MODIFIED ARRAITA FLOW AND
WASSERSTEIN DIFFUSION

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Extending previous work [19] by the first author we present a variant of the Arratia flow of a field of coalescing Brownian motions starting from every point of the unit interval. The important new feature of the model is that individual particles carry mass which aggregated upon colasence and which regulates the diffusivity of each particle in inverse proportional way. – As our main results we show a large deviation principle for small times, leading to a Varadhan-formula for the induced measure valued flow with the quadratic Wasserstein distance as rate function. Hence this model is proposed as another candidate for an intrinsic Brownian motion on the Wasserstein space of probability measures.

1. Introduction and statement of main results. Since its introduction in [24] Otto’s formal infinite dimensional Riemannian calculus for optimal transportation has been the inspiration for numerous new results both in pure and applied mathematics, see e.g. [25, 35, 22, 30, 1]. It can be considered a lift of conventional calculus of points to the case of point ensembles resp. spatially continuous mass distributions. It is therefore natural to ask whether this lifting procedure from points to mass configurations has a probabilistic counterpart. The fundamental object of such a theory would need to be an analogue of Brownian motion on space of probability measures adapted to Otto’s Riemannian structure of optimal transportation. In [36] the second author together with Sturm proposed a first candidate of such a measure valued Brownian motion (with drift), calling it Wasserstein Diffusion, and showed among other things that its short time asymptotics are indeed governed by the geometry of optimal transport in the sense of Varadhan’s formula with the Wasserstein distance as rate function. – However, the construction in [36] has several limitations since it is strictly restricted to diffusing measures on the real line, it brings about additional seemingly
non-physical correction/renormalization terms and lastly it is obtained by abstract Dirichlet form methods which e.g. do not allow for generic starting points of the evolution. Hence, in spite of several ad-hoc finite dimensional approximations [29, 2, 31] the process remained a rather obscure object.

In this paper we give a different and very explicit construction of another diffusion process in the probabilities on the real line which shares some crucial features of the Wasserstein diffusion. The construction is based on a modification of the so-called Arratia flow of coalescing Brownian motions, which was introduced in [3] and which was later extensively studied by Dorogovtsev and coauthors [8, 10, 9, 23, 11] resp. Le Jan-Raimond [21]. The modified Arratia flow was introduced in [18] and is obtained by assigning a mass to each particle and which is aggregated when particles coalesce. Secondly, the diffusivity of each particle scales inverse proportional to its mass. In [19] it was shown that such a system can be constructed starting with a (zero mass) particle at each point of the unit interval.

The resulting model, which we shall call modified Arratia flow, can best be described in terms of a stochastic flow of martingales for the motion of the particles. Letting $D([0,1], C[0,T])$ denote the Skorokhod space of càdlàg-functions from $[0,1]$ into the metric space of continuous real valued trajectories over the time interval $[0,T]$ the main result of [19] reads as follows.

**Theorem 1.1.** There is a process $y \in D([0,1], C[0,T])$ such that

- (C1) for all $u \in [0,1]$, the process $y(u, \cdot)$ is a continuous square integrable martingale with respect to the filtration

  $\mathcal{F}_t = \sigma(y(u,s), u \in [0,1], s \leq t), \quad t \in [0,T]$;

- (C2) for all $u \in [0,1]$, $y(u,0) = u$;

- (C3) for all $u < v$ from $[0,1]$ and $t \in [0,T]$, $y(u,t) \leq y(v,t)$;

- (C4) for all $u, v \in [0,1]$,

  $$\left[y(u,\cdot), y(v,\cdot)\right]_t = \int_0^t \frac{\mathbb{1}_{\{\tau_{u,v} \leq s\}} ds}{m(u,s)},$$

  where $m(u,t) = \text{Leb}\{v : \exists s \leq t \ y(v,t) = y(u,t)\}$, $\tau_{u,v} = \inf\{t : y(u,t) = y(v,t)\} \wedge T$ and Leb denotes Lebesgue measure on $[0,1]$.

By (C3) the map $[0,1] \ni u \to y(u,t)$ is monotone (and cadlag), hence the one-to-one map between probability measures on $\mathbb{R}$ and their quantile
functions on $[0, 1]$ yields an equivalent parametrization of $y$ by the induced measure valued flow

$$\mu_t := y(\cdot, t)\#\text{Leb}, \quad t \in [0, T],$$

and this process is our main interest for the connection to the Wasserstein diffusion. But before, let us state our main results for the process $y$ first.

1.1. Main results on the modified Arratia flow. Our main result on $y$ is a large deviation principle for the family of time-rescaled processes $y^\epsilon(\cdot) = y(\cdot \epsilon)$. For a proper statement let us introduce some notation, starting with $L_2(\mu) = L_2([0, 1], \mu),$ $\mu(du) = \kappa(u)du,$ where $\kappa : [0, 1] \to [0, 1]$ for some fixed $\beta > 1,$

$$D^\uparrow = \{ h \in D([0, 1], \mathbb{R}) : h \text{ is non-decreasing}\},$$

and finally

$$H = \{ \varphi \in C([0, T], L_2([0, 1], du) \cap D^\uparrow) : \varphi(0) = \text{id} \text{ and } t \mapsto \varphi(t) \in L_2([0, 1], du) \text{ is absolutely continuous}\},$$

$$I(\varphi) = \begin{cases} \frac{1}{2} \int_0^T \|\dot{\varphi}(t)\|^2_{L_2(du)}dt, & \varphi \in H, \\ +\infty, & \text{otherwise}. \end{cases}$$

**Theorem 1.2.** The family of processes $\{y^\epsilon\}_{\epsilon > 0}$ satisfies a large deviations principle in the space $C([0, T], L_2(\mu))$ with the good rate function $I.$

For an illustration of this result we refer to section 4 below, where we show that for each $g \in L_2(du),$ $s \mapsto J(g)(s) := (g, y(s))_{L_2(du)}$ is a continuous square integrable martingale with quadratic variation process

$$[J(g)]_t = \int_0^t \|\text{pr}_{y(s)}g\|^2_{L_2(du)}ds, \quad t \in [0, T],$$

where $(\cdot, \cdot)_{L_2(du)}$ denotes the inner product in $L_2(du), \text{pr}_{y(s)}g := \mathbb{E}_{du}[g|\sigma(y(s))].$ Showing that the process $y(\cdot)$ is a weak martingale solution to the infinite dimensional SDE

$$dy(s) = \text{pr}_{y(s)}dW_s.$$
where $dW$ is space-time white noise in $L_2(du)$. Equivalently the processes $y^\varepsilon(\cdot)$ solve
\[
dy^\varepsilon(s) = \text{pr}_{y^\varepsilon(s)} \sqrt{\varepsilon} dW_s.
\]
The LDP above therefore appears as an instance of the classical Freidlin-Wentzel LDP for solutions of SDE, but here we have to deal with additional difficulties since the diffusion operator $y \to \sigma(y) = \text{pr}_y$ is not continuous as an operator-valued map on $L_2(\mu)$, and generally little is known about such large deviation principles for solutions of SDE with non-smooth coefficients even in finite dimensions. In our case we can overcome these difficulties with additional arguments, using the fact that $\sigma$ is continuous on strictly monotone $y \in L_2(\mu)$.

1.2. A relative to the Wasserstein Diffusion. Let us collect the previous results in terms of the induced measure valued process $\mu_t := y(\cdot, t) \# \text{Leb}$, $t \in [0, T]$, in order to identify it as a close relative of the Wasserstein diffusion. First we claim that $\mu_t$ solves a martingale problem very similar to its counterpart for the Wasserstein diffusion. In fact we may use stochastic calculus w.r.t. the $L_2(\mu)$-valued martingale $y$ (op. cit. and section 3 below) to obtain the following immediate result.

**Corollary 1.1.** Let $\mu_t := y(\cdot, t) \# \text{Leb}$, then for all $f \in C_b^2(\mathbb{R})$
\[
M^f_t := \langle f, \mu_t \rangle - \int_0^t \langle f, \Gamma(\mu_s) \rangle \, ds
\]
is a continuous local martingale with quadratic variation process
\[
\left[ M^f \right]_t = \int_0^t \langle (f')^2, \mu_s \rangle \, ds,
\]
where
\[
\langle f, \Gamma(\mu) \rangle = \sum_{x \in \text{supp}(\mu)} f''(x).
\]
Consequently, $\mu_t$ is a probability valued martingale solution to the SPDE
\[
d\mu_t = \Gamma(\mu_t) dt + \text{div}(\sqrt{\mu} dW_t),
\]
which should be compared to the corresponding SPDE for the Wasserstein diffusion [36, 2]
\[
d\mu_t = \beta \Delta \mu_t dt + \hat{\Gamma}(\mu_t) dt + \text{div}(\sqrt{\mu} dW_t),
\]
with
\[
\langle f, \hat{\Gamma}(\mu) \rangle = \sum_{I \in \text{gaps}(\mu)} \left[ \frac{f''(I_+) + f''(I_-)}{2} - \frac{f'(I_+) - f'(I_-)}{I} \right].
\]

Thus, besides the apparent similarity of the second order part in the drift operators \( \Gamma \) and \( \hat{\Gamma} \), both models share the same singular multiplicative noise which gives rise to the characteristic density \( \langle (f')^2, \mu \rangle \) in the quadratic variation process. Of course, this is the same expression appearing in Otto’s definition of the Riemannian energy of a variation of a measure resp. in the Benamou-Brenier Formula for optimal transportation.

This connection is made more rigorous by the following variant of the Varadhan formula which is a straightforward combination of Theorem 1.2, the contraction principle applied to the endpoint map \( C([0, 1], L^2(\mu)) \to L_2(\mu), \mu_{t \in [0, 1]} \to \mu_1 \) and the fact that the map
\[
i : D^\uparrow([0, 1]) \ni y \mapsto y_# \text{Leb} \in \mathcal{P}(\mathbb{R})
\]
is an isometry from the \( L^2(du) \)-metric to the quadratic Wasserstein metric \( d_{W} \). Furthermore, let \( \tau_\mu \) denote the image topology on \( \mathcal{P}(\mathbb{R}) \) of the \( L^2(\mu) \)-topology on \( \mathcal{P}(\mathbb{R}) \) induced from the bijection \( i \). Recall also that a set \( A \subset \mathcal{P}(\mathbb{R}) \) is called displacement convex if it is the image of a convex subset of \( D^\uparrow([0, 1]) \) under the map \( i \).

**Corollary 1.2.** Let \( A \subset \mathcal{P}(\mathbb{R}) \) \( \tau_\mu \)-closed with nonempty interior and displacement convex, then
\[
\lim_{\varepsilon \to 0} \varepsilon \log P(\mu_\varepsilon \in A) = -\frac{(d_W(\text{Leb}, A))^2}{2}.
\]

**Organisation of the Paper.** We conclude the introduction by an outline of the organization of the paper. In section 2 we give a streamlined review of the construction of the modified Arratia flow from [19]. In sections 3 we introduce some elements of a stochastic calculus relative to \( y \) to the extent needed in the sequel. The final sections 4 are devoted to the proof of the large deviations principle Theorem 1.2.

**2. Construction of a system of coalescing heavy diffusion particles.** The goal of this chapter is the construction of a random element \( \{y(u, t), \ u \in [0, 1], \ t \in [0, T]\} \) in the Skorokhod space \( D([0, 1], C[0, T]) \) that describes the evolution of coalescing heavy diffusion particles on the real
line, where \( y(u, t) \) is the position of particles at time \( t \) starting from \( u \). Since such a mathematical model was constructed in [19], we only give a brief exposition of the construction.

2.1. A finite number of particles. In this section we describe a finite system of particles which start from a finite number of points, move independently up to the moment of the meeting and coalesce, and change their diffusion. Since a system of particles starting from all points of the interval \([0, 1]\) will be approximated by a finite system, we will consider the case where particles start from the points \( \frac{k}{n}, k = 1, \ldots, n \), with the mass \( \frac{1}{n} \). So, let \( n \in \mathbb{N} \) be fixed.

**Proposition 2.1.** There exists a set of the processes \( \{x^n_k(t), k = 1, \ldots, n, t \in [0, T]\} \) that satisfies the following conditions

1. **(F1)** for each \( k \), \( x^n_k \) is a continuous square integrable martingale with respect to the filtration
   \[
   \mathcal{F}^n_t = \sigma(x^n_l(s), s \leq t, l = 1, \ldots, n);
   \]
2. **(F2)** for all \( k \), \( x^n_k(0) = \frac{k}{n} \);
3. **(F3)** for all \( k < l \) and \( t \in [0, T] \), \( x^n_k(t) \leq x^n_l(t) \);
4. **(F4)** for all \( k \) and \( l \),
   \[
   [x^n_k, x^n_l]_t = \int_0^t \frac{\mathbb{1}_{\{x^n_k \leq s\}}}{m^n_k(s)} ds,
   \]
   where \( m^n_k(t) = \frac{1}{n} \# \{ j : \exists s \leq t \ x^n_j(s) = x^n_k(s) \} \), \( \tau^n_{k,l} = \inf \{ t : x^n_k(t) = x^n_l(t) \} \wedge T \) and \( \# A \) denotes the number of points of \( A \).

Such a system of processes can be constructed from a family of independent Wiener processes, coalescing their trajectories [18]. Moreover, in [18] there was proved that conditions \((F1)-(F4)\) of Proposition 2.1 uniquely determine the distribution of \((x^n_k)\) in the space of continuous functions from \([0, T]\) to \( \mathbb{R}^n \).

2.2. Tightness of a finite system in the space \( D([0, 1], C[0, T]) \). In this section we give a brief sketch of the proof that the sequence of processes

\[
y_n(u, t) = \begin{cases} 
  x^n_{\lfloor un \rfloor + 1}(t), & u \in [0, 1), \\
  x^n_n(t), & u = 1,
\end{cases} \quad t \in [0, 1],
\]

is tight in the Skorokhod space \( D([0, 1], C[0, T]) \). It will follow from the following lemmas.
**Lemma 2.1.** For all $n \in \mathbb{N}$, $u \in [0,2]$, $h \in [0,u]$ and $\lambda > 0$

$$\mathbb{P}\{ \| y_n(u+h,\cdot) - y_n(u,\cdot) \|_{C[0,T]} > \lambda, \| y_n(u,\cdot) - y_n(u-h,\cdot) \|_{C[0,T]} > \lambda \} \leq \frac{4h^2}{\lambda^2}.$$  

Here $y_n(u,\cdot) = y_n(1,\cdot)$, $u \in [1,2]$, and $\| \cdot \|_{C[0,T]}$ is the uniform norm on $[0,T]$.

**Proof.** Let $(\mathcal{F}^y_t)$ be the filtration generated by $y_n$. Consider the $(\mathcal{F}^y_t)$-stopping times

$$\sigma^+ = \inf\{ t : y_n(u+h,t) - y_n(u,t) \geq \lambda \} \wedge T,$$

$$\sigma^- = \inf\{ t : y_n(u,t) - y_n(u-h,t) \geq \lambda \} \wedge T.$$

and the process

$$M(t) = (y_n(u+h,t \wedge \sigma^+) - y_n(u,t \wedge \sigma^+)) - (y_n(u,t \wedge \sigma^-) - y_n(u-h,t \wedge \sigma^-)), \quad t \in [0,T].$$

Using conditions (F1)–(F4), we can show that $M(\cdot)$ is a supermartingale (ref. to [19]). Consequently, since $M(T) \geq \lambda^2 \mathbb{I}_{\{\sigma^+ \wedge \sigma^- < T\}}$,

$$\mathbb{P}\{ \| y_n(u+h,\cdot) - y_n(u,\cdot) \|_{C[0,T]} > \lambda, \| y_n(u,\cdot) - y_n(u-h,\cdot) \|_{C[0,T]} > \lambda \} \leq \mathbb{P}\{ \sigma^+ \wedge \sigma^- < T \} \leq \frac{\mathbb{E}M(T)}{\lambda^2} \leq \frac{\mathbb{E}M(0)}{\lambda^2} \leq \frac{4h^2}{\lambda^2}.$$  

The lemma is proved.

