Harmonic maps of affine manifolds

Fatma Muazzez Şimşir, Jürgen Jost*

Hitit University, Department of Mathematics

***Max Planck Institute for Mathematics in the Sciences**

fmuazzezsimsir@hitit.edu.tr, jost@mis.mpg.de

Abstract

A manifold is said to be affine flat if it admits local coordinate systems whose transition maps are affine transformations. For affine flat manifolds it is natural to ask the following question: "What are the harmonic maps of this structure?" Harmonic map equation from an affine domain manifold to a target Riemannian manifold constitutes a semi-linear elliptic system of partial differential equations. In this work, we will establish an existence and uniqueness result for affine harmonic maps.

1. Introduction

Basic tools of Riemannian geometry are the geodesics and their higher dimensional generalizations, the harmonic maps. They are the critical points of an energy integral that involves the metric. Therefore, they are backed by a variational structure. This depends on the Levi-Cività connection underlying the Riemannian metric. A Hermitian manifold, however, naturally possesses a different connection, the complex one that respects the complex structure. This connection is different from the Levi-Cività connection unless the manifold is Kähler. Similarly, an affine manifold carries a flat affine connection that has nothing to do with the Levi-Cività connection of the auxiliary Riemannian metric.

Thus, harmonic maps are not naturally defined on such manifolds, and the main point of this paper is to discuss suitable substitutes. Thus, Hermitian harmonic maps, as introduced and studied in [JY], are defined through the complex connection, and affine harmonic maps, as introduced and studied in [JŞ], are determined by the affine connection, and the resulting equations do not satisfy a variational principle. The absence of a variational structure makes the analysis more difficult. Therefore, we need an additional global non-triviality condition to guarantee the existence of an affine harmonic map in a given homotopy class. As in the case of ordinary harmonic maps, nonpositive curvature of the target manifold is also required.

2. Affine harmonic maps

As described above, on an affine manifold M with metric tensor $\gamma_{\alpha\beta}$, we can define an affinely invariant differential operator, $L := \gamma^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}$. A function $f : M \longrightarrow \mathbb{R}$ that satisfies $Lf = 0$ is called affine harmonic. More generally, a map $f : M \longrightarrow N$ where N is a Riemannian manifold with metric g_{ij} and Christoffel symbols Γ^i_{jk} is called affine harmonic if it satisfies

$$\gamma^{\alpha\bar{\beta}}\left(\frac{\partial^2 f^i}{\partial x^\alpha \partial x^{\bar{\beta}}} + \Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^{\bar{\beta}}}\right) = 0. \quad (1)$$

in local coordinates on N. In invariant notation (1) can be written as

$$\gamma^{\alpha\beta} D_\alpha D_\beta f = 0 \quad (2)$$

where D is the connection on the bundle $T^*M \otimes f^{-1}TN$ induced by the flat connection on M and the Levi-Cività connection on N. Jost and Şimşir and Yau obtained the following general existence result for affine harmonic maps, [JŞ].

Theorem 1 (Jost - Simsir) *Let M be a compact affine manifold, N a compact Riemannian manifold of nonpositive sectional curvature. Let $g : M \to N$ be continuous, and suppose g is not homotopic to a map $g_0 : M \to N$ for which there is a nontrivial parallel section of $g_0^{-1}TN$. Then g is homotopic to an affine harmonic map $f : M \to N$.*

In fact, this result is stronger than the one stated in [JŞ]; the latter was formulated only for the special case of *Kähler* affine manifolds in the sense of [CY]. However, in the next section, we shall describe the analytic scheme for showing existence in such a way that it applies to any compact affine manifold M.
Again, one may construct examples to show that the global topological condition is needed in general, see [JŞ]. Using the argument of Al'ber [A], one can also show that the affine harmonic map is unique in its homotopy class under the assumptions of the above theorem. In fact, here, we also need the global condition.

References

[A] S. I. Al'ber, *Spaces of mappings into a manifold with negative curvature* ,Sov.Math.Dokl., 6-9, 1967 **9** (1967), 6–9.

[CY] S. Y. Cheng, S. T. Yau, *The real Monge-Ampère equation and affine flat structures*, Differential geometry and differential equations, Proc.Beijing Symp.1980, 339–370, 1982.

[JY] J. Jost, S. T. Yau, *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry*, Acta Math. **170**, (1993), 221–254.

[JŞ] J. Jost, F. M. Şimşir, *Affine harmonic maps*, Analysis, **29**, (2009), 185–197.