CONFORMAL GRADIENT FIELDS

PARVANEH

Where the question comes from:

A well-known theorem of Lichnerowicz states that a *(n>1)-dimensional* Riemannian manifold of constant scalar curvature is isometric to a sphere, if it admits a conformal gradient field.

If a Riemannian manifold obtains a concircular vector field, then *gradient of the conformal characteristic function ρ is a conformal vector field. Therefore, ρ satisfies the* following second order differential equation:

$$\nabla_k \rho_l = \phi(x) g_{kl}.$$

This equation helps to define a special coordinate system around every ordinary point of *ρ*, in which the metric has a warped product structure.

Why this method doesn't work in Finsler case:

$$F(u,v) := \sqrt{\sqrt{u^4+v^4}+\lambda(u^2+v^2)}$$

$$(g_{ij}) = \begin{pmatrix} \lambda + \frac{u^2(u^4+3v^4)}{(u^4+v^4)^{\frac{3}{2}}} & \frac{-2u^3v^3}{(u^4+v^4)^{\frac{3}{2}}} \\ \frac{-2u^3v^3}{(u^4+v^4)^{\frac{3}{2}}} & \lambda + \frac{v^2(v^4+3u^4)}{(u^4+v^4)^{\frac{3}{2}}} \end{pmatrix}$$

$$\rho(x,y) := ax + b$$

$$\Rightarrow grad(\rho)(x,y) = \left(\frac{a}{\lambda+1}, 0\right)$$

Adopted coordinates coincide with the initial one.

Propositions:

- If (M,g) be a Finsler manifold and V a C-concircular vector field, then
  $$\mathcal{L}_{\hat{V}}(Ric)_{ij} = -2(n-1)\phi(x)g_{ij}$$
- If grad ρ be a conformal vector field, then

The integral curves of grad ρ are geodesics of Finsler structure

Main results:

Theorem 1:

Let (M,g) be a compact Einstein-Finsler manifold of non-positive constant Ricci curvature. If M admits a C-concircular vector field V, then

V is homothetic.

Proof:

By computing the Lie derivative of Ricci tensor, with the help of Ricci constant assumption,

$$\nabla_i \rho_j = -k\rho(x) g_{ij}$$

ON EINSTEIN-RANDERS SPACES

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This equation changes to the following ODE along integral curves of liouville vector field:

$$\frac{d^2\rho}{dt^2} + k\rho = 0.$$

$k = 0 \Rightarrow \rho(t) = C_1 t + C_2.$

$k < 0 \Rightarrow \rho(t) = C_1 e^{-\sqrt{-k}t} + C_2 e^{\sqrt{-k}t}.$

Theorem 2:

If M is connected compact of positive constant Ricci curvature and admits a C-concircular vector field, then

M is homeomorphic to sphere

Proof:

Suppose $\gamma(s)$ be a geodesic starting from p_0 with initial velocity X_0 ,

$\frac{d^2\rho}{ds^2} + k^2\rho = 0$.

- $\rho(s) = Acos(ks) + Bsin(ks)$, where $A = \rho(p_0)$ and $B = \frac{1}{k}X_0(\rho)$.

Now suppose that γ is the integral curve of gradρ at p_0 and p_+ and p_- maximum and minimum points of ρ on γ. W.l.o.g $\rho(p_+) = 1$. Take p_+ as initial point.

$\Rightarrow$ all geodesics issuing from p_+ meet again at:

- ✓ Q is a minimum point of ρ.
- ✓ $Q = exp_{p_+}\left(S^n\left(\frac{\pi}{k}\right)\right)$.
- ✓ Q is conjugate to p_+.

for unit vector X, γ_X the geodesic with initial velocity X and C(X) the moment γ_X reaches the first conjugate point to p_+

- ✓ By Bonnet-Myres theorem :
  $t(X) \leq C(X) \leq \frac{\pi}{k}$.
- ✓ By Hopf-Rinow theorem:
  Q is reachable from p_+ by a minimal geodesic with unit velocity.
- ✓ On the other hand every such a geodesic reaches Q at $s = \frac{\pi}{k}$.
  $\Longrightarrow t(X) \geq \frac{\pi}{k}$.

$\Rightarrow t(X) = C(X) = \frac{\pi}{k} \Rightarrow Cut(p_+) = \{Q\}$

- ✓ again by Bonnet-Myres theorem :

$$diam(M) \leq \frac{\pi}{k},$$

$\Longrightarrow$ **There is no point more distant from p_+ than Q and so it is the only minimum point of ρ.**

By theorem of Reeb, M is homeomorphic to sphere.