

Where the question comes from:

A well-known theorem of Lichnerowicz states that a $(n>1)$ -dimensional Riemannian manifold of constant scalar curvature is isometric to a sphere, if it admits a conformal gradient field.

If a Riemannian manifold obtains a concircular vector field, then *gradient of the conformal characteristic function ρ is a conformal vector field*. Therefore, ρ satisfies the following second order differential equation:

$$\nabla_k \rho_l = \phi(x) g_{kl}.$$

This equation helps to define a special coordinate system around every ordinary point of ρ , in which the metric has a warped product structure.

Why this method doesn't work in Finsler case:

$$F(u, v) := \sqrt{\sqrt{u^4 + v^4} + \lambda(u^2 + v^2)}$$

$$(g_{ij}) = \begin{pmatrix} \lambda + \frac{u^2(u^4 + 3v^4)}{(u^4 + v^4)^{\frac{3}{2}}} & \frac{-2u^3v^3}{(u^4 + v^4)^{\frac{3}{2}}} \\ \frac{-2u^3v^3}{(u^4 + v^4)^{\frac{3}{2}}} & \lambda + \frac{v^2(v^4 + 3u^4)}{(u^4 + v^4)^{\frac{3}{2}}} \end{pmatrix}$$

$$\rho(x, y) := ax + b$$

$$\Rightarrow \text{grad}(\rho)(x, y) = \left(\frac{a}{\lambda + 1}, 0 \right)$$

Adopted coordinates coincide with the initial one.

Propositions:

✚ If (M, g) be a Finsler manifold and V a C-concircular vector field, then

$$\mathcal{L}_V(\text{Ric})_{ij} = -2(n-1)\phi(x)g_{ij}$$

✚ If $\text{grad } \rho$ be a conformal vector field, then

The integral curves of $\text{grad } \rho$ are geodesics of Finsler structure

Main results:

Theorem 1:

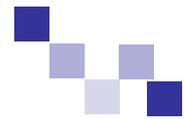
Let (M, g) be a compact Einstein-Finsler manifold of non-positive constant Ricci curvature. If M admits a C-concircular vector field V , then

V is homothetic.

Proof:

By computing the Lie derivative of Ricci tensor, with the help of Ricci constant assumption,

$$\nabla_i \rho_j = -k\rho(x)g_{ij}$$



This equation changes to the following ODE along integral curves of liouville vector field:

$$\frac{d^2\rho}{dt^2} + k\rho = 0.$$

$$k = 0 \Rightarrow \rho(t) = C_1 t + C_2.$$

$$k < 0 \Rightarrow \rho(t) = C_1 e^{-\sqrt{-k}t} + C_2 e^{\sqrt{-k}t}.$$

Theorem 2:

If M is connected compact of positive constant Ricci curvature and admits a C -conircular vector field, then

M is homeomorphic to sphere

Proof:

Suppose $\gamma(s)$ be a geodesic starting from p_0 with initial velocity X_0 ,

$$\frac{d^2\rho}{ds^2} + k^2\rho = 0.$$

$$\blacktriangleright \rho(s) = A \cos(ks) + B \sin(ks),$$

where $A = \rho(p_0)$ and $B = \frac{1}{k} X_0(\rho)$.

Now suppose that γ is the integral curve of $\text{grad}\rho$ at p_0 and p_+ and p_- maximum and minimum points of ρ on γ . W.l.o.g $\rho(p_+) = 1$. Take p_+ as initial point.

\Rightarrow all geodesics issuing from p_+ meet again at:

✓ Q is a minimum point of ρ .

$$\checkmark Q = \exp_{p_+} \left(S^n \left(\frac{\pi}{k} \right) \right).$$

✓ Q is conjugate to p_+ .

for unit vector X , γ_X the geodesic with initial velocity X and $C(X)$ the moment γ_X reaches the first conjugate point to p_+

✓ By Bonnet-Myres theorem :

$$t(X) \leq C(X) \leq \frac{\pi}{k}.$$

✓ By Hopf-Rinow theorem:

Q is reachable from p_+ by a minimal geodesic with unit velocity.

✓ On the other hand every such a geodesic reaches Q at $s = \frac{\pi}{k}$.

$$\Rightarrow t(X) \geq \frac{\pi}{k}.$$

$$\Rightarrow t(X) = C(X) = \frac{\pi}{k} \Rightarrow \text{Cut}(p_+) = \{Q\}$$

✓ again by Bonnet-Myres theorem :

$$\text{diam}(M) \leq \frac{\pi}{k},$$

\Rightarrow There is no point more distant from p_+ than Q and so it is the only minimum point of ρ .

By theorem of Reeb, M is homeomorphic to sphere.