\rangle ds \leq CTe^{(2+\frac{2}{\alpha})\operatorname{diam} f(E)}\|\Gamma f\|_\infty < \infty,$$

it is in fact a martingale. Therefore, upon choosing $s = t$ we find

$$\mathbb{E}^{\mu_0} e^{-\langle \mu_t, f\rangle} = e^{-\langle \mu_0, V_t f\rangle}.$$

By density of $\mathcal{A}$ in $C_b(E)$ the previous equation also holds for $f \in C_b(E)$, which yields the claim. □

Observe in particular, that the duality in Theorem 2 determines the Laplace-transform of μ_t and by the same argument as in Theorem 11 of [19], also the finite dimensional distributions of μ uniquely. Hence, we obtain uniqueness (in law) for solutions to the Dean-Kawasaki equation.

In order to develop our non-existence statement, we insert a short intermezzo on generating functions.

Whereas of course, for the probability-generating function

$$g(s) := \mathbb{E}s^X = \sum_{k=0}^{\infty} p_k s^k \tag{9}$$

of a some discrete random variable X with values in $\mathbb{N}^0 = \{0,1,2,\dots\}$, one knows that $p_k = P(X=k)$, we are interested in the opposite direction and establish

Lemma 3. *Let X be a non negative random variable, such that for each $n \in \mathbb{N}^0$*

$$g(s) = \mathbb{E}s^X = \sum_{k=0}^{n} s^k p_k + o(s^n), \quad \text{as} \;\; s \to 0+, \tag{10}$$

for some sequence $\{p_k,\ k \in \mathbb{N}^0\}$. Then $X \in \mathbb{N}^0$ a.s., $p_k \geq 0$ and

$$\mathbb{P}\{X = k\} = p_k$$

for each $k \in \mathbb{N}$.

Proof. We will prove the statement by induction. Due to (10),

$$\mathbb{E}s^X \to p_0 \quad \text{as} \;\; s \to 0+.$$

On the other hand, since $s^X \to \mathbb{1}_{\{X=0\}}$ a.s. as $s \to 0+$, we infer by dominated convergence

$$\mathbb{E}s^X \to \mathbb{P}\{X=0\} \quad \text{as} \;\; s \to 0+.$$

Hence, p_0 is non negative and equals $\mathbb{P}\{X=0\}$.

For the induction step, let us assume that $\mathbb{P}\{X = k\} = p_k$ for $k = 1, \ldots, n-1$ and $X \in \{0, 1, \ldots, n-1\} \cup (n-1, \infty)$ a.s. Again, by (10),

$$\frac{1}{s^n}\left(\mathbb{E}s^X - \sum_{k=0}^{n-1} p_k s^k\right) \to p_n \quad \text{as} \quad s \to 0+ .$$

Using the induction assumption, we can write

$$\begin{aligned}\frac{1}{s^n}\left(\mathbb{E}s^X - \sum_{k=0}^{n-1} p_k s^k\right) &= \frac{1}{s^n}\mathbb{E}\left[s^X - \sum_{k=0}^{n-1} \mathbb{1}_{\{X=k\}} s^X\right] = \frac{1}{s^n}\mathbb{E}\left[\mathbb{1}_{\{\xi > n-1\}} s^X\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\{X \in (n-1,n)\}} s^{X-n}\right] + \mathbb{E}\left[\mathbb{1}_{\{X \geq n\}} s^{X-n}\right].\end{aligned}$$

Again, by the dominated convergence theorem,

$$\mathbb{E}\left[\mathbb{1}_{\{X \geq n\}} s^{X-n}\right] \to \mathbb{P}\{X = n\} \quad \text{as} \quad s \to 0+ .$$

Thus, $\mathbb{E}\left[s^{X-n}\mathbb{1}_{\{X \in (n-1,n)\}}\right]$ is bounded for $s \in (0, 1]$, which implies

$$\mathbb{P}\{X \in (n-1, n)\} = 0.$$

This finishes the proof of the lemma. □

We can now return to our main question and prove, that the trivial solutions we found for $\alpha \in \mathbb{N}$ and atomic μ_0 must in fact be the only possible ones.

Proof of Theorem 1. Take $\alpha > 0$, $\mu_0 \in \mathcal{M}_1(E)$ and a solution μ to $(\mathrm{MP})^{\alpha}_{\mu_0}$. Also, for $A \in \mathcal{B}(E)$ and fixed $t \in [0, T]$, let us abbreviate $h(x) = P_t \mathbb{1}_A(x)$. Note that for bounded A, we can find $\delta > 0$ with $0 \leq h(x) \leq 1 - \delta$ for all $x \in E$.

For such an A and fixed $t > 0$, let us consider the generating function g of the real-valued random variable $X = \alpha\mu_t(A)$. The Laplace-duality of Theorem 2 yields

$$\mathbb{E}e^{-r\alpha\mu_t(A)} = \mathbb{E}e^{-\langle \mu_t, r\alpha\mathbb{1}_A\rangle} = e^{\langle \mu_0, \ln(P_t e^{-r\mathbb{1}_A})\rangle}. \tag{11}$$

But since $P_t e^{-r\mathbb{1}_A} = 1 + (e^{-r} - 1)h$ and setting $s = e^{-r}$, the previous display reads

$$g(s) = e^{\alpha\langle \mu_0, \ln(1+(s-1)h)\rangle} = e^{\alpha\left\langle \mu_0, \ln(1-h) + \ln\left(1 + \frac{h}{1-h}s\right)\right\rangle}. \tag{12}$$

By the boundedness of h, the function g is well-defined on $(-\delta, \infty)$. Moreover, it is infinitely differentiable. Thus, for each $n \in \mathbb{N}^0$

$$g(s) = \sum_{k=0}^{n} p_k s^k + o(s^n) \quad \text{on} \quad (-\delta, \delta),$$

by Taylor's theorem. Consequently, by Lemma 3 we know that $\alpha\mu_t(A) \in \mathbb{N}^0$ a.s. So, we have proved that for each bounded $A \in \mathcal{B}(\mathbb{R}^d)$ and $t > 0$

$$\mu_t(A) \in \left\{0, \frac{1}{\alpha}, \ldots, \frac{\lfloor \alpha \rfloor}{\alpha}\right\} \quad \text{a.s.}$$

Here, we also used the fact that μ_t is a probability measure a.s. Next, making $A \uparrow E$, we obtain that

$$1 = \mu_t(E) \leq \frac{\lfloor \alpha \rfloor}{\alpha} \leq 1 \quad \text{a.s.}$$

This implies that $\alpha \in \mathbb{N}$. In order to make a conclusion about μ_0, we take a bounded set A with μ_0-zero boundary and use the continuity of the process μ. So, we obtain

$$\mu_0(A) \in \left\{ 0, \frac{1}{\alpha}, \dots, \frac{\lfloor \alpha \rfloor}{\alpha} \right\}.$$

Hence, there exist x_k, $k \in \{1, \dots, \alpha\}$ such that

$$\mu_0 = \frac{1}{\alpha} \sum_{k=1}^{\alpha} \delta_{x_k}.$$

$\square$

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