AN EQUIVARIANT CW-COMPLEX FOR THE FREE LOOP SPACE OF A FINSLER MANIFOLD

HANS-BERT RADEMACHER

Abstract. We consider a compact manifold M with a bumpy Finsler metric. The free loop space Λ of M carries a canonical action of the group S^1. Using Morse theory for the energy functional $E:\Lambda\to\mathbb{R}$ we construct with the help of a space of geodesic polygons an equivariant CW complex which is S^1-homotopy equivalent to the free loop space.

1. Statement of the Result

For a compact differentiable manifold M with Finsler metric F we denote by $\Lambda=\Lambda M$ the *free loop space* of absolutely continuous closed curves $\gamma: S^1\to M$ with finite energy $E(\gamma)=\frac{1}{2}\int_0^1 F^2\left(\gamma'(t)\right)\,dt\;<\infty$, here $S^1=[0,1]/\{0,1\}$ denotes the 1-dimensional sphere. The free loop space Λ carries a canonical S^1-action leaving the energy functional $E:\Lambda\to\mathbb{R}$ invariant. For $a\in\mathbb{R}$ we use the following notation for the sublevel set: $\Lambda^a:=\{\gamma\in\Lambda\,|\,E(\gamma)\le a\}$.

Morse introduced for the investigation of geodesics a *finite-dimensional approximation* by a space of geodesic polygons, cf. [7, ch.16]. Assume that $\eta>0$ is the *injectivity radius* of (M,F), i.e. η is the maximal positive number such that any geodesic $c:[0,1]\to M$ of length $L(c)\le\eta$ is minimal. We call the geodesic c *minimal* if the distance $d(c(0),c(1))$ between its end points equals its length $L(c)=\int_0^1 F(c'(t))\,dt$. For a positive number a one can choose a positive integer $k>2a/\eta^2$ and one defines the space

$$\Lambda(k,a):=\{c\in\Lambda^a\,;\,c\,|[i/k,(i+1)/k]\;\text{ is a geodesic };i=0,1,2,\ldots,k-1\}$$

consisting of *geodesic polygons* with k vertices $c(0),c\left(1/k\right),c\left(2/k\right),\ldots,c\left((k-1)/k\right)$ (i.e. *geodesic k-gons*) of energy $\le a$. Since $d\left(c\left(i/k\right),c\left(\left(i+1\right)/k\right)\right)<\eta$ the geodesic k-gon c can be identified with the set $c(0),c(1/k),c(2/k),\ldots,c((k-1)/k)$ of vertices. On the other hand the space $\Lambda(k,a)$ has the structure of a submanifold with boundary of the free loop space of dimension $\dim\Lambda(k,a)=k\cdot\dim M$. The space $\Lambda(k,a)$ can be viewed as a *finite-dimensional approximation* of the space Λ^a in the following sense: The critical points of the restriction of the energy functional $E':\Lambda(k,a)\to\mathbb{R}$ coincide with the critical points of the energy functional $E:\Lambda^a\to\mathbb{R}$, in particular they are the closed geodesics of energy $\le a$. In addition it is well known that the indices and nullities of the hessian $d^2E'(c)$ and $d^2E(c)$ coincide, cf. [10, p.55]. Therefore for existence results for closed geodesics

Date: 2009-11-15.

Key words and phrases. closed geodesics, geodesic polygons, free loop space, equivariant CW-complex, equivariant homotopy type.

Dedicated to Paul Rabinowitz with best wishes on his 70th birthday.

one can study the critical point theory (resp. Morse theory) of the energy functional on the finite-dimensional and compact subspace $\Lambda(k,a)$. But there is one disadvantage of this finite-dimensional approximation. The space $\Lambda(k,a)$ is not closed under the canonical S^1-action, but it carries a canonical $\mathbb{Z}_k$-action induced from the S^1-action. Here for $c \in \Lambda(k,a), u \in [0,1]/\{0,1\} = S^1$ let $u.c \in S^1.\Lambda(k,a)$ be defined by $u.c(t) = c(t+u)$; i.e. $u.c$ is a geodesic polygon with k vertices $c(u), c(u+1/k), c(u+2/k), \ldots, c\,(u+(k-1)/k)$. Following the concepts developped by the author in [10, sec.6] and [9, §4] and by Bangert & Long in [2, Sec.3] one can find a candidate for a finite-dimensional approximation of the free loop space which is closed under the canonical S^1-action:

Theorem 1.1. *Let F be a bumpy Finsler metric on a compact differentiable manifold M and let $(a_j)_{j\geq 0}$ be a strictly increasing sequence of regular values of the energy functional $E:\Lambda\to\mathbb{R}$ on the free loop space Λ.*
Then there is a S^1-CW complex X which is S^1-homotopy equivalent to Λ induced from the Morse theory of the energy functional. In addition there is a filtration $(X_j)_{j\geq 1}$ by finite S^1-CW subcomplexes of X which are S^1-homotopy equivalent to Λ^{a_j}.