**Lemma 2.2.** For all $\beta > 1$

$$\lim_{\delta \to 0} \sup_{n \geq 1} \mathbb{E} \left[ \| y_n(\delta,\cdot) - y_n(0,\cdot) \|_{C[0,T]}^{\beta} \wedge 1 \right] = 0.$$

**Proof.** Set

$$\sigma_\delta = \inf\{ t : y_n(\delta,t) - y_n(0,t) = 1 \} \wedge T.$$

The assertion of the lemma follows from the inequalities

$$\mathbb{E} \left[ \sup_{t \in [0,T]} (y_n(\delta,t) - y_n(0,t))^{\beta} \wedge 1 \right] = \mathbb{E} \sup_{t \in [0,T]} (y_n(\delta,t \wedge \sigma_\delta) - y_n(0,t \wedge \sigma_\delta))^{\beta} \leq C_\beta \mathbb{E}(y_n(\delta,T \wedge \sigma_\delta) - y_n(0,T \wedge \sigma_\delta))^{\beta} \leq C_\beta \mathbb{E}(y_n(\delta,T \wedge \sigma_\delta) - y_n(0,T \wedge \sigma_\delta)) \leq C_\beta \delta.$$  

\[ \square \]
Lemma 2.3. For all \( u \in [0, 1] \) the sequence \( \{ y_n(u, t), \ t \in [0, T] \}_{n \geq 1} \) is tight in \( C[0, T] \).

Proof. To prove the lemma we use the Aldous tightness criterion (see e.g. Theorem 3.6.4. [4]), namely we show that

(A1) for all \( t \in [0, T] \) the sequence \( \{ y_n(u, t) \}_{n \geq 1} \) is tight in \( \mathbb{R} \);

(A2) for all \( r > 0 \), a set of stopping times \( \{ \sigma_n \}_{n \geq 1} \) taking values in \( [0, T] \) and a sequence \( \delta_n \downarrow 0 \)

\[
\lim_{n \to \infty} \mathbb{P}\{|y_n(u, \sigma_n + \delta_n) - y_n(u, \sigma_n)| \geq r\} = 0.
\]

Note that (A1) follows from Chebyshev’s inequality and the estimation

\[
\mathbb{E}|y_n(u, t)| \leq \mathbb{E}|y_n(u, t) - \int_0^1 y_n(q, t) dq| + \mathbb{E}\left| \int_0^1 y_n(q, t) dq \right|
\leq \mathbb{E}(y_n(1, t) - y_n(0, t)) + \mathbb{E}\left| \int_0^1 y_n(q, t) dq \right|
= 1 + \mathbb{E}\left| \int_0^1 y_n(q, t) dq \right|
\]

where \( \int_0^1 y_n(q, t) dq \) is a Wiener process.

Condition (A2) can be checked by the following way. Similarly as in the proof of Lemma 2.16 [19], we have that for each \( \alpha \in (0, \frac{3}{2}) \) there exists a constant \( C \) such that for all \( u \in [0, 1] \) and \( n \geq 1 \)

\[
\mathbb{E} \frac{1}{m_n^\alpha(u, t)} \leq \frac{C}{\sqrt{t}},
\]

where

\[
m_n(u, t) = \begin{cases} 
m_{\lfloor un \rfloor + 1}^n(t), & u \in [0, 1), \\
m_n^n(t), & u = 1,
\end{cases} \quad t \in [0, 1].
\]
Thus, one can estimate
\[
\lim_{n \to \infty} \mathbb{P}\{|y_n(u, \sigma_n + \delta_n) - y_n(u, \sigma_n)| \geq r\}
\leq \frac{1}{r^2} \lim_{n \to \infty} \mathbb{E}(y_n(u, \sigma_n + \delta_n) - y_n(u, \sigma_n))^2
= \frac{1}{r^2} \lim_{n \to \infty} \mathbb{E} \int_{\sigma_n}^{\sigma_n + \delta_n} \frac{1}{m_n(u, s)} \, ds
= \frac{1}{r^2} \lim_{n \to \infty} \mathbb{E} \int_0^T \mathbb{I}_{(\sigma_n, \sigma_n + \delta_n]}(u, s) \frac{1}{m_n(u, s)} \, ds
\leq \frac{1}{r^2} \lim_{n \to \infty} \left( \mathbb{E} \int_0^T \mathbb{I}_{(\sigma_n, \sigma_n + \delta_n]}(u, s) \, ds \right)^{\frac{1}{4}} \left( \mathbb{E} \int_0^T \frac{1}{m_n(u, s)} \, ds \right)^{\frac{3}{4}}
\leq \frac{C}{r^2} \sqrt{T} \lim_{n \to \infty} \delta_n^{\frac{1}{4}} = 0.
\]

It proves the lemma.

Now, using Lemmas 2.1, 2.2 and 2.3, Theorems 3.8.6 and 3.8.8 [13], Remark 3.8.9 [13], we obtain the tightness of \{y_n\}.

**Proposition 2.2.** The sequence \{y_n(u, t), u \in [0, 1], t \in [0, T]\} is tight in \(D([0, 1], C[0, T])\).

**Remark 2.1.** Since the space \(D([0, 1], C[0, T])\) is Polish, the tightness implies the relative compactness of \{y_n(u, t), u \in [0, 1], t \in [0, T]\} in \(D([0, 1], C[0, T])\).

2.3. Martingale characterization of limit points (proof of Theorem 1.1).

In this section we prove that every limit point of \{y_n\} satisfies \(C1\)–\(C4\), which proves Theorem 1.1.

Let \{y_{n'}\} converge to \(y\) weakly in the space \(D([0, 1], C[0, T])\), for some subsequence \{n'\}. By Skorokhod’s theorem (see Theorem 3.1.8 [13]) we may suppose that \{y_{n'}\} converging to \(y\) a.s. For convenience of notation we will suppose that \{y_n\} converging to \(y\) a.s. Next, to prove the theorem, first we show that \(y_n(u, \cdot)\) tends to \(y(u, \cdot)\) in \(C[0, T]\) a.s. Note that, in general case, it does not follow from convergence in the space \(D([0, 1], C[0, T])\). So we need some continuity property of \(y(u, \cdot), u \in [0, 1]\), in \(u\).

**Lemma 2.4.** For all \(u \in [0, 1]\) one has
\[
\mathbb{P}\{y(u, \cdot) \neq y(u-, \cdot)\} = 0.
\]
Proof. Let \( u \in [0,1] \) be fixed. Since for all \( n \geq 1 \), \( y_n \) is non-decreasing in the first argument, \( y \) is non-decreasing too. So, for \( 0 < \gamma < 1 \), \( \delta > 0 \) and \( \beta > 1 \) we have

\[
P\{\|y(u, \cdot) - y(u - \delta, \cdot)\|_{C[0,T]} > \gamma\}
\leq P\{\|y(u + \delta, \cdot) - y(u - \delta, \cdot)\|_{C[0,T]} > \gamma\}
\leq \lim_{n \to \infty} P\left\{\bigcup_{k=n}^{\infty} \left\{\|y_k(u + 2\delta, \cdot) - y_k(u - 2\delta, \cdot)\|_{C[0,T]} > \gamma\right\}\right\}
= \lim_{n \to \infty} P\left\{\bigcap_{k=n}^{\infty} \{\|y_k(u + 2\delta, \cdot) - y_k(u - 2\delta, \cdot)\|_{C[0,T]} > \gamma\}\right\}
\leq \lim_{n \to \infty} P\{\|y_n(u + 2\delta, \cdot) - y_n(u - 2\delta, \cdot)\|_{C[0,T]} > \gamma\}
\leq \frac{1}{\gamma^{\beta}} \lim_{n \to \infty} E\left[\|y_n(u + 2\delta, \cdot) - y_n(u - 2\delta, \cdot)\|_{C[0,T]}^{\beta} \wedge 1\right] \leq \frac{4\delta C^{\beta}}{\gamma^{\beta}}.
\]

Next, passing to the limit as \( \delta \) tends to 0 and using the monotonicity of \( \{\|y(u, \cdot) - y(u - \delta, \cdot)\|_{C[0,T]} > \gamma\} \) in \( \delta \) we obtain

\[
P\{\|y(u, \cdot) - y(u - \cdot)\|_{C[0,T]} > \gamma\} = 0.
\]

This proves the lemma.

\[\]

Corollary 2.1. For all \( u \in [0,1] \)

\( y_n(u, \cdot) \to y(u, \cdot) \) in \( C[0,T] \) a.s.

Corollary 2.1 and Proposition 9.1.17 [15] immediately imply properties \((C1) - (C3)\). Property \((C4)\) will follow from the following lemma and the representation of \( m(u, t) \) and \( m_n(u, t) \) via \( \tau_{u,v} \), \( v \in [0,1] \), and \( \tau_{u,v}^n \), \( v \in [0,1] \), i.e.

\[
m(u, t) = \int_0^1 I_{\{\tau_{u,v} \leq t\}} dv,
\]

\[
m_n(u, t) = \int_0^1 I_{\{\tau_{u,v}^n \leq t\}} dv.
\]

**Lemma 2.5.** Let \( z_n(t), t \in [0,T] \), \( n \geq 1 \), be a set of continuous local square integrable martingales such that for all \( n \geq 1 \) and \( t \in [0,\tau_n] \)

\[
(2.1) \quad [z_n(\cdot)]_t \geq pt,
\]
where \( \tau_n = \inf\{t : z_n(t) = 0\} \land T \) and \( p \) is a non-random positive constant. Let \( z(t), t \in [0, T], \) be a continuous process such that

\[
z(\cdot \land \tau) = \lim_{n \to \infty} z_n(\cdot \land \tau_n) \quad (\text{in } C([0, T], \mathbb{R})) \text{ a.s.,}
\]

where \( \tau = \inf\{t : z(t) = 0\} \land T. \) Then

\[
\tau = \lim_{n \to \infty} \tau_n \text{ in probability.}
\]

**Proof.** The proof of this technical lemma can be found in [19, Lemma 2.10] \( \square \)

**Remark 2.2.** Unfortunately, we do not know whether (C1) – (C4) uniquely determined the distribution in the space \( D([0, 1], C[0, T]) \) and whether \( \{y_n\} \) has only one limit point. Since we have proved that every limit point of \( \{y_n\} \) satisfies (C1) – (C4), henceforth we will suppose that \( y \) only satisfies (C1) – (C4) and not necessarily be a limit point of \( \{y_n\} \).

2.4. Some properties of the modified Arratia flow. Let \( y \) satisfy (C1) – (C4). Then the following property holds

\[
\begin{align*}
\text{P1} & \text{ For each } \beta \in (0, \frac{3}{2}) \text{ the exists a constant } C \text{ such that for all } u \in [0, 1] \\
& \quad \mathbb{E} \frac{1}{m^\beta(u, t)} \leq \frac{C}{\sqrt{t}}, \quad t \in (0, T].
\end{align*}
\]

\[
\begin{align*}
\text{P2} & \text{ There exists a constant } C \text{ such that for all } u \in [0, 1] \\
& \quad \mathbb{E} \int_0^t \frac{ds}{m(u, s)} \leq C\sqrt{t}, \quad t \in [0, T].
\end{align*}
\]

\[
\begin{align*}
\text{P3} & \text{ There exists a constant } C \text{ such that for all } u \in [0, 1] \\
& \quad \mathbb{E}(y(u, t) - u)^2 \leq C\sqrt{t}, \quad t \in [0, T].
\end{align*}
\]

\[
\begin{align*}
\text{P4} & \text{ For all } t \in (0, T] \text{ the function } y(u, t), \ u \in [0, 1], \text{ is a step function in } D([0, 1], \mathbb{R}) \text{ with a finite number of jumps. Moreover,} \\
& \quad \mathbb{P}\{\forall u, v \in [0, 1], \ y(u, t) = y(v, t) \implies y(u, t + \cdot) = y(v, t + \cdot)\} = 1.
\end{align*}
\]

It should be noted that the fact that \( y(\cdot, t) \) is a step function with a finite number of jumps easily follows from (P1). Indeed, let \( N(t) \) be a number of distinct points of \( \{y(u, t), \ u \in [0, 1]\}, t \in [0, T]. \) Then under (2.3) it is easy to see that

\[
N(t) = \int_0^1 \frac{du}{m(u, t)}.
\]

Consequently, \( N(t) \) must be finite for all \( t \in (0, T], \) since \( \mathbb{E}N(t) < \infty. \)
3. Some elements of stochastic analysis for the system of heavy diffusion particles. The main result of this chapter is the construction of some stochastic analysis for the process $y$, namely we introduce the stochastic integral on $L_2$-valued predictable functions with respect to the flow $y$, study its properties, prove the analog of Girsanov’s theorem and construct the flow of particles with drift. In this chapter $L_2$ will denote the space of square integrable measurable functions on $[0, 1]$ with respect to Lebesgue measure and $\| \cdot \|_{L_2}$ the usual norm in $L_2$.

3.1. Predictable $L_2$-valued processes. We will construct the stochastic integral with respect to the flow of particles by the standard way. First we introduce it for simple functions and then we pass to the limit. So in this section we show that each predictable $L_2$-valued process can be approximated by simple processes. We need some characterization of Borel $\sigma$-field on $L_2$.

Lemma 3.1. The Borel $\sigma$-algebra $\mathcal{B}(L_2)$ coincides with the $\sigma$-algebra generated by functionals $(\cdot, a), \ a \in L_2,$ where $(\cdot, \cdot)$ denotes the inner product on $L_2$.

Proof. The assertion follows from Proposition 1.1.1 [26].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space with a filtration $(\mathcal{F}_t)_{t \in [0,T]}$.

Definition 3.1. The $\sigma$-algebra $\mathcal{S}$ of subsets of $[0, T] \times \Omega$ generated by sets of the form,

$$(s, t] \times F, \ 0 \leq s < t \leq T, \ F \in \mathcal{F}_s \quad \text{and} \quad \{0\} \times F, \ F \in \mathcal{F}_0,$$

is called a predictable $\sigma$-field.

Note that $\mathcal{S}$ is generated by left-continuous adapted processes on $\mathbb{R}$ (see Lemma 25.1 [17]).

Definition 3.2. An $L_2$-valued process $X(t), \ t \in [0, T]$, is called predictable if the map

$$X : [0, T] \times \Omega \rightarrow L_2$$

is $\mathcal{S}/\mathcal{B}(L_2)$-measurable.

Lemma 3.1 immediately implies the following assertion.
Lemma 3.2. An $L_2$-valued process $X(t), t \in [0, T]$, is predictable if and only if for each $a \in L_2$ the process $(X(t), a), t \in [0, T]$, is predictable.

Next we prove a proposition that is similar to Lemma 25.1 [17], namely, $S$ is generated by $L_2$-valued processes which are continuous in some sense. For this purpose we introduce the following definition.

Definition 3.3. An $L_2$-valued process $X(t), t \in [0, T]$, is weakly left-continuous if $(X(t), a), t \in [0, T]$, is left-continuous for all $a \in L_2$.

Proposition 3.1. The $\sigma$-algebra $S$ coincides with the $\sigma$-algebra generated all weakly left-continuous $L_2$-valued processes.