For a bumpy Riemannian metric this result is contained in [9, Thm.4.2]. The proof does not directly extend to the Finsler case due to a lack of regularity of the energy functional on the free loop space. In several papers it is claimed that the energy functional on the free loop space of a compact Finsler manifold is twice differentiable at critical points. But Abbondandolo & Schwarz show that the energy functional on the free loop space of a compact Finsler manifold is twice differentiable at a critical point only if the metric is Riemannian along this closed geodesic, cf. [1, Remark 2.4]. But for the use of the Morse Lemma this differentiability is needed. The finite-dimensional and equivariant approximation by spaces of geodesic polygons offers one way out of the problem with the low regularity of the energy functional in the Finsler case. For certain applications as in the work of Bangert & Long [2] and the author's work [8] and [11] an equivariant version of the Morse Lemma has to be used.
The definition of an equivariant CW-complex resp. G-CW complex can be found in [12, II.1]. For the group $G = S^1$ an r-dimensional equivariant cell $e^r := \Phi\left(S^1/\mathbb{Z}_m \times D^r\right)$ of an S^1-CW complex X with r-skeleton X^r and with isotropy subgroup $I(x) \cong \mathbb{Z}_m$ for $x \in \Phi\left(D^r - S^{r-1}\right)$ is described by an S^1-equivariant characteristic map

$$(\Phi,\phi) : S^1/\mathbb{Z}_m \times \left(D^r, S^{r-1}\right) \to \left(X^r, X^{r-1}\right).$$

Let $\dot{e}^r := \phi\left(S^1/\mathbb{Z}_m \times S^{r-1}\right)$, then the restriction $\Phi : S^1/\mathbb{Z}_m \times \left(D^r - S^{r-1}\right) \to e^r - \dot{e}^r$ is a homeomorphism. The restriction $\phi = \Phi\left|S^1/\mathbb{Z}_m \times S^{r-1} : S^1/\mathbb{Z}_m \times S^{r-1} \to \dot{e}^r \subset X^{r-1}\right.$ is also called *attaching map* of the r-cell e^r. The complex is *finite* if it consists of finitely many equivariant cells. It also follows that the quotient space Λ/S^1 has the homotopy type of an ordinary CW complex. The subcomplexes X_j also carry the finer structure of a $(\mathbb{Z}_{m_j}, S^1)$-CW complex introduced by the author [9, §2] where m_j is a multiple of all multiplicities of closed geodesics of energy $\leq a_j$. For any orbit $S^1.c$ of a closed geodesic c of multiplicity m there is a subcomplex which is of the form $S^1 \times_{\mathbb{Z}_m} D^-(c)$, here $D^-(c)$ is a negative disc of the closed geodesic c, cf. Proposition 2.4 and Remark 2.1.

2. Proofs:

Let F be a bumpy Finsler metric on a compact differentiable manifold with injectivity radius η. The metric is bumpy if all closed geodesics are non-degenerate. Then the S^1-orbit $S^1.c$ of closed geodesic is an isolated critical orbit. Let $i = \operatorname{ind}(c)$ resp. $m = \operatorname{mul}(c)$ be its *index* resp. *multiplicity.* Here the index of a closed geodesic is the maximal dimension of a subspace of the tangent space $T_c\Lambda$ on which the index form $d^2E(c)$ is negative definite. A closed geodesic c has multiplicity m if $c(t) = c_0(mt)$ for all $t \in S^1$ for a closed curve c_0 which is injective up to possibly finitely many selfintersection points. The closed geodesic c_0 is also called *prime.* The crucial observation by Morse is that the critical points of the restriction $E' : \Lambda(k,a) \to \mathbb{R}$ coincide with the critical points of $E : \Lambda^a \to \mathbb{R}$ and there indices and nullities coincide, too. The space $\Lambda(k,a)$ carries as a subspace of Λ a canonical $\mathbb{Z}_k$-action induced by the S^1-action. The strong deformation retraction of Λ^a onto $\Lambda(k,a)$ given in [7, 16.2] can be modified in the category of $\mathbb{Z}_k$-equivariant maps:

Proposition 2.1. [10, sec.6.2] *There is a strong $\mathbb{Z}_k$-deformation retraction $r_u : \Lambda^a \to \Lambda^a, u \in [0,1]$ onto the subspace $\Lambda(k,a)$.*

Proof. For $u \in [0,1]; i = 0,1,\ldots,k-1$ one defines:

$$\begin{aligned} r_u(c)\,|[i/k,(i+u)/k] &= \text{minimal geodesic} \\ &\quad\ \text{joining } c(i/k) \text{ and } c((i+u)/k) \\ r_u(c)\,|[(i+u)/k,(i+1)/k] &= c\,|[(i+u)/k,(i+1)/k] \end{aligned}$$

Then $r_u(c) = c$ for all $c \in \Lambda(k,a)$ and $u \in [0,1]$, and $r_1(c) \in \Lambda(k,a)$ for all $c \in \Lambda^a$. □

The energy functional $E : \Lambda \to \mathbb{R}$ satisfies the Palais-Smale condition, cf. [1, Proposition 2.5] resp. [3, Thm.3.1]. Therefore we conclude:

Proposition 2.2. *(a) If for two numbers $a < b$ the closed interval $[a,b]$ does not contain a critical value of the energy functional $E : \Lambda \to \mathbb{R}$ then the sublevel set Λ^a is an strong S^1-deformation retract of the sublevel set Λ^b.*

(b) Let c be a closed geodesic of energy $a = E(c)$ and multiplicity m such that the S^1-orbit $S^1.c$ is the set of all closed geodesics with energy in $[a-\epsilon_1, a+\epsilon_1]$ for some $\epsilon_1 > 0$. Then there is a $\mathbb{Z}_m$-invariant hypersurface $\Sigma_c \subset \Lambda$ with $c \in \Sigma_c$ which is transversal to the orbit $S^1.c$ at c such that for sufficiently small $\epsilon \in (0,\epsilon_1)$ the subset $\Lambda^{a-\epsilon} \cup S^1.\Sigma_c$ is a strong S^1-deformation retract of the sublevel set $\Lambda^{a+\epsilon}$.

The hypersurface Σ_c is also called a *slice,* cf. [6, Lem. 2.2.8]. The tubular neighborhood $S^1.\Sigma_c \subset \Lambda$ is S^1-homeomorphic to $S^1 \times_{\mathbb{Z}_m} \Sigma_c$. Here we use the following notation: For a $\mathbb{Z}_m$-space Y we denote by $S^1 \times_{\mathbb{Z}_m} Y$ the quotient $\left(S^1 \times Y\right)/\mathbb{Z}_m$ (also called *twist product*) with respect to the $\mathbb{Z}_m$-action $(u,(v,y)) \in \mathbb{Z}_m \times \left(S^1 \times Y\right) \mapsto (vu^{-1}, u.y) \in S^1 \times Y$ where we consider $\mathbb{Z}_m$ as subgroup of S^1.

An S^1-subspace $A \subset X$ of the S^1-space X is called *strong S^1-deformation retract,* if there is an S^1-map $H : [0,1] \times X \to X$ (called a *strong S^1-deformation retraction* from X onto A) which satisfies the following conditions: $H(0,x) = x$ for all $x \in X$; $H(1,x) \in A$ for all $x \in X$ and $H(t,a) = a$ for all $t \in [0,1], a \in A$. In particular the inclusion $A \to X$ is a S^1-homotopy equivalence.

Proposition 2.3. *Let c be a closed geodesic of multiplicity $m \geq 1$, energy $a = E(c)$ and $\Lambda(k,b) \subset \Lambda$ a finite-dimensional approximation with $a < b$ such that m divides*

k. Choose a $\mathbb{Z}_m$-invariant hypersurface $\Sigma_c \subset \Lambda$ as above and choose a $\mathbb{Z}_m$-invariant hypersurface $V_c \subset \Lambda(k,a)$ transversal to the orbit $S^1.c$ at $c \in V_c$ with $V_c \subset \Sigma_c$.
If the orbit $S^1.c$ consists of all closed geodesics with energy in $[a-\epsilon_1, a+\epsilon_1]$ for some $\epsilon_1 > 0$ then for sufficiently small $\epsilon \in (0,\epsilon_1)$ the set $\Lambda^{a-\epsilon} \cup S^1.V_c$ is a strong S^1-deformation retract of $\Lambda^{a+\epsilon}$.

Proof. Following the Proof of [2, Lem.3.3] we consider the map $G : S^1 \times \Lambda(k, a+\epsilon_1) \to \Lambda(k, a+\epsilon_1)$ with $G(\gamma, s) = r_1(u.\gamma)$ which does not increase the energy and satisfies $G(0,\gamma) = \gamma$ for all $\gamma \in \Lambda(k, a+\epsilon_1)$. The map r_1 is defined in the proof of Proposition 2.1. For a sufficiently small neighborhood $U \subset \Lambda(k, a+\epsilon_1)$ of c there is an $\delta > 0$ and a smooth function $\sigma : U \to (-\delta, \delta)$ uniquely defined by $G(\sigma(\gamma), \gamma) \in V_c$. Then we define $h : [0,1] \times U \to \Lambda(k, a+\epsilon_1)$ by: $h(t,\gamma) = G(t\sigma(\gamma),\gamma) = r_1((t\sigma(\gamma)).\gamma)$. Let $h_t(\gamma) = h(t,\gamma)$ then $h_0(\gamma) = \gamma, h_1(\gamma) \in V_c$ for all $\gamma \in V_c$; $h_t(\gamma) = \gamma$ for all $\gamma \in V_c \cap U$ and $E(h_t(\gamma)) \le E(\gamma)$ for all $t \in [0,1]$ and $\gamma \in U$. Therefore one can define for sufficiently small $\epsilon \in (0,\epsilon_1)$ an S^1-map $H_t : \Lambda^{a-\epsilon} \cup S^1.\Sigma_c \to \Lambda^{a-\epsilon} \cup S^1.\Sigma_c$ with $H_1(\gamma) \in \Lambda^{a-\epsilon} \cup S^1.V_c$ for all $\gamma \in \Lambda^{a-\epsilon} \cup S^1.\Sigma_c$ and $H_t(\gamma) = \gamma$ whenever $E(\gamma) \le a