Proof. Let $S'$ denotes the $\sigma$-algebra generated by all weakly left-continuous $L_2$-valued processes. Lemma 3.1 implies that $S' \subseteq S$. To proof the inclusion $S' \supseteq S$, it is enough to note that every left-continuous process on $\mathbb{R}$ can be considered as the weakly left-continuous $L_2$-valued process. The proposition is proved.

For an $L_2$-valued process $X(t), t \in [0, T]$, set

$$
\|X\|_{A^2} = \left( \mathbb{E} \int_0^T \|X(t)\|_{L_2}^2 dt \right)^{\frac{1}{2}}.
$$

Let $A^2$ denote the set of all predictable $L_2$-valued processes $X$ with

$$
(3.1) \quad \|X\|_{A^2} < \infty.
$$

The map $\| \cdot \|_{A^2}$ is a norm on $A^2$.

Consider an $L_2$-valued process $Y$ of the form

$$
(3.2) \quad Y(t) = \varphi_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{m-1} \varphi_k \mathbb{1}_{(t_k, t_{k+1}]}(t), \quad t \in [0, T],
$$

where $0 = t_0 < t_1 < \ldots < t_m = T$ and $\varphi_k, k = 0, \ldots, m - 1$, are $L_2$-valued random elements. It is easy to see that $Y \in A^2$ if and only if $\varphi_k$ is $\mathcal{F}_{t_k}$-measurable and $\mathbb{E}\|\varphi_k\|_{L_2}^2 < \infty$ for all $k = 0, \ldots, m - 1$. Denote by $A_0^2$ the subset of $A^2$ that contains all processes of the form (3.2).
Proposition 3.2. The set $A_0^2$ is dense in $A^2$.

Proof. Let $\{e_k\}_{k \geq 1}$ be dense in $L_2$ and $X \in A^2$. Set

$$
\rho_n(t, \omega) = \min \{\|X(t, \omega) - e_k\|_{L_2}, k = 1, \ldots, n\},
$$

$$
k_n(t, \omega) = \min \{k = 1, \ldots, n : \rho_n(t, \omega) = \|X(t, \omega) - e_k\|_{L_2}\},
$$

$$
Y_n(t, \omega) = e_{k_n(t, \omega)}, \quad t \in [0, T], \ \omega \in \Omega, \ n \geq 1.
$$

Show that $X_n \in A^2$. Consider maps:

$$
\tilde{\rho}_n : L_2 \to \mathbb{R},
$$

$$
\tilde{\rho}_n(a) = \min \{\|a - e_k\|_{L_2}, k = 1, \ldots, n\},
$$

$$
\tilde{k}_n : \mathbb{R} \times L_2 \to \mathbb{N},
$$

$$
\tilde{k}_n(r, a) = \inf \{k = 1, \ldots, n : r = \|a - e_k\|_{L_2}\} \land (m + 1),
$$

$$
\tilde{Y}_n : \mathbb{N} \to L_2,
$$

$$
\tilde{Y}_n(k) = e_k.
$$

Since $\tilde{\rho}_n, \tilde{k}_n$ and $\tilde{Y}_n$ are $\mathcal{B}(L_2)/\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(L_2)/\mathcal{B}(\mathbb{N})$ and $\mathcal{B}(\mathbb{N})/\mathcal{B}(L_2)$-measurable, respectively,

$$
Y_n(t, \omega) = \tilde{Y}_n(\tilde{k}_n(\tilde{\rho}_n(X(t, \omega))), X(t, \omega))
$$

is $\mathcal{S}/\mathcal{B}(L_2)$-measurable. So, $Y_n$ is predictable. Moreover

$$
\|Y_n\|_{A^2} \leq T \max \{\|e_k\|_{L_2}, k = 1, \ldots, n\}.
$$

Note that $Y_n$ takes on values in $\{e_k, k = 1, \ldots, n\}$. Consequently, there exist the sets $A_1, \ldots, A_n \in \mathcal{S}$ such that

$$
Y_n = \sum_{k=1}^{n} e_k \mathbb{1}_{A_k}.
$$

Next, for $\varepsilon > 0$ choose sets $\Gamma_k^\varepsilon$ of the form $(s, t] \times F$, $0 \leq s < t \leq T$, $F \in \mathcal{F}_s$ or $\{0\} \times F$, $F \in \mathcal{F}_0$, such that

$$
\mathbb{P} \otimes \text{Leb}(A_k \Delta \Gamma_k^\varepsilon) < \varepsilon, \quad k = 1, \ldots, n,
$$

Set

$$
Y_n^\varepsilon = \sum_{k=1}^{n} e_k \mathbb{1}_{\Gamma_k^\varepsilon}.
$$
The choice of $Y^\varepsilon_n$ easily implies that $Y^\varepsilon_n \in A^2_0$. Let us estimate

$$\|Y_n - Y^\varepsilon_n\|^2_{A^2} \leq \sum_{k=1}^{n} \|e_k\|^2_{L^2} \int_0^T \|\mathbb{I}_{A_k}(t) - \mathbb{I}_{\Gamma_k}(t)\|^2_{L^2} dt < \varepsilon T \sum_{k=1}^{n} \|e_k\|^2_{L^2}.$$

Since $\{e_k\}_{k \geq 1}$ is dense in $L^2$,

$$\rho_n(t, \omega) = \|X(t, \omega) - Y_n(t, \omega)\|_{L^2}$$

monotonically tends to 0, for all $(t, \omega) \in [0, T] \times \Omega$. By the monotone convergence theorem,

$$\|X - Y_n\|_{A^2} \to 0, \quad n \to \infty.$$

Setting $X_n = Y^\varepsilon_n$, where $\varepsilon_n = \sum_{k=1}^{n} \|e_k\|^2_{L^2}$, we obtain that $\|X - X_n\|_{A^2} \to 0$ as $n \to \infty$. This proves the proposition.

**Lemma 3.3.** The space $A^2$ is complete.

**Proof.** The assertion of the lemma follows from the completeness of $L_2([0,1] \times [0,T] \times \Omega)$ and Fubini’s theorem. \qed

3.2. **Stochastic integral on $A^2_0$.** In the present section we construct the stochastic integral with respect to the system of coalescing heavy diffusion particles for simple functions (from $A^2_0$) and prove that it is a square integrable martingale. Consider the random element in $D([0,1], C([0,T]))$ which satisfies $(C1)-(C4)$. Hereafter we will assume that $\sigma$-algebra $\mathcal{S}$ is predictable with respect to $(\mathcal{F}_t)$, where

$$\mathcal{F}_t = \sigma(y(u, s), u \in [0,1], s \leq t), \quad t \in [0,T].$$

Let $f$ belong to $A^2_0$ and can be written as

$$f(t) = \varphi_0 \mathbb{I}_{[0]}(t) + \sum_{k=0}^{n-1} \varphi_k \mathbb{I}_{(t_k, t_{k+1})}(t), \quad t \in [0,T],$$

where $0 = t_0 < t_1 < \ldots < t_n = T$ and $\varphi_k$ is $\mathcal{F}_{t_k}$-measurable $L^2$-valued random elements, $k = 0, \ldots, n - 1$. Set

$$I_t(f) = \int_0^t \int_0^1 f(u, s) dy(u, s) du = \sum_{k=0}^{n} (\varphi_k(u, t_{k+1}) - y(u, t_k)) =$$

$$= \sum_{k=0}^{n} \int_0^1 \varphi_k(u)(y(u, t_{k+1}) - y(u, t_k)) du, \quad t \in [0,T].$$
Remark 3.1. Since \( y(t) = y(\cdot, t) \), \( t \in [0, T] \), is \( L_2 \)-valued random process, usually we will use the notation \( I_t(f) = \int_0^1 f(s) \, dy(s) \).

For \( a, b \in L_2 \) denote the projection of \( a \) into the space of \( \sigma(b) \)-measurable functions from \( L_2 \) by \( \text{pr}_b a \).

Theorem 3.1. For each \( f \in A_0^2 \), \( I(f) \) is a continuous square integrable \((\mathcal{F}_t)\)-martingale with the characteristic

\[
[I(f)]_t = \int_0^t \| \text{pr}_{y(s)} f(s) \|_{L_2}^2 \, ds, \quad t \in [0, T].
\]

To prove the theorem we will use integration in a Banach spaces. So, let \( H \) be a Banach space, \( \xi : \Omega \to H \) be a random element such that \( \mathbb{E} \| \xi \|_H < \infty \) and \( \sigma \)-algebra \( \mathcal{R} \) be contained in \( \mathcal{F} \). Then there exists the unique \( \mathcal{R} \)-measurable random element \( \eta \) in \( H \) such that \( \mathbb{E} \| \eta \|_H \leq \mathbb{E} \| \xi \|_H \)

and for all \( A \in \mathcal{R} \), \( h^* \in H^* \)

\[
\int_A (\xi, h^*) \, d\mathbb{P} = \int_A (\eta, h^*) \, d\mathbb{P}.
\]

This element \( \eta \) is called a conditional expectation of \( \eta \) with respect to \( \mathcal{R} \) and we will denote it by \( \mathbb{E}^{\mathcal{R}} \xi \). More detailed information about conditional expectation for \( H \)-valued random elements can be found in [32].

Lemma 3.4. If \( L \) is a bounded linear operator from \( H \) to a Banach space \( K \) then

\[
\mathbb{E}^{\mathcal{R}} L \xi = L \mathbb{E}^{\mathcal{R}} \xi.
\]

The assertion of the lemma easily follows from the definition of the conditional expectation. Let us state the following lemma that is used to prove Theorem 3.1.

Lemma 3.5. Let \( \xi(u), \eta(u), u \in [0, 1] \), be measurable (in \( (u, \omega) \)) random processes such that \( \mathbb{E} \int_0^1 |\xi(u)| \, du < \infty \), \( \eta(u) = \mathbb{E}(\xi(u)|\mathcal{R}) \) for almost all \( u \in [0, 1] \) and for each \( a \in L_\infty([0, 1]) \), \( \int_0^1 a(u) \eta(u) \, du \) be an \( \mathcal{R} \)-measurable random variable. Then \( \xi, \eta \) are random elements in \( L_1([0, 1]) \) and \( \mathbb{E}^{\mathcal{R}} \xi = \eta \).

Proof. The measurability of \( \xi, \eta \) as maps from \( \Omega \) to \( L_1([0, 1]) \) follows from the measurability of the processes \( \xi(u), \eta(u), u \in [0, 1] \). Next, take
A ∈ ℜ, a ∈ L∞([0, 1]) and consider
\[
\int_A (a, \eta) dP = \int_A \left( \int_0^1 a(u) \eta(u) du \right) dP = \int_0^1 \left( \int_A (a(u) \eta(u) dP) \right) du
\]
\[
= \int_0^1 \left( \int_A (a(u) \xi(u) dP) \right) du = \int_A (a, \xi) dP.
\]
The lemma is proved.

**Proof of Theorem 3.1.** Let \( f \in \mathcal{A}_0^2 \). The continuity of \( I(f) \) follows from its construction. Next, check the martingale property of the integral. Fix \( s < t \) and note that \( f \) on \((s, t]\) has the form
\[
f(r) = \sum_{k=0}^{p-1} \varphi_k \bar{y}(s_k, s_{k+1})(r), \quad r \in (s, t],
\]
where \( s = s_0 < s_1 < \ldots < s_p = t \) and \( \varphi_k \) is \( \mathcal{F}_{s_k} \)-measurable, \( k = 0, \ldots, p-1 \). Let us calculate
\[
\mathbb{E}(I_t(f) - I_s(f) | \mathcal{F}_s) = \mathbb{E} \left( \sum_{k=0}^{p-1} (\varphi_k, y(s_{k+1}) - y(s_k)) \middle| \mathcal{F}_s \right)
\]
\[
= \sum_{k=0}^{p-1} \mathbb{E} \left( \int_s^t \varphi_k(u)(y(u, s_{k+1}) - y(u, s_k)) du \middle| \mathcal{F}_s \right)
\]
\[
= \sum_{k=0}^{p-1} \int_s^t \mathbb{E}^{\mathcal{F}_s} \left( \varphi_k \cdot (y(s_{k+1}) - y(s_k)) \right)(u) du
\]
\[
= \sum_{k=0}^{p-1} \int_s^t \mathbb{E}^{\mathcal{F}_s} \left[ \mathbb{E}^{\mathcal{F}_{s_k}} (\varphi_k \cdot (y(s_{k+1}) - y(s_k))) \right](u) du
\]
\[
= \sum_{k=0}^{p-1} \int_s^t \mathbb{E}^{\mathcal{F}_s} \left[ \varphi_k \mathbb{E}^{\mathcal{F}_{s_k}} (y(s_{k+1}) - y(s_k)) \right](u) du = 0.
\]
Therefore \( I(f) \) is a martingale. It should be noted that \( I(f) \) is a square integrable martingale. It follows from \( \mathbb{E}(\int_0^1 y^2(u, t)) du < \infty \) (See Lemma 2.18 [19] or \((P3))

Let us calculate the quadratic characteristic of \( I(f) \). Denote
\[
M(t) = I_t^2(f) - \int_0^t ||pr_{y(s)}f(s)||^2_{L^2} ds, \quad t \in [0, T].
\]
\[ E( M(t) - M(s) \mid \mathcal{F}_s) = E\left( (I_t(f) - I_s(f))^2 \mid \mathcal{F}_s \right) \]

\[- E\left( \int_s^t \left\| \text{pr}_{y(r)} f(r) \right\|^2_{L^2} dr \right) \mathcal{F}_s \]

\[ + 2I_s(f)E \left( (I_t(f) - I_s(f)) \mid \mathcal{F}_s \right). \]

The third term in the left hand side of the latter relation equals 0 since \( I(f) \) is a martingale. Next calculate

\[ E((I_t(f) - I_s(f))^2 \mid \mathcal{F}_s) \]

\[ = E\left[ \left( \sum_{k=0}^{p-1} \int_0^1 \varphi_k(u)(y(u, s_{k+1}) - y(u, s_k)) du \right)^2 \right. \mathcal{F}_s \]

\[ = \sum_{k,l=0}^{p-1} E \left[ \left( \int_0^1 \varphi_k(u)(y(u, s_{k+1}) - y(u, s_k)) du \right) \cdot \left( \int_0^1 \varphi_l(u)(y(u, s_{l+1}) - y(u, s_l)) du \right) \right] \mathcal{F}_s \]

\[ = \sum_{k=0}^{p-1} E \left[ \left( \int_0^1 \varphi_k(u)(y(u, s_{k+1}) - y(u, s_k)) du \right)^2 \right] \mathcal{F}_s \].

Here, for \( l > k \), we have used the equality

\[ E \left[ \left( \int_0^1 \varphi_k(u)(y(u, s_{k+1}) - y(u, s_k)) du \right) \cdot \left( \int_0^1 \varphi_l(u)(y(u, s_{l+1}) - y(u, s_l)) du \right) \right] \mathcal{F}_s \]

\[ = E \left[ \int_0^1 \varphi_k(u)(y(u, s_{k+1}) - y(u, s_k)) du \right] \mathcal{F}_{s_l} \left| \mathcal{F}_s \right] = 0. \]

Let \( N(t) \) is a number of distinct points of \( \{y(u, t), \ u \in [0, 1]\}, t \in [0, T] \). \( N(t) \) is finite by property (P4). Observe that \( N(t) = \int_0^1 \frac{1}{m(u,t)} \ so \ N(t), \ t \in [0, T], \) is an \( (\mathcal{F}_t) \)-adapted càdlàg process. Denote

\[ \tau_0^k = s_k, \]

\[ \tau_l^k = \inf\{t > \tau_{l-1}^k : N(t) < N(\tau_{l-1}^k) \} \wedge s_{k+1}, \ l \in \mathbb{N}, \ k = 0, \ldots, p - 1, \]
and \( \pi(u, t) = \{ v : \exists s \leq t \ y(v, s) = y(u, s) \} \). By Proposition 2.1.5 [13], \( \tau^k_l \) is an \((\mathcal{F}_t)\)-stopping time for all \( l, k \). Consequently, we can write

\[
\mathbb{E} \left[ \left( \int_0^1 \varphi_k(u)(y(u, s_{k+1}) - y(u, s_k))du \right)^2 \bigg| \mathcal{F}_s \right]
\]

\[
= \mathbb{E} \left[ \left( \sum_{l=0}^{\infty} \int_0^1 \varphi_k(u)(y(u, \tau^k_{l+1}) - y(u, \tau^k_l))du \right)^2 \bigg| \mathcal{F}_s \right]
\]

\[
= \sum_{l=0}^{\infty} \mathbb{E} \left[ \left( \int_0^1 \varphi_k(u)(y(u, \tau^k_{l+1}) - y(u, \tau^k_l))du \right)^2 \bigg| \mathcal{F}_s \right]
\]

\[
= \sum_{l=0}^{\infty} \mathbb{E} \left[ \int_0^1 \frac{1}{m(v, \tau^k_l)} \left( \int_{\pi(v, \tau^k_l)} \varphi_k(u)(y(u, \tau^k_{l+1}) - y(u, \tau^k_l))du \right)^2 dv \bigg| \mathcal{F}_s \right]
\]

\[
= \sum_{l=0}^{\infty} \mathbb{E} \left[ \int_0^1 \frac{1}{m(v, \tau^k_l)} \left( \int_{\pi(v, \tau^k_l)} \varphi_k(u)du \right)^2 (y(v, \tau^k_{l+1}) - y(v, \tau^k_l))^2 dv \bigg| \mathcal{F}_s \right]
\]

\[
= \sum_{l=0}^{\infty} \mathbb{E} \left[ \int_0^1 m(v, \tau^k_l) \left( \operatorname{pr}_{y(\tau^k_l)} \varphi_k(\tau^k_l) \right)^2 (v)(y(v, \tau^k_{l+1}) - y(v, \tau^k_l))^2 dv \bigg| \mathcal{F}_s \right]
\]

\[
= \sum_{l=0}^{\infty} \mathbb{E} \left[ \int_0^1 m(v, \tau^k_l) \left( \operatorname{pr}_{y(\tau^k_l)} \varphi_k(\tau^k_l) \right)^2 (v)\int_{\tau^k_l}^{\tau^k_{l+1}} \frac{1}{m(v, \tau^k_l)} dr dv \bigg| \mathcal{F}_s \right]
\]

\[
= \sum_{l=0}^{\infty} \mathbb{E} \left[ \int_0^1 \left( \operatorname{pr}_{y(\tau^k_l)} \varphi_k(\tau^k_l) \right)^2 (v)\int_{\tau^k_l}^{\tau^k_{l+1}} \frac{1}{m(v, \tau^k_l)} dr dv \bigg| \mathcal{F}_s \right]
\]

\[
= \mathbb{E} \left[ \int_{s_k}^{s_{k+1}} \int_0^1 \left( \operatorname{pr}_{y(r)} f(r) \right)^2 (u)dr du \bigg| \mathcal{F}_s \right].
\]

Therefore,

\[
\mathbb{E} \left[ (I_t(f) - I_s(f))^2 \bigg| \mathcal{F}_s \right] = \mathbb{E} \left[ \int_s^t \| \operatorname{pr}_{y(r)} f(r) \|^2_{L^2} dr \bigg| \mathcal{F}_s \right]
\]

that proves the theorem. \( \square \)

3.3. Stochastic integral for predictable functions. Let \( \mathcal{M}_2 \) denote the space of continuous square integrable \((\mathcal{F}_t)\)-martingales \( M(t), t \in [0, T] \) with
the norm
\[ \|M\|_{\mathcal{M}_2} = (\mathbb{E}M^2(T))^{\frac{1}{2}}. \]

It is well-known that \( \mathcal{M}_2 \) is a complete and separable metric space \([14]\).

In the previous section we have proved that \( I : \mathcal{A}_0^2 \to \mathcal{M}_2 \) is a linear operator. Moreover,

\[
\|I(f)\|_{\mathcal{M}_2}^2 = \mathbb{E} \left( \int_0^T \int_0^1 f(u,t) dy(u,t) \right)^2 = \mathbb{E} \int_0^T \|\text{pr}_y(t)f(t)\|_{L_2}^2 dt \\
\leq \mathbb{E} \int_0^T \|f(t)\|_{L_2}^2 dt = \|f\|_{\mathcal{A}_2}^2.
\]

Hence \( I \) is bounded on \( \mathcal{A}_0^2 \) and consequently it can be extended to the bounded operator on \( \mathcal{A}^2 \). We will denote this extension by \( \int \cdot 0 \int_0^1 f(u,t) dy(u,t) \) or \( \int_0^t (f(t), dy(t)) \).

**Proposition 3.3.** For all \( f \in \mathcal{A}^2 \),

\[
(3.3) \quad \left[ \int_0^t (f(s), dy(s)) \right]_t = \int_0^t \|\text{pr}_y(s)f(s)\|_{L_2}^2 ds, \quad t \in [0,T].
\]

The proof of the proposition follows from the standard argument.

**Corollary 3.1.** For each \( f \in \mathcal{A}^2 \)

\[
\left[ \int_0^t (f(s), dy(s)) \right]_t = \left[ \int_0^t \left( \text{pr}_y(s)f(s), dy(s) \right) \right]_t, \quad t \in [0,T].
\]

**Corollary 3.2.** For each \( f, g \in \mathcal{A}^2 \),

\[
\left[ \int_0^t (f(s), dy(s)), \int_0^t (g(s), dy(s)) \right]_t = \int_0^t \left( \text{pr}_y(s)f(s), \text{pr}_y(s)g(s) \right) ds.
\]

Let \( \mathcal{A} \) denotes a set of predictable \( L_2 \)-valued processes \( f(t), t \in [0,T] \), such that

\[
(3.4) \quad \int_0^T \|f(t)\|_{L_2}^2 dt < \infty \quad \text{a.s.}
\]

Then by the standard argument one can construct the stochastic integral \( \int_0^t (f(s), dy(s)) \) for all \( f \in \mathcal{A} \). Moreover such an integral is a continuous local square integrable martingale with respect to \( (\mathcal{F}_t) \) and it has the characteristic of the form (3.3).
3.4. Girsanov’s theorem. In this section we construct a system of coalescing diffusion particles with drift. So, fix $\varphi \in \mathcal{A}$ and consider on $(\Omega, \mathcal{F})$ the new measure

$$
\mathbb{P}^\varphi(A) = \mathbb{E}_A \exp \left\{ \int_0^T (\varphi(s), dy(s)) - \frac{1}{2} \int_0^T \|pr_{y(t)} \varphi(t) \|^2_{L_2} dt \right\}, \ A \in \mathcal{F}.
$$

If $\mathbb{E} \exp \left\{ \int_0^T (\varphi(s), dy(s)) - \frac{1}{2} \int_0^T \|pr_{y(t)} \varphi(t) \|^2_{L_2} dt \right\} = 1$ then $\mathbb{P}^\varphi$ is a probability measure.

**Theorem 3.2.** The random element $\{y(u, t), \ u \in [0, 1], \ t \in [0, T]\}$ in $D([0, 1], C[0, T])$ satisfies the following properties under $\mathbb{P}^\varphi$.

(D1) for all $u \in [0, 1]$, the process

$$
\eta(u, \cdot) = y(u, \cdot) - \int_0^t \left( pr_{y(s)} \varphi(s) \right)(u) ds
$$

is a continuous local square integrable $(\mathcal{F}_t)$-martingale;

(D2) for all $u \in [0, 1]$, $y(u, 0) = u$;

(D3) for all $u < v$ from $[0, 1]$ and $t \in [0, T]$, $y(u, t) \leq y(v, t)$;

(D4) for all $u, v \in [0, 1]$ and $t \in [0, T]$,

$$
[\eta(u, \cdot), \eta(v, \cdot)]_t = \int_0^t \frac{I_{\{u, v \leq s\}} ds}{m(u, s)}.
$$

To prove the theorem we state an auxiliary lemma.

**Lemma 3.6.** For each $u \in [0, 1]$

$$
y(u, t) = u + \int_0^t \int_0^1 \frac{I_{\{u, s-\}}(q)}{m(u, s-)} dy(q, s) dq.
$$

**Proof.** Set $f(q, s) = \frac{I_{\{u, s-\}}(q)}{m(u, s-)}$ and using $(P2)$, we have

$$
\mathbb{E} \int_0^T \|f(s)\|^2_{L_2} ds = \mathbb{E} \int_0^T \int_0^1 \frac{I_{\{u, s-\}}(q)}{m^2(u, s-)} ds dq d s
$$

$$
= \mathbb{E} \int_0^T \left( \frac{1}{m^2(u, s)} \int_{\pi(u, s)} dq \right) ds
$$

$$
= \mathbb{E} \int_0^T \frac{1}{m(u, s)} ds < \infty.
$$
Next, put

\[ \sigma_k = \inf \{ s : N(s) \leq k \} \land t, \quad k \in \mathbb{N}. \]

and note that \( \sigma_k \) is an \((\mathcal{F}_t)\)-stoping time, \( \sigma_k \geq \sigma_{k+1} \). Consider

\[
\int_0^t \int_0^1 f(q,s) dy(q,s) dq = \sum_{k=1}^{\infty} \int_{\sigma_k}^{\sigma_{k+1}} \int_0^1 \frac{\pi(u,s-)}{m(u,s-)} dy(q,s) dq
\]

\[
= \sum_{k=1}^{\infty} \int_0^{\sigma_k} \int_0^1 \frac{\pi(u,s-)}{m(u,s-)} (y(q,\sigma_k) - y(q,\sigma_{k+1})) dq
\]

\[
= \sum_{k=1}^{\infty} \int_{\sigma_k}^{\sigma_{k+1}} \frac{1}{m(u,\sigma_{k+1})} (y(u,\sigma_k) - y(u,\sigma_{k+1}))
\]

\[
= \sum_{k=1}^{\infty} (y(u,\sigma_k) - y(u,\sigma_{k+1})) = y(u,t) - u.
\]

The lemma is proved. \( \square \)

**Corollary 3.3.** For each \( f \in \mathcal{A} \) and \( u \in [0,1] \)

\[
\left[ \int_0^t (f(s), dy(s)), y(u, \cdot) \right]_t = \int_0^t \left( \text{pr}_{y(s)} f(s) \right) (u) ds, \quad t \in [0,T].
\]

**Proof of Theorem 3.2.** The proof of the assertion follows from Girsanov’s theorem (see Theorem 5.4.1 [14]) and Corollary 3.3. \( \square \)

**Remark 3.2.** For the functions \( f \in \mathcal{A}^2 \) (resp. \( \mathcal{A} \)) we can construct the stochastic integral with respect to the flow \( \{y(u,t), u \in [0,1], t \in [0,T]\} \) satisfying conditions \((D1)-(D4)\) by the same way as in the case of conditions \((C1)-(C4)\). Moreover,

\[
\int_0^t (f(s), dy(s)) = \int_0^t (f(s), \text{pr}_{y(s)} \varphi(s)) ds + \int_0^t (f(s), d\eta(s))
\]

and \( \int_0^t (f(s), d\eta(s)) \) is a continuous square integrable (resp. local square integrable) \((\mathcal{F}_t)\)-martingale with the characteristic

\[
\left[ \int_0^t (f(s), d\eta(s)) \right]_t = \int_0^t \| \text{pr}_{y(s)} f(s) \|^2_{L^2} ds.
\]

4. Large deviation principle for the Arratia flow.
4.1. Exponential tightness. In this section we prove exponential tightness of the modified Arratia flow. In order to prove this we will use “exponentially fast” version of Jakubowski’s tightness criterion (see Theorem 1 [27]). So, let \( \{ y(u,t), u \in [0,1], t \in [0,T] \} \) be a random element in \( D([0,1], C[0,T]) \) satisfying (C1) – (C4), \( \kappa : [0,1] \to [0,1] \)

\[
\kappa(u) = \begin{cases} 
  u^\beta, & u \in [0,1/2] \\
  (1-u)^\beta, & u \in (1/2,1],
\end{cases}
\]

for fixed \( \beta > 1 \) and

\[
\mu(du) = \kappa(u)du.
\]

**Remark 4.1.** \( y(\cdot,t), t \in [0,T], \) is a continuous \( L^2(\mu) \)-valued random process.

We will establish exponential tightness of \( \{ y^\varepsilon \}_{\varepsilon>0} \) in the space \( C([0,T], L^2(\mu)) \), where \( y^\varepsilon(t) = y(\cdot, \varepsilon t), t \in [0,T] \).

By Theorem 1 [27], \( \{ y^\varepsilon \}_{\varepsilon>0} \) is exponential tight in \( C([0,T], L^2(\mu)) \), i.e. for every \( M > 0 \) there exists a compact \( K_M \subset C([0,T], L^2(\mu)) \), such that

\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{ y^\varepsilon \notin K_M \} \leq -M;
\]

if and only if

(E1) for every \( M > 0 \) there exists a compact \( K_M \subset L^2(\mu) \), such that

\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{ \exists \ t \in [0,T] : y^\varepsilon(t) \notin K_M \} \leq -M;
\]

(E2) for every \( h \in L^2(\mu) \) the sequence \( \{(h, y^\varepsilon(t))_{L^2(\mu)} : t \in [0,T]\} \) is exponentially tight in \( C([0,T], \mathbb{R}) \), where \( (\cdot, \cdot)_{L^2(\mu)} \) denotes the inner product in \( L^2(\mu) \).

Since \( y(u,t), u \in [0,1], \) is non-decreasing for all \( t \in [0,T] \), to find the compact \( K_M \subset L^2(\mu) \) satisfying (4.1) we have to control the behavior of processes \( y^\varepsilon(u,t), t \in [0,T] \), for \( u \) equal 0 or 1. Note that the diffusion of the process \( y^\varepsilon(u,t), t \in [0,T], \) tends to infinity as \( t \to 0 \). But

\[
M_*(u,t) \leq y(u,t) \leq M^*(u,t), \quad t \in [0,T],
\]

where

\[
M^*(u,t) = \frac{1}{1-u} \int_u^1 y(v,t)dv,
\]

\[
M_*(u,t) = \frac{1}{u} \int_0^u y(v,t)dv.
\]
and
\begin{equation}
\frac{d[M^*(u, \cdot)]_t}{dt} \leq \frac{1}{1 - u}, \quad \frac{d[M_*(u, \cdot)]_t}{dt} \leq \frac{1}{u}.
\end{equation}

The latter inequalities follow from the simple relation

\[
M_*(u, t) = \frac{1}{u} \int_0^t (\mathbb{1}_{[0, u]}, dy(s))_{L^2(du)}
\]

and the formula for the characteristic of the stochastic integral (3.3).

Inequalities (4.2) allows to find a compact \(K_L \subset L^2(\mu)\) satisfying (4.1) for the measure \(\mu\).

Since \(y(u, t), \ u \in [0, 1]\), belongs to \(D([0, 1], \mathbb{R})\) and is a non-decreasing function, we will often work with non-decreasing functions so denote

\[
D^\uparrow = \{h \in D([0, 1], \mathbb{R}) : h \text{ is non-decreasing}\}.
\]

**Lemma 4.1.** The set

\[
A_M = \left\{h \in D^\uparrow : h(1/n) \geq -Mn \text{ and } h(1 - 1/n) \leq Mn, \ n \in \mathbb{N}\right\}.
\]

is compact in \(L^2(\mu)\) for all positive \(M\).

**Proof.** First we prove that \(A_M \subset L^2(\mu)\). Let \(h \in A_M\). Without loss of generality let \(h\) be positive on \([1/2, 1]\) and negative on \([0, 1/2]\). Then

\[
\int_0^1 h^2(u) \mu(du) = \int_0^1 h^2(u)\kappa(u) du \leq C \sum_{n=2}^{\infty} \frac{M^2 n^2}{n^3} \left( \frac{1}{n-1} - \frac{1}{n} \right) < C_1
\]

and \(C_1\) is independent of \(h\).

Next, take a sequence \(\{h_k\}_{k \geq 1}\) of \(A_M\). Since \(\{h_k\}_{k \geq 1} \subset D^\uparrow\), there exists a subsequence \(\{h_{k'}\}\) that converges to \(h \in D^\uparrow\) \(\mu\)-a.e. Since \(|h_k(u)| \leq f(u), \ u \in [0, 1]\), where

\[
f(u) = \begin{cases} 
Mn, & u \in [1 - 1/(n-1), 1 - 1/n), \\
Mn, & u \in [1/n, 1/(n-1)]
\end{cases}
\]

and \(f \in L^2(\mu), \|h_{k'}\|_{L^2(\mu)} \to \|h\|_{L^2(\mu)}\), by dominated convergence theorem. Consequently, this and Lemma 1.32 [17] implies \(h_{k'} \to h\) in \(L^2(\mu)\). The lemma is proved. \(\square\)
Lemma 4.2. The sequence of processes \( \{y^\varepsilon\}_{\varepsilon > 0} \) satisfies (E1), i.e. for every \( M > 0 \) there exists a compact \( K_M \subset L_2(\mu) \), such that

\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{\exists t \in [0, T] : y^\varepsilon(t) \notin K_M\} \leq -M;
\]

Proof. By Lemma 4.1, we can take \( K_M = A_L \) and show that for some \( L > 0 \) (4.4) holds. So,

\[
\mathbb{P}\{\exists t \in [0, T] : y^\varepsilon(t) \notin A_L\} \leq \sum_{n=1}^{\infty} \mathbb{P}\{\exists t \in [0, T] : y^\varepsilon(1/n, t) < -Ln\} + \sum_{n=1}^{\infty} \mathbb{P}\{\exists t \in [0, T] : y^\varepsilon(1 - 1/n, t) > Ln\}
\]

Using (4.2) and (4.3), estimate

\[
\mathbb{P}\{\exists t \in [0, T] : y(1 - 1/n, \varepsilon t) > Ln\} = \mathbb{P}\left\{ \sup_{t \in [0, T]} y(1 - 1/n, \varepsilon t) > Ln \right\} 
\]

\[
\leq \mathbb{P}\left\{ \sup_{t \in [0, T]} M^* (1 - 1/n, \varepsilon t) > Ln \right\} 
\]

\[
\leq \mathbb{P}\left\{ \sup_{t \in [0, T]} (w(n\varepsilon t) + 1) > Ln \right\} 
\]

\[
\leq \frac{2}{\sqrt{2\pi n\varepsilon T}} \int_{Ln-1}^{\infty} e^{-x^2/2n\varepsilon t} dx \leq C \exp \left\{ -\frac{L^2 n}{2\varepsilon T} + \frac{L}{\varepsilon T} \right\}.
\]

Here \( w \) is a Wiener process and \( C \) is independent of \( \varepsilon, L \) and \( n \).

Similarly

\[
\mathbb{P}\{\exists t \in [0, T] : y(1/n, \varepsilon t) < -Ln\} \leq C \exp \left\{ -\frac{L^2 n}{2\varepsilon T} + \frac{L}{\varepsilon T} \right\}.
\]

Now, for \( M > 0 \) we can estimate

\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{\exists t \in [0, T] : y^\varepsilon(t) \notin A_L\} 
\]

\[
\leq \lim_{\varepsilon \to 0} \varepsilon \ln \left( 2C \sum_{n=1}^{\infty} \exp \left\{ -\frac{L^2 n}{2\varepsilon T} + \frac{L}{\varepsilon T} \right\} \right) 
\]

\[
\leq -\frac{L^2}{2T} + \frac{L}{T} < -M,
\]

where \( L \) is taken large enough. The lemma is proved. \( \square \)
Lemma 4.3. The sequence of processes \( \{y^\varepsilon\}_{\varepsilon>0} \) satisfies (E2), i.e. for every \( h \in L_2(\mu) \) the sequence \( \{(h, y^\varepsilon(t))_{L_2(\mu)}; \ t \in [0,T]\} \) is exponentially tight in \( C([0,T], \mathbb{R}) \).

Proof. To prove the lemma we will use Theorem 3 [27]. For each \( h \) we will show that there exist positive constants \( \alpha, \gamma \) and \( k \) such that for all \( s, t \in [0,T], \ s < t \)

\[
\mathbb{E} \exp \left\{ \frac{\gamma}{\varepsilon(t-s)^\alpha} |M_h(\varepsilon t) - M_h(\varepsilon s)| \right\} \leq k^{1/\varepsilon}, \ \forall \varepsilon \geq \varepsilon_0.
\]

where \( M_h(t) = (h, y(t))_{L_2(\mu)}, \ t \in [0,T] \).
Using (3.3), for

\[
M_h(t) = \int_0^1 h(u) y(u, t) \kappa(u) du = \int_0^t (h \kappa, dy(s))_{L_2(du)},
\]
we have

\[
[M_h]_t = \int_0^t \|pr_{y(s)}(h \kappa)\|_{L_2(du)}^2 ds \leq \int_0^t \|h \kappa\|_{L_2(du)}^2 ds \leq \|h\|_{L_2(\mu)}^2 t.
\]

The inequality for the characteristic of \( M_h \) and Novikov’s theorem implies

\[
\mathbb{E} \exp \left\{ \beta \int_s^t (h \kappa, dy(r))_{L_2(du)} - \frac{\beta^2}{2} \int_s^t \|pr_{y(r)}(h \kappa)\|_{L_2(du)}^2 dr \right\} = 1.
\]

So, for \( \delta > 0 \)

\[
\mathbb{E} \exp \{ \delta |M_h(\varepsilon t) - M_h(\varepsilon s)| \} \leq \mathbb{E} \exp \{ \delta (M_h(\varepsilon t) - M_h(\varepsilon s)) \}
\]

\[
+ \mathbb{E} \exp \{ \delta (M_h(\varepsilon s) - M_h(\varepsilon t)) \}
\]

\[
= \mathbb{E} \exp \left\{ \delta \int_s^t (h \kappa, dy(r))_{L_2(du)} \right\}
\]

\[
- \frac{\delta^2}{2} \int_s^t \|pr_{y(r)}(h \kappa)\|_{L_2(du)}^2 dr
\]

\[
+ \frac{\delta^2}{2} \int_s^t \|pr_{y(r)}(h \kappa)\|_{L_2(du)}^2 dr
\]

\[
+ \mathbb{E} \exp \{ \delta (M_{-h}(\varepsilon t) - M_{-h}(\varepsilon s)) \}
\]

\[
\leq 2 \mathbb{E} \exp \left\{ \frac{\varepsilon \delta^2}{2} \|h\|_{L_2(\mu)}^2 (t-s) \right\}
\]

Taking \( \delta = \frac{\sqrt{2}}{e(t-s)^{1/2} \|h\|_{L_2(\mu)}} \), we have

\[
\mathbb{E} \exp \left\{ \frac{\sqrt{2} \|M_h(\varepsilon t) - M_h(\varepsilon s)|}{\varepsilon \|h\|_{L_2(\mu)}(t-s)^{1/2}} \right\} \leq 2e^{1/\varepsilon} \leq (2e)^{1/\varepsilon}.
\]

It finishes the proof of the lemma.
From the two previous lemmas we obtain the exponential tightness of \( \{y^\varepsilon\}_{\varepsilon > 0} \).

**Proposition 4.1.** The sequence \( \{y^\varepsilon\}_{\varepsilon > 0} \) is exponentially tight in \( C([0,T], L_2(\mu)) \).

**4.2. Proof of the large deviation principle.** In this section we will establish the large deviation principle (LDP) for the sequence \( \{y^\varepsilon\}_{\varepsilon > 0} \). To define the rate function we have to differentiate functions with values in a Hilbert space.

**Definition 4.1.** A function \( f(t), t \in [0,T] \), taking values in a Hilbert space \( H \) is called **absolutely continuous** if there exist an integrable function \( t \mapsto h(t) \in H \) (in Bochner sense) such that

\[
f(t) = f(0) + \int_0^t h(s)ds,
\]

and we will denote the function \( h \) by \( \dot{f} \).

Set

\[
L_2^+(\mu) = \{g \in L_2(\mu) : \exists \tilde{g} \in D_2^+, g = \tilde{g}, \mu \text{ a.e.} \}
\]

and let

\[
\mathcal{H} = \{\varphi \in C([0,T], L_2^+(du)) : \varphi(0) = \text{id and } \varphi \text{ is absolutely continuous} \}
\]

and

\[
I(\varphi) = \begin{cases} 
\frac{1}{2} \int_0^T \|\dot{\varphi}(t)\|_{L_2(du)}^2 dt, & \varphi \in \mathcal{H}, \\
+\infty, & \text{otherwise.}
\end{cases}
\]

The main result of this section is the following theorem.

**Theorem 1.2.** The family \( \{y^\varepsilon\}_{\varepsilon > 0} \) satisfies a LDP in the space \( C([0,T], L_2(\mu)) \) with the good rate function \( I \).

Set \( C_{\text{id}}([0,T], L_2^+(\mu)) = \{\varphi \in C([0,T], L_2^+(\mu)) : \varphi(0) = \text{id} \} \).

**Remark 4.2.** Since the set \( C_{\text{id}}([0,T], L_2^+(\mu)) \) is closed in \( C([0,T], L_2(\mu)) \), it is enough to state LDP for \( \{y^\varepsilon\}_{\varepsilon > 0} \) in the metric space \( C_{\text{id}}([0,T], L_2^+(\mu)) \).

To prove the theorem we will use the method proposed in [7, 6].

Since we have proved the exponential tightness, for the upper bound it is enough to consider compact sets. According to [5] (see Theorem 4.1.11), for this it is enough to show that \( I \) is a lower-semicontinuous function and
i) weak upper bound:
\[
\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \leq -I(\varphi),
\]
where \(B_r(\varphi)\) is the open ball in \(C_{id}([0, T], L^2_1(\mu))\) with center \(\varphi\) and radius \(r\);
ii) lower bound: for every open set \(A \in C_{id}([0, T], L^2_1(\mu))\)
\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in A\} \geq -\inf_{\varphi \in A} I(\varphi).
\]

4.2.1. The upper bound. First we show the following bound
\[
\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \leq -I(\varphi).
\]
Set
\[H = \{h \in C([0, T], L_2(\mu^{-1})) : \dot{h} \in L_2([0, T], L_2(\mu^{-1}))\},\]
where \(\mu^{-1}(du) = \frac{1}{\kappa(u)} du\).
For \(h \in H\) let
\[
M^{\varepsilon, h}_t = \exp \left\{ \frac{1}{\varepsilon} \left[ \int_0^t (h(s), dy^\varepsilon(s))_{L_2(du)} - \frac{1}{2} \int_0^t \|pr_{y^\varepsilon(s)} h(s)\|_{L_2(du)}^2 ds \right] \right\}.
\]
By Novikov's theorem, \(M^{\varepsilon, h}_t, t \in [0, T]\), is a martingale with \(\mathbb{E}M^{\varepsilon, h}_T = 1\).
By an integration by parts (Lemma A.1), we can write
\[
M^{\varepsilon, h}_T = \exp \left\{ \frac{1}{\varepsilon} F(y^\varepsilon, h) \right\},
\]
where
\[
F(\varphi, h) = (h(T), \varphi(T))_{L_2(du)} - (h(0), \text{id})_{L_2(du)}
- \int_0^T (\dot{h}(s), \varphi(s))_{L_2(du)} ds
- \frac{1}{2} \int_0^T \|pr_{\varphi(s)} h(s)\|^2_{L_2(du)} ds, \quad \varphi \in C_{id}([0, T], L^2_1(\mu)).
\]
For \(\varphi \in C_{id}([0, T], L^2_1(\mu))\), we have
\[
\mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} = \mathbb{E} \left[ \mathbb{1}_{\{y^\varepsilon \in B_r(\varphi)\}} \frac{M^{\varepsilon, h}_T}{M^{\varepsilon, h}_T} \right]
\leq \exp \left\{ -\frac{1}{\varepsilon} \inf_{\psi \in B_r(\varphi)} F(\psi, h) \right\} \mathbb{E}M^{\varepsilon, h}_T
= \exp \left\{ -\frac{1}{\varepsilon} \inf_{\psi \in B_r(\varphi)} F(\psi, h) \right\} \cdot
Using the inequality $\|pr\phi(s)h(s)\|_{L^2(du)}^2 \leq \|h(s)\|_{L^2(du)}^2$, we obtain
\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \leq - \inf_{\psi \in B_r(\varphi)} F(\psi, h) \leq - \inf_{\psi \in B_r(\varphi)} \Phi(\psi, h),
\]
where
\[
\Phi(\varphi, h) = (h(T), \varphi(T))_{L^2(du)} - (h(0), \text{id})_{L^2(du)} - \int_0^T (\dot{h}(s), \varphi(s))_{L^2(du)} ds - \frac{1}{2} \int_0^T \|h(s)\|_{L^2(du)}^2 ds,
\]
$\varphi \in C_{\text{id}}([0, T], L^1_L(\mu))$.

Since the map $\Phi(\varphi, h)$, $\varphi \in C_{\text{id}}([0, T], L^1_L(\mu))$, is continuous,
\[
\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \leq - \Phi(\varphi, h).
\]

Minimizing in $h \in H$, we obtain
\[
\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \leq - \sup_{h \in H} \Phi(\varphi, h).
\]

**Proposition 4.2.** For each $\varphi \in C_{\text{id}}([0, T], L^1_L(\mu))$
\[
\sup_{h \in H} \Phi(\varphi, h) = I(\varphi).
\]

**Proof.** First we prove that the assertion of the proposition for $\varphi$ satisfying
\[
J(\varphi) := \sup_{h \in H} \Phi(\varphi, h) < \infty.
\]
Replacing $h$ by $\lambda h$, $\lambda \in \mathbb{R}$, we get
\[
J(\varphi) = \frac{1}{2} \sup_{h \in H} \int_0^T \frac{G^2(\varphi, h)}{\|h(s)\|_{L^2(du)}^2} ds < \infty,
\]
where
\[
G(\varphi, h) = (h(T), \varphi(T))_{L^2(du)} - (h(0), \text{id})_{L^2(du)} - \int_0^T (\dot{h}(s), \varphi(s))_{L^2(du)} ds.
\]
We can consider $H$ as linear subspace of $L_2([0, T], L_2(du))$. By Lemma A.2, it is a dense subspace and consequently the linear form

$$G_{\varphi} : h \rightarrow G(\varphi, h),$$

which is continuous on $H$, by (4.5), can be extended to the space $L_2([0, T], L_2(du))$. Using the Riesz theorem, there exists a function $k_{\varphi} \in L_2([0, T], L_2(du))$ such that

$$(4.6) \quad G(\varphi, h) = \int_0^T (k_{\varphi}(s), h(s)) ds.$$

Thus, by Lemma A.3, $\varphi$ is absolutely continuous and $\dot{\varphi} = k_{\varphi}$. Applying the Cauchy-Schwarz inequality to (4.6) we get

$$G(\varphi, h)^2 \leq \int_0^T \|k_{\varphi}(s)\|_{L_2(du)}^2 ds \cdot \int_0^T \|h(s)\|_{L_2(du)}^2 ds$$

$$= 2I(\varphi) \int_0^T \|h(s)\|_{L_2(du)}^2 ds,$$

with equality for $h$ proportional to $k_{\varphi}$. The latter inequality yields $J(\varphi) \leq I(\varphi)$ and since $H$ is dense in $L_2([0, T], L_2(du))$, we get the equality $J(\varphi) = I(\varphi)$.

If $I(\varphi) < \infty$, $\varphi$ is absolutely continuous and $k_{\varphi} = \dot{\varphi}$ in (4.6). So, $J(\varphi) \leq I(\varphi) < \infty$ and consequently we have $J(\varphi) = I(\varphi)$. This completes the proof of the proposition.

**Corollary 4.1.** $I$ is lower-semicontinuous as supremum of continuous functions.

4.2.2. The lower bound. In order to obtain the lower bound, it is enough to find a subset $\mathcal{R} \subset C_{\text{id}}([0, T], L_2^\uparrow(\mu))$ such that for each $\varphi \in \mathcal{R}$,

$$\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \geq -I(\varphi),$$

and prove that for each $\varphi$ satisfying $I(\varphi) < \infty$, there exists a sequence $\{\varphi_n\} \subset \mathcal{R}$ such that $\varphi_n \rightarrow \varphi$ in $C_{\text{id}}([0, T], L_2^\uparrow(\mu))$ and $I(\varphi_n) \rightarrow I(\varphi)$.

Denote

$$D^{\uparrow\uparrow} = \{g \in D([0, 1], \mathbb{R}) : \forall u < v \in [0, 1], \ g(u) < g(v)\}$$

and define $L_2^{\uparrow\uparrow}(\mu)$ in the same way as $L_2^\uparrow(\mu)$, replacing $D^\uparrow$ with $D^{\uparrow\uparrow}$. Set

$$\mathcal{R} = \{\varphi \in C([0, T], L_2^{\uparrow\uparrow}(du)) : I(\varphi) < \infty, \ \dot{\varphi} \in H_{L_2(du)},$$

$\dot{\varphi}$ is continuous in $(u, t)$ and $\varphi_u(u, t)$ is bounded $\forall u, t\},$
where
\[ H_{L_2(du)} = \{ h \in C([0,T], L_2(du)) : \dot{h} \in L_2([0,T], L_2(du)) \}. \]

For \( h \in H_{L_2(du)} \) define the new probability measure \( \mathbb{P}^{\varepsilon,h} \) with density
\[
\frac{d\mathbb{P}^{\varepsilon,h}}{d\mathbb{P}} = M^{\varepsilon,h}_T.
\]

By Theorem 3.2, the random element \( y^\varepsilon \) in \( D([0,1], C[0,T]) \) satisfies (w.r.t. \( \mathbb{P}^{\varepsilon,h} \)) the following property

\( (D^\varepsilon 1) \) for all \( u \in [0,1], \) the process
\[
\eta^\varepsilon(u, \cdot) = y^\varepsilon(u, \cdot) - \int_0^u \left( \Pr_{y^\varepsilon(s)} (h(s)) \right) (u) ds
\]
is a continuous local square integrable \( (\mathcal{F}_s) \)-martingale;

\( (D^\varepsilon 2) \) for all \( u \in [0,1], \) \( y^\varepsilon(u,0) = u; \)

\( (D^\varepsilon 3) \) for all \( u < v \) from \( [0,1] \) and \( t \in [0,T], \) \( y^\varepsilon(u,t) \leq y^\varepsilon(v,t); \)

\( (D^\varepsilon 4) \) for all \( u,v \in [0,1] \) and \( t \in [0,T], \)
\[
[\eta^\varepsilon(u, \cdot), \eta^\varepsilon(v, \cdot)]_t = \varepsilon \int_0^t \frac{I_{\{\tau_{\varepsilon u} \leq s\}} ds}{m^\varepsilon(u,s)},
\]
where \( \tau^\varepsilon \) and \( m^\varepsilon \) is defined in the same way as \( \tau \) and \( m \), replacing \( y \) with \( y^\varepsilon \).

Note that if \( \varphi \in \mathcal{R} \) then \( \lim_{\varepsilon \to 0} \mathbb{P}^{\varepsilon,\varphi}\{ y^\varepsilon \in B_r(\varphi)\} = 1 \) for all \( r > 0 \), by Proposition C.1.

Set \( h = \dot{\varphi} \) and noting that \( y^\varepsilon \in L_2([0,T], L_2(du)) \) a.s., we estimate
\[
\mathbb{P}\{ y^\varepsilon \in B_r(\varphi)\} = \mathbb{E}^{\varepsilon,h}\{ y^\varepsilon \in B_r(\varphi)\}
\geq \exp \left\{ -\frac{1}{\varepsilon} \sup_{\psi \in B_r(\varphi) \cap L_2([0,T], L_2(du))} \mathbb{E}^{\varepsilon,h}\{ y^\varepsilon \in B_r(\varphi)\} \right\}
\geq \exp \left\{ -\frac{1}{\varepsilon} \sup_{\psi \in B_r(\varphi) \cap L_2([0,T], L_2(du))} \mathbb{E}^{\varepsilon,h}\{ y^\varepsilon \in B_r(\varphi)\} \right\} F(\psi, h)
\]
where \( \mathbb{E}^{\varepsilon,h} \) denotes the expectation w.r.t. \( \mathbb{P}^{\varepsilon,h} \). Thus
\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{ y^\varepsilon \in B_r(\varphi)\} \geq -\sup_{\psi \in B_r(\varphi) \cap L_2([0,T], L_2(du))} F(\psi, h),
\]
and by Corollary B.1, we have
\[
\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{ y^\varepsilon \in B_r(\varphi)\} \geq -F(\varphi, h) = -I(\varphi).
\]
Proposition 4.3. For each \( \varphi \) satisfying \( I(\varphi) < \infty \), there exists a sequence \( \{\varphi_n\} \subset \mathcal{R} \) such that \( \varphi_n \to \varphi \) in \( C_{id}([0,T], L^2_\mu(\mu)) \) and \( I(\varphi_n) \to I(\varphi) \).

Remark 4.3. To prove the proposition, it is enough to check that for all \( \varepsilon > 0 \) there exists \( \psi \in \mathcal{R} \) such that

\[
(4.7) \quad \int_0^T \|\dot{\varphi}(t) - \dot{\psi}(t)\|_{L^2(\mu)}^2 dt < \varepsilon.
\]

Remark 4.4. From (4.7) it follows that the statement is enough to check only for functions \( \varphi \) with a bounded derivative, i.e.

\[
(4.8) \quad \sup_{(u,t) \in [0,1] \times [0,T]} |\dot{\varphi}(u,t)| < \infty.
\]

For measurable functions \( f, g \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \) let \( f \ast g \) denote the convolution of \( f \) and \( g \), i.e.

\[
f \ast g(u,t) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(u-v,t-r)g(v,r)du dr, \quad u, t \in \mathbb{R}.
\]

Proof of Proposition 4.3. Let \( \varphi \) satisfy \( I(\varphi) < \infty \) and (4.8) hold. Set

\[
C = \sup_{(u,t) \in [0,1] \times [0,T]} |\dot{\varphi}(u,t)|
\]

and extend \( \varphi \) to the function on \( \mathbb{R}^2 \) as follows

\[
\tilde{\varphi}(u,t) = \begin{cases} 
\varphi(u,t) & u \in [0,1], \\
C & u \in (1,2], \\
-C & u \in [-1,0), \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
\varphi(\cdot,t) = \text{id} \cdot \mathbb{1}_{[-1,2]}(\cdot) + \int_{-\infty}^t \tilde{\varphi}(\cdot,s)ds,
\]

where \( \int_{-\infty}^t \tilde{\varphi}(\cdot,s)ds \) is the Bochner integral in the space of square integrable functions on \( \mathbb{R} \) w.r.t. the Lebesgue measure. Note, that it is well-defined extension of \( \varphi \).

Let \( \varsigma \) be a positive twice continuously differentiable symmetric function on \( \mathbb{R} \) with \( \text{supp} \varsigma \subset [-1,1] \) and \( \int_{\mathbb{R}} \varsigma(u)du = 1 \). Set for \( 0 < \delta < 1 \) and \( \alpha > 0 \)

\[
\varsigma_\delta(u,t) = \frac{1}{\delta^2} \varsigma \left( \frac{u}{\delta} \right) \varsigma \left( \frac{t}{\delta} \right), \quad u, t \in \mathbb{R}.
\]
and
\[ \psi(u, t) = u + \int_0^t \tilde{\varphi} \ast \varsigma \delta(u, s) ds + \alpha tu, \quad u \in [0, 1], \quad t \in [0, T]. \]

Note that the continuity property of the convolution (see Theorem 11.21 [12]) in the space of square integrable functions implies that (4.3) holds for small enough \( \delta \) and \( \alpha \). Thus, to prove the proposition, we need to show that \( \psi \) belongs to \( \mathcal{R} \). Since \( \tilde{\varphi} \ast \varsigma \delta \) is smooth because \( \varsigma \) is smooth and the function \( \tilde{\varphi} \) is bounded with compact support, the statement will hold if we check that \( \psi(\cdot, t) \in D^{1\uparrow} \) for all \( t \in [0, T] \). So, let \( f_0(u, t) = u \) for \( u \in [-1, 2], t \in \mathbb{R} \) and \( f_0 = 0 \) for \( u \in \mathbb{R} \setminus [-1, 2], t \in \mathbb{R} \). Using the symmetry of \( \varsigma \) we have
\[ u = f_0 \ast \varsigma \delta(u, t), \quad u \in [0, 1], \quad t \in \mathbb{R}. \]

Thus we may rewrite the integral
\[
\begin{align*}
&u + \int_0^t \tilde{\varphi} \ast \varsigma \delta(u, s) ds = u + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{\varphi}(u - v, s - r) \varsigma \delta(v, r) ds dr dv \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ f_0(u - v) + \int_0^t \tilde{\varphi}(u - v, s - r) ds \right] \varsigma \delta(v, r) dr dv \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ f_0(u - v) + \int_0^{t - r} \tilde{\varphi}(u - v, s) ds \right] \varsigma \delta(v, r) dr dv \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(u - v, t - r) \varsigma \delta(v, r) dr dv.
\end{align*}
\]
Since \( \varphi(v_1, t) \leq \varphi(v_2, t) \), for all \( v_1 < v_2 \) from \([-1, 2]\) and \( t \in \mathbb{R} \), we obtain
\[
u_1 + \int_0^t \tilde{\varphi} \ast \varsigma \delta(u_1, s) ds \leq u_2 + \int_0^t \tilde{\varphi} \ast \varsigma \delta(u_2, s) ds, \quad u_1, u_2 \in [0, 1], \ u_1 < u_2,
\]
which proves the proposition.

Collecting the previous results establishes the LDP for the family \( \{y^\epsilon\}_{\epsilon > 0} \). The Varadhan formula for the measure-valued flow is then a direct consequence of the properties of the push forward map from monotone functions to measures on the real line.
APPENDIX A: SOME PROPERTIES OF ABSOLUTELY CONTINUOUS FUNCTIONS

Lemma A.1. For every absolutely continuous function $f(t)$, $t \in [0,T]$, with values in $L_2(du)$

$$\int_0^t (f(s), dy(s))_{L_2(du)} = \left( f(t), y(t) \right)_{L_2(du)} - \left( f(0), y(0) \right)_{L_2(du)}$$

$$- \int_0^t (\dot{f}(s), y(s))_{L_2(du)} ds.$$

Proof. Observe that, for an absolutely continuous function $\varphi$ and $t \in [0,T]$

$$\int_0^t (f(s), dy(s))_{L_2(du)} = \lim_{n \to \infty} \sum_{k=1}^n (f(t_{k-1}), y(t_k) - y(t_{k-1}))_{L_2(du)},$$

where $t_k = \frac{kt}{n}$, $k = 0, \ldots, n$. Thus, the statement follows from the standard argument.

Lemma A.2. The set $H$, which is defined in Subsection 4.2.1, i.e.

$$H = \{ h \in C([0,T], L_2(\mu^{-1})) : \dot{h} \in L_2([0,T], L_2(\mu^{-1})) \},$$

where $\mu^{-1}(du) = \frac{1}{\kappa(u)} du$, is dense in $L_2([0,T], L_2(du))$.

Lemma A.3. If for $\varphi \in L_2([0,T], L_2(\mu))$

$$(h(T), \varphi(T))_{L_2(du)} - (h(0), \varphi(0))_{L_2(du)} - \int_0^T (\dot{h}(s), \varphi(s))_{L_2(du)} ds$$

$$= \int_0^T (h(s), k(s))_{L_2(du)} ds, \quad \forall h \in H,$$

then $\varphi$ is absolutely continuous with $\dot{\varphi} = k$ and $\dot{\varphi} \in L_2([0,T], L_2(du))$.

APPENDIX B: CONTINUITY PROPERTY OF THE PROJECTION $PR_G$

Lemma B.1. Let $g \in L_2^{\uparrow\uparrow}(\mu)$, $f \in L_2(du)$ and $f$ be piecewise continuous. If a sequence $\{g_n\}_{n \geq 1}$ of elements $L_2^\uparrow(\mu)$ converges to $g$ a.e. then $\{pr_{g_n} f\}_{n \geq 1}$ converges to $f$ a.e.
PROOF. Without any restriction of generality let \( g_n \in D^\uparrow \) and \( g \in D^{\uparrow \uparrow} \). First we find the view of \( \text{pr}_{g_n} f \). Set

\[
\mathcal{A}_n = \{(a, b) \subset [0, 1] : g_n(a) = g_n(b)\},
\]

\[
K_n = \cup \mathcal{A}_n.
\]

For \( u \in K_n \) denote

\[
J_u = \bigcup_{u \in A \in \mathcal{A}_n} A
\]

Since every set \( A \) from \( \mathcal{A}_n \) is open interval, \( J_u = (a_n, b_n) \), for some \( a_n < b_n \). Set \( I_u = [a_n, b_n] \) and \( K_n = \bigcup_{u \in K_u} I_u \). Note that the function \( g_n \) is constant on \( I_u \), for all \( u \in K_u \), and \( g_n(u) \neq g_n(v) \) if \( u \) and \( v \) belong to separate \( I_u \). Hence, there exist a countable number of the distinct intervals \( I_u \) and consequently \( K_u \) is a Borel set. Now we can describe the filtration \( \sigma(g_n) \) generated by \( g_n \) via the sets \( I_u \), \( u \in K_u \) and obtain \( \text{pr}_{g_n} f \). So,

\[
\sigma(g_n) = (B([0, 1]) \cap (K_u^*)^c) \cup \{I_u^* : x \in K_u^*\}
\]

and consequently

\[
(\text{pr}_{g_n} f)(u) = \begin{cases} f(u), & u \notin K_u^* \\ \frac{1}{\text{Leb}(I_u^*)} \int_{I_u^*} f(v)dv, & u \in K_u^*. \end{cases}
\]

Let \( U = \{u \in [0, 1] : g_n(u) \rightarrow g(u)\} \) and \( D_f \) denotes the set of discontinuous points of \( f \). We show that for each \( u \in U \cap D_f \cap (0, 1) \), \( (\text{pr}_{g_n} f)(u) \rightarrow f(u) \). Since \( \text{Leb}(U \cap D_f) = 1 \), we will obtain the assertion of the lemma.

Fix \( u \in U \cap D_f \cap (0, 1) \) and prove that for an infinite sequence \( \{n'\} \) satisfying \( u \in K_{n'}^*, a_{n'} \rightarrow u \) and \( b_{n'} \rightarrow u \), where \( I_{n'} = [a_{n'}, b_{n'}] \). Let \( \varepsilon > 0 \) and \( v \in (0 \vee (u - \varepsilon), u) \). Since \( g_n(u) \rightarrow g(u), g_n(v) \rightarrow g(v) \) and \( g(v) < g(u) \), there exists \( N \in \mathbb{N} \) such that for all \( n' \geq N \), \( g_n(v) < g_n(u) \). By choosing of \( I_{n'} \), \( v \notin I_{n'} \) and consequently \( v < a_{n'} \), for all \( n' \geq N \). So, we obtain \( u - a_{n'} < \varepsilon \). Similarly \( b_{n'} - u < \varepsilon \) for all \( n' \geq N' \), where \( N' \) is large enough.

Next, since \( f \) is continuous at \( u \), \( \frac{1}{\text{Leb}(I_{n'})} \int_{I_{n'}} f(v)dv \rightarrow f(u) \). It finishes the proof of the lemma.

COROLLARY B.1. Let \( g \in L^\uparrow_{2, \mu}, f \in L_2(du) \) and \( f \) be piecewise continuous. If a sequence \( \{g_n\}_{n \geq 1} \) of elements \( L^\uparrow_{2, \mu} \) converges to \( g \) in \( L_2(\mu) \), then \( \{\text{pr}_{g_n} f\}_{n \geq 1} \) converges to \( f \) in \( L_2(du) \) and consequently

\[
\lim_{n \rightarrow \infty} \|\text{pr}_{g_n} f\|_{L_2(du)} = \|f\|_{L_2(du)}.
\]
Proof. By Proposition 4.12 [17], Lemma 4.2 [17] and Lemma B.1, to prove the assertion of the corollary, it is enough to show that (B.1) holds. Since 
\[ \|\text{pr}_{g_n} f\|_{L^2(du)} \leq \|f\|_{L^2(du)}, \]

let us prove that
\[ \lim_{n \to \infty} \|\text{pr}_{g_n} f\|_{L^2(du)} \leq \|f\|_{L^2(du)}. \]

Suppose there exists a sequence \( \{n'\} \) such that
\[ \lim_{n' \to \infty} \|\text{pr}_{g_{n'}} f\|_{L^2(du)} < \|f\|_{L^2(du)}. \]

Using Lemma 4.2 [17], we can choose subsequence \( \{n''\} \subset \{n'\} \) such that
\[ g_{n''} \to g \text{ a.e.} \] by Lemma B.1 and Fatou’s lemma
\[ \lim_{n'' \to \infty} \|\text{pr}_{g_{n''}} f\|_{L^2(du)} \geq \|f\|_{L^2(du)}, \]

which contradicts the choosing of \( \{n'\} \). The corollary is proved.

APPENDIX C: CONVERGENCE OF THE FLOW OF PARTICLES WITH DRIFT

In this section we prove that the process \( \{z^\varepsilon\}_{\varepsilon > 0} \) satisfying \((D^\varepsilon 1) - (D^\varepsilon 4)\) with \( h = \dot{\varphi} \) tends to \( \varphi \). Note that \( z^\varepsilon \) is a weak martingale solution to the equation
\[ dz^\varepsilon(t) = \text{pr}_{z^\varepsilon} \dot{\varphi}(t)dt + \sqrt{\varepsilon}\text{pr}_{z^\varepsilon}dW_t. \]

If we show that \( z^\varepsilon \) converges to a process \( z \) taking values from \( L^\perp_2(\mu) \), then by Lemma B.1, \( z \) should be a solution of the equation
\[ dz(t) = \dot{\varphi}(t)dt, \]

It gives \( z = \varphi \).

Thus, we prove first that the family \( \{z^\varepsilon\}_{\varepsilon > 0} \) is tight. Then we show that any limit point \( z \) of \( \{z^\varepsilon\}_{\varepsilon > 0} \) is \( L^\perp_2(\mu) \)-valued process. As we noted, it immediately gives \( z = \varphi \). Since \( \{z^\varepsilon\}_{\varepsilon > 0} \) has only one nonrandom limit point, we obtain that \( \{z^\varepsilon\}_{\varepsilon > 0} \) tends to \( \varphi \) in probability (not only in distribution).

Proposition C.1. Let \( \varphi \in \mathcal{R} \) and a family of random elements \( \{z^\varepsilon\}_{\varepsilon > 0} \) satisfies properties \((D^\varepsilon 1) - (D^\varepsilon 4)\) with \( h = \dot{\varphi} \), then \( z^\varepsilon \) tends to \( \varphi \) in the space \( C([0,T], L_2(\mu)) \) in probability.
To prove the proposition, we first establish tightness of \( \{ z^\varepsilon \} \) in \( C([0, T], L_2(\mu)) \), using the boundedness of \( \dot{\varphi} \) and the same argument as in the proof of exponential tightness of \( \{ y^\varepsilon \} \). Next, testing the convergent subsequence \( \{ z^\varepsilon \} \) by functions \( l \) from \( C([0, 1] \times [0, T], \mathbb{R}) \) and using integration by parts we will obtain

\[
\int_0^T \int_0^1 l(u, t)(z^\varepsilon(u, t) - u) dt du = \int_0^T \int_0^1 L(u, t)(\text{pr}_{z^\varepsilon(t)}\dot{\varphi}(t))(u) dt du + \int_0^T \int_0^1 L(u, t)\dot{\varphi}(u, t) dt du,
\]

where \( L(u, t) = \int_t^1 l(u, s) ds \) and \( z^\varepsilon \rightarrow z \). If \( z(t) \) belongs to \( L_2^T(\mu) \), for all \( t \in [0, T] \), then passing to the limit and using Corollary B.1 we obtain

\[
\int_0^T \int_0^1 l(u, t)(z(u, t) - u) dt du = \int_0^T \int_0^1 L(u, t)\dot{\varphi}(u, t) dt du,
\]

which implies \( z = \dot{\varphi} \).

The fact that \( z(t) \in L_2^T(\mu) \) will follow from the following lemma.

**Lemma C.1.** Let \( \varphi \) and \( \{ z^\varepsilon \} \) be such as in Proposition C.1. Then for each \( u < v \) there exists \( \delta > 0 \) such that

\[
\lim_{\varepsilon \to 0} \mathbb{P}\{ z^\varepsilon(v, t) - z^\varepsilon(u, t) \leq \delta \} = 0.
\]

Let \( \mathcal{S}(u, v, t) \) be a finite set of intervals contained in \([u, v]\), for all \( t \in (0, T] \), such that

1) if \( \pi_1, \pi_2 \in \mathcal{S}(u, v, t) \) and \( \pi_1 \neq \pi_2 \) then \( \pi_1 \cap \pi_2 = \emptyset \);
2) \( \bigcup \mathcal{S}(u, v, t) = [u, v] \);
3) for all \( s < t \) and \( \pi_1 \in \mathcal{S}(u, v, s) \) there exists \( \pi_2 \in \mathcal{S}(u, v, t) \) that contains \( \pi_1 \);
4) there exists decreasing sequence \( \{ t_n \}_{n \geq 1} \) on \((0, T]\) that tends to 0 and \( \mathcal{S}(u, v, t) = \mathcal{S}(u, v, t_n), \ t \in [t_n, t_{n-1}), \ n \in \mathbb{N}, \ t_0 = T \);
5) for each monotone sequence \( \pi(t) \in \mathcal{S}(u, v, t), \ t > 0, \bigcap_{t > 0} \pi(t) \) is a one-point set.

**Lemma C.2.** Let \( \varphi \in \mathcal{R} \) and \([\bar{u}, \bar{v}] \subset (0, 1) \). Then there exists \( \gamma > 0 \) such that for each interval \((u, v) \supset [\bar{u}, \bar{v}]\) there exist \( u_0 \in (u, \bar{u}) \) and \( v_0 \in (\bar{v}, v) \):

\[
\inf_{t \in [0, T]} \left[ v_0 - u_0 + \int_0^t (\text{pr}_{\mathcal{S}(s)}\dot{\varphi}(s))(v_0) ds - \int_0^t (\text{pr}_{\mathcal{S}(s)}\dot{\varphi}(s))(u_0) ds \right] = \delta > 0,
\]
for all \( \mathcal{G}(t) = \mathcal{G}(0 \vee (u - \gamma), (v + \gamma) \wedge 1, t) \), \( t \in (0, T) \), such that \( u_0 \) and \( v_0 \) belong to separate intervals from \( \mathcal{G}(T) \), and \( \text{pr}_{\mathcal{G}(t)} \) denotes the projection in \( L_2(du) \) on the space of \( \sigma(\mathcal{G}(t)) \)-measurable functions.

**Proof.** Let \( u \in [0, 1] \) and \( \mathcal{G}(t) = \mathcal{G}(0, 1, t), t \in [0, T] \). Then we can choose a sequence of intervals \( \{\pi_n\}_{n \geq 1} \) and a decreasing sequence \( \{s_n\}_{n \geq 1} \) from \( (0, T) \) converging to 0 such that \( \pi_{n+1} \subseteq \pi_n \subseteq [0, 1], \{u\} = \bigcap_{n=1}^{\infty} \pi_n \) and

\[
\begin{align*}
u + \int_{0}^{t} (\text{pr}_{\mathcal{G}(s)}) \varphi(s))(u))ds &= u + \sum_{n=1}^{\infty} \int_{s_{n-1} \wedge t}^{s_n \wedge t} \left( \frac{1}{|\pi_n|} \int_{\pi_n} \varphi(q, r)dr \right) dq \\
&= u + \sum_{n=1}^{\infty} \frac{1}{|\pi_n|} \int_{\pi_n} (\varphi(q, s_{n-1} \wedge t) - \varphi(q, s_n \wedge t)) dq \\
&= \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q, t) dq + \sum_{n=k}^{\infty} \frac{1}{|\pi_{n+1}|} \int_{\pi_{n+1}} \varphi(q, s_n) dq \\
&- \frac{1}{|\pi_n|} \int_{\pi_n} \varphi(q, s_n) dq,
\end{align*}
\]

where \( t \in [s_k, s_{k-1}) \).

Estimate the \( n \)-th term of the sum. For convenience of calculations, let \( [a, b] \subseteq [c, d] \) and \( f : [0, 1] \rightarrow \mathbb{R} \) be non-decreasing absolutely continuous function with bounded derivative. So,

\[
\frac{1}{b-a} \int_{a}^{b} f(x) dx - \frac{1}{d-c} \int_{c}^{d} f(x) dx \leq \frac{1}{b-a} \int_{a}^{b} f(x) dx - \frac{1}{b-c} \int_{c}^{b} f(x) dx
\]

\[
= \frac{a-c}{(b-a)(b-c)} \int_{a}^{b} f(x) dx - \frac{1}{b-c} \int_{c}^{a} f(x) dx
\]

\[
\leq \frac{a-c}{b-c} f(b) - \frac{a-c}{b-c} f(c) = \frac{a-c}{b-c} (f(b) - f(c))
\]

\[
\leq \sup_{x \in [0,1]} \dot{f}(x)(a-c).
\]

Taking \( c = a_n, d = b_n, a = a_{n+1}, b = b_{n+1} \) and \( f = \varphi(\cdot, s_n) \), where \( a_n < b_n \) are the ends of \( \pi_n \), we get

\[
u + \int_{0}^{t} (\text{pr}_{\mathcal{G}(s)}) \varphi(s))(u))ds \leq \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q, t) dq + \sum_{n=k}^{\infty} \bar{C}(a_{n+1} - a_n)
\]

\[
\leq \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q, t) dq + \bar{C}(u - a_k),
\]

where \( \bar{C} = \sup_{(u,t) \in [0,1] \times [0,T]} \frac{\partial \varphi}{\partial t}(u, t) \).
Similarly, we can obtain
\[ u + \int_0^t (\text{pr}_{\mathcal{E}(s)} \dot{\varphi}(s))(u) ds \geq \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q, t) dq - \tilde{C}(b_k - u). \]

Next, let \( \tilde{u} \) and \( \tilde{v} \) is from the statement of the lemma. Since \( \varphi \) is continuous on \([0, 1] \times [0, T]\) and increasing by the first argument, the function
\[ G(a, b, t) = \frac{1}{\tilde{v} - b} \int_b^{\tilde{v}} \varphi(q, t) dq - \frac{1}{a - \tilde{u}} \int_{\tilde{u}}^a \varphi(q, t) dq \]
is positive and continuous on \( E = \{(a, b, t) : \tilde{u} \leq a \leq b \leq \tilde{v}, \ t \in [0, T]\} \).
Hence
\[ \delta_1 = \inf_E G > 0. \]
Take \( \gamma = \frac{\delta_1}{8\tilde{C}} \) and for \( u, v \in [0, 1] \) satisfying \( (u, v) \supset [\tilde{u}, \tilde{v}] \) set
\[ u_0 = (u + \gamma) \land \tilde{u}, \]
\[ v_0 = (v - \gamma) \lor \tilde{v}. \]

Let \( \mathcal{G}(t) = \mathcal{G}(0 \lor (u - \gamma), (v + \gamma) \land 1, t), \ t \in (0, T], \) such that \( u_0 \) and \( v_0 \) belong to separate intervals from \( \mathcal{G}(T) \) and \( t \in (0, T] \) be fixed. For \( \pi_1, \pi_2 \) belonging to \( \mathcal{G}(t) \) and containing \( u_0, \ v_0 \) respectively, we obtain
\[ v_0 - u_0 + \int_0^t (\text{pr}_{\mathcal{E}(s)} \dot{\varphi}(s))(v_0) ds - \int_0^t (\text{pr}_{\mathcal{E}(s)} \dot{\varphi}(s))(u_0) ds \]
\[ \geq \frac{1}{|\pi_2|} \int_{\pi_2} \varphi(q, t) dq - \frac{1}{|\pi_1|} \int_{\pi_1} \varphi(q, t) dq \]
\[ - \tilde{C}(d - v_0) - \tilde{C}(u_0 - a) \]
\[ \geq G(b \lor \tilde{u}, c \land \tilde{v}, t) - \tilde{C}(d - v_0) - \tilde{C}(u_0 - a), \]
where \( c < d \) and \( a < b \) are the ends of \( \pi_1 \) and \( \pi_2 \) respectively and \( b \leq c \) because \( u_0 \) and \( v_0 \) belong to separate intervals from \( \mathcal{G}(T) \). Since \( u - \gamma \leq a < u_0 \leq u + \gamma \) and \( v - \gamma \leq v_0 < b \leq v + \gamma \), we have
\[ v_0 - u_0 + \int_0^t (\text{pr}_{\mathcal{E}(s)} \dot{\varphi}(s))(v_0) ds - \int_0^t (\text{pr}_{\mathcal{E}(s)} \dot{\varphi}(s))(u_0) ds \]
\[ \geq \delta_1 - \tilde{C}(v + \gamma - v + \gamma) - \tilde{C}(u + \gamma - u + \gamma) \]
\[ = \delta_1 - 4\tilde{C}\gamma = \frac{\delta_1}{2} > 0. \]
It finishes the proof of the lemma.
Proof of Lemma C.1. Let \( u < v \) be a fixed points from \((0, 1)\) and \(\delta, u_0, v_0, \gamma\) be defined in Lemma C.2 for some \([\bar{u}, \bar{v}] \subset (u, v)\). Suppose that

\[
\lim_{\varepsilon \to 0} \mathbb{P} \left\{ z^\varepsilon(v, t) - z^\varepsilon(u, t) \leq \frac{\delta}{2} \right\} > 0.
\]

Set

\[
B_1^\varepsilon = \{ z^\varepsilon(u, t) = z^\varepsilon((u - \gamma) \lor 0, t) \},
\]

\[
B_2^\varepsilon = \{ z^\varepsilon(v, t) = z^\varepsilon((v + \gamma) \land 1, t) \},
\]

\[
A^\varepsilon = \left\{ z^\varepsilon(v, t) - z^\varepsilon(u, t) \leq \frac{\delta}{2} \right\}.
\]

Since the diffusion of \( z^\varepsilon(u, \cdot) \) grows to infinity when the time tends to 0, it is convenient to work with the mean of \( z^\varepsilon \) because we can control the growing of diffusion in this case. So, denote

\[
\xi^\varepsilon_u(t) = \frac{1}{u_0 - u} \int_0^{u_0} z^\varepsilon(q, t) dq - u,
\]

\[
\xi^\varepsilon_v(t) = \frac{1}{v_0 - v} \int_0^{v_0} z^\varepsilon(q, t) dq - v.
\]

It is easy to see that

\[
\bar{A}^\varepsilon = \left\{ \xi^\varepsilon_v(t) - \xi^\varepsilon_u(t) \leq \frac{\delta}{2} \right\} \supseteq A^\varepsilon.
\]

Next, using the processes \( \xi^\varepsilon_u \) and \( \xi^\varepsilon_v \), we want to show that

\[
\lim_{\varepsilon \to 0} \mathbb{P} \{ A^\varepsilon \cap (B_1^\varepsilon \cup B_2^\varepsilon)^c \} = 0.
\]

Note that \( \xi^\varepsilon_u \) and \( \xi^\varepsilon_v \) are diffusion processes, namely

\[
\xi^\varepsilon_u(t) = \frac{u_0 + u}{2} + \int_0^t a^\varepsilon_u(s) ds + \chi^\varepsilon_u(t),
\]

\[
\xi^\varepsilon_v(t) = \frac{v_0 + v}{2} + \int_0^t a^\varepsilon_v(s) ds + \chi^\varepsilon_v(t),
\]
where

\[
\begin{align*}
    a^\varepsilon_u(t) &= \frac{1}{u_0 - u} \int_{u}^{u_0} \left( \Pr_{z^\varepsilon(t)}(\hat{\varphi}(t))(q) \right) dq, \\
    a^\varepsilon_v(t) &= \frac{1}{v - v_0} \int_{v_0}^{v} \left( \Pr_{z^\varepsilon(t)}(\hat{\varphi}(t))(q) \right) dq, \\
    \chi^\varepsilon_u(t) &= \frac{1}{u_0 - u} \int_{0}^{t} \int_{u_0}^{u} \eta_\varepsilon(q,s) dq ds, \\
    \chi^\varepsilon_v(t) &= \frac{1}{v - v_0} \int_{0}^{t} \int_{v_0}^{v} \eta_\varepsilon(q,s) dq ds.
\end{align*}
\]

By choosing of \( u_0, \ v_0 \) and \( \delta \), we have

\[
L^\varepsilon(t) = \frac{v_0 + v}{2} - \frac{u_0 + u}{2} \\
+ \int_{0}^{t} a^\varepsilon_u(s,\omega) ds - \int_{0}^{t} a^\varepsilon_v(s,\omega) ds \geq \delta, \quad t \in [0,T], \ \omega \in (B^\varepsilon_1 \cup B^\varepsilon_2)^c.
\]

Denote the difference \( \xi^\varepsilon_u - \xi^\varepsilon_u \) by \( \xi^\varepsilon \). Note that the characteristic of the martingale part \( \chi^\varepsilon \) of \( \xi^\varepsilon \) satisfies

\[
[\chi^\varepsilon]_t \leq \varepsilon C t,
\]

where \( C = \frac{1}{u_0 - u} + \frac{1}{v - v_0} \). So, denoting

\[
\sigma^\varepsilon = \inf \left\{ t : \xi^\varepsilon(t) = \frac{\delta}{2} \right\},
\]

we get

\[
\mathbb{P}\{A^\varepsilon \cap (B^\varepsilon_1 \cup B^\varepsilon_2)^c\} \leq \mathbb{P}\{\{\sigma^\varepsilon \leq t\} \cap (B^\varepsilon_1 \cup B^\varepsilon_2)^c\},
\]

which implies (C.1). Indeed, by Theorem 2.7.2 [14], there exists a standard Wiener process \( w^\varepsilon(t) \), \( t \geq 0 \), such that

\[
\chi^\varepsilon(t) = w^\varepsilon([\chi^\varepsilon]_t).
\]

Define \( \tau^\varepsilon = \inf \left\{ t : w^\varepsilon(\varepsilon C t) = -\frac{\delta}{2} \right\} \). Since

\[
\xi^\varepsilon(t) = \delta + (L^\varepsilon(t) - \delta) + \chi^\varepsilon(t)
\]

and the term \( L^\varepsilon - \delta \) is non-negative on \( (B^\varepsilon_1 \cup B^\varepsilon_2)^c \), the process \( \delta + w^\varepsilon(\varepsilon C \cdot) \) hits at the point \( \frac{\delta}{2} \) sooner than \( \xi^\varepsilon \). Consequently, \( \{\sigma^\varepsilon \leq t\} \cap (B^\varepsilon_1 \cup B^\varepsilon_2)^c \subseteq \{\tau^\varepsilon \leq t\} \cap (B^\varepsilon_1 \cup B^\varepsilon_2)^c \). This yields (C.1).
Next, the relation \( P\{A^{\varepsilon} \cap (B_1^{\varepsilon} \cup B_2^{\varepsilon})\} = P\{A^{\varepsilon}\} - P\{A^{\varepsilon} \cap (B_1^{\varepsilon} \cup B_2^{\varepsilon})\} \), (C.1) and supposing that \( \lim_{\varepsilon \to 0} P\{A^{\varepsilon}\} > 0 \) imply

\[
\lim_{\varepsilon \to 0} P\{A^{\varepsilon} \cap (B_1^{\varepsilon} \cup B_2^{\varepsilon})\} > 0.
\]

Thus, we obtain

\[
\lim_{\varepsilon \to 0} P\{A^{\varepsilon} \cap B_1^{\varepsilon}\} > 0 \quad \text{or} \quad \lim_{\varepsilon \to 0} P\{A^{\varepsilon} \cap B_2^{\varepsilon}\} > 0.
\]

It means that we can extend the interval \([u, v]\) to \([\tilde{u}, \tilde{v}]\) or \([u, (v + \gamma) \wedge 1]\), i.e.

\[
\lim_{\varepsilon \to 0} P\left\{z^{\varepsilon}(v, t) - z^{\varepsilon}((u - \gamma) \vee 0, t) \leq \frac{\delta}{2}\right\} > 0
\]

or

\[
\lim_{\varepsilon \to 0} P\left\{z^{\varepsilon}((v + \gamma) \wedge 1, t) - z^{\varepsilon}(u, t) \leq \frac{\delta}{2}\right\} > 0.
\]

Noting that \( \gamma \) only depends on \([\tilde{u}, \tilde{v}]\) and applying the same argument for new start points of the particles in finitely many steps, we obtain

\[
\lim_{\varepsilon \to 0} P\left\{z^{\varepsilon}(v_1, t) - z^{\varepsilon}(u_1, t) \leq \frac{\delta}{2}\right\} > 0,
\]

where \((u_1, v_1) \supset [\tilde{u}, \tilde{v}]\) and \(u_1 = 0\) or \(v_1 = 1\). Next, applying the same argument for new start points of the particles, but replacing \(B_1^{\varepsilon} \cup B_2^{\varepsilon}\) with \(B_1^{\varepsilon}\), if \(v_1 = 1\) or \(B_2^{\varepsilon}\), if \(u_1 = 0\), in finitely many steps we get

\[
\lim_{\varepsilon \to 0} P\left\{z^{\varepsilon}(1, t) - z^{\varepsilon}(0, t) \leq \frac{\delta}{2}\right\} > 0.
\]

But it is not possible because the same argument (without \(B_1^{\varepsilon}\) and \(B_2^{\varepsilon}\)) gives

\[
\lim_{\varepsilon \to 0} P\left\{z^{\varepsilon}(1, t) - z^{\varepsilon}(0, t) \leq \frac{\delta}{2}\right\} = 0.
\]

The lemma is proved. \(\square\)

Proof of Proposition C.1. Using Jakubowski’s tightness criterion (see Theorem 3.1 [16]) and boundedness of \(\hat{\varphi}\), as in the proof of exponential tightness of \(\{y^{\varepsilon}\}\) (see Proposition 4.1), we can prove that \(\{z^{\varepsilon}\}_{\varepsilon > 0}\) is tight in \(C([0, T], L_2(\mu)).\)

Let \(\{z^{\varepsilon}\}\) be a convergent subsequence and \(z\) is its limit. By Skorokhod’s theorem (see Theorem 3.1.8 [13]), we can define a probability space and a
sequence of random elements \( \{ \tilde{z}' \} \), \( \tilde{z} \) on this space such that \( \text{Law}(z') = \text{Law}(\tilde{z}') \), \( \text{Law}(z) = \text{Law}(\tilde{z}) \) and \( \tilde{z}' \to \tilde{z} \) in \( C([0,T], L_2(\mu)) \) a.s. If we show that \( \tilde{z} = \varphi \), we finish the proof because this implies that \( \tilde{z}' \to \varphi \) in \( C([0,T], L_2(\mu)) \) in probability and since \( \varphi \) is non-random, \( z' \to \varphi \) in probability. Thus, it will easily yield that \( z \to \varphi \) in probability.

So, for convenience of notation we will assume that \( z' \to z \) a.s., instead \( \tilde{z}' \to \tilde{z} \). First we check that \( z(t) \in L_2(\mu) \), for all \( t \in [0,T] \). Let \( t \) is fixed.

One can show that

\[
\tilde{z}'(t) \to z(t) \quad \text{in measure } \mathbb{P} \otimes \mu.
\]

By Lemma 4.2 [17], there exists subsequence \( \{ \varepsilon' \} \) such that

\[
z'(t) \to z(t) \quad \mathbb{P} \otimes \mu \text{- a.e.}
\]

Set \( A = \{ (\omega, u) : z'(u,t,\omega) \to z(u,t,\omega) \} \). Since \( \mathbb{P} \otimes \mu(A^c) = 0 \), it is easy to see that there exists the set \( U \subseteq [0,1] \) such that \( \mu(U^c) = 0 \) and \( \mathbb{P}(A_u) = 1 \), for all \( u \in U \), where \( A_u = \{ \omega : (\omega,u) \in A \} \). Note, it implies that for each \( u \in U \)

\[
z'(u,t) \to z(u,t) \quad \text{a.s.}
\]

Let \( U_{\text{count}} \) is a countable subset of \( U \) which is dense in \([0,1]\). From Lemma C.1 it follows that

\[
z(u,t) < z(v,t) \quad \text{a.s. for all } u, v \in U_{\text{count}}, u < v.
\]

Denote

\[
\Omega' = \bigcap_{u<v, u,v\in U_{\text{count}}} \{ z(u,t) < z(v,t) \}.
\]

Since \( U_{\text{count}} \) is countable, \( \mathbb{P}(\Omega') = 1 \). Next, define

\[
\tilde{z}(u,t,\omega) = \inf_{u\leq v, v\in U_{\text{count}}} z(v,t,\omega), \quad u \in [0,1], \ \omega \in \Omega'.
\]

Then for all \( \omega \in \Omega' \), \( \tilde{z}(\cdot,t,\omega) \in D^{\uparrow\uparrow} \). Let us show that \( \mu\{ u : \tilde{z}(u,t) \neq z(u,t) \} = 0 \) a.s.

Denote

\[
\tilde{\Omega} = \left( \bigcap_{u\in U_{\text{count}}} A_u \right) \cap \Omega' \cap \{ z'(t) \to z(t) \text{ in } L_2(\mu) \}.
\]

Then

(C.2) \[
z'(u,t,\omega) \to z(u,t,\omega) = \tilde{z}(u,t,\omega),
\]
for all \( u \in U_{\text{count}} \) and \( \omega \in \tilde{\Omega} \). Fix \( \omega \in \tilde{\Omega} \). Since \( \tilde{z}(\cdot, t, \omega) \) is nondecreasing, it has a countable set \( D_{\tilde{z}(\cdot, t, \omega)} \) of discontinuous points. The countability implies that \( \mu(D_{\tilde{z}(\cdot, t, \omega)}) = 0 \). Take \( u \in D_{\tilde{z}(\cdot, t, \omega)} \), then from monotonicity of \( \tilde{z}(\cdot, t, \omega) \) and \( z^\varepsilon(\cdot, t, \omega) \), density of \( U_{\text{count}} \) and (C.2) we can obtain

\[
 z^\varepsilon(u, t, \omega) \rightarrow \tilde{z}(u, t, \omega).
\]

Thus,

\[
 z^\varepsilon(\cdot, t, \omega) \rightarrow \tilde{z}(\cdot, t, \omega) \quad \mu \text{ - a.e.}
\]

On the other hand,

\[
 z^\varepsilon(\cdot, t, \omega) \rightarrow z(\cdot, t, \omega) \quad \text{in} \ L_2^2(\mu).
\]

Consequently, \( z(\cdot, t, \omega) = \tilde{z}(\cdot, t, \omega) \) \( \mu \)-a.e., for all \( \omega \in \tilde{\Omega} \). So, it means that \( z(t) \in L_2^{||}(\mu) \) a.s., for all \( t \in [0, T] \).

Now we can prove that \( z = \varphi \). Take \( l \in C([0, 1] \times [0, T], \mathbb{R}) \). By the dominated convergence theorem, \( \int_0^T \int_0^1 l(u, t)(z^\varepsilon(u, t) - u)dtdu \) converges to \( \int_0^T \int_0^1 l(u, t)(z(u, t) - u)dtdu \) a.s. Integrating by parts, we get

\[
 \int_0^T \int_0^1 l(u, t)(z^\varepsilon(u, t) - u)dtdu = \int_0^T \int_0^1 L(u, t)(\text{pr}_{z^\varepsilon(t)} \varphi(t))(u)dtdu
\]

\[
 + \int_0^T \int_0^1 L(u, t)d\eta^\varepsilon(u, t)du,
\]

where \( L(u, t) = \int_t^T l(u, s)ds \). The first term in the right hand side of the previous relation converges to \( \int_0^T \int_0^1 L(u, t)\varphi(u, t)dtdu \), by Corollary B.1. The second term is the stochastic integral so we can estimate the expectation of its second moment

\[
 \mathbb{E} \left( \int_0^T \int_0^1 L(u, t)d\eta^\varepsilon(u, t)du \right)^2 \leq \varepsilon \mathbb{E} \int_0^T \int_0^1 (\text{pr}_{z^\varepsilon(t)} L(t))^2(u)dtdu
\]

\[
 \leq \varepsilon \mathbb{E} \int_0^T \int_0^1 L^2(u, t)dtdu \rightarrow 0, \quad \varepsilon \rightarrow 0.
\]

Consequently, we obtain

\[
 \int_0^T \int_0^1 l(u, t)(z(u, t) - u)dtdu = \int_0^T \int_0^1 L(u, t)\varphi(u, t)dtdu,
\]

which easily implies \( z = \varphi \). The proposition is proved. \( \square \)
REFERENCES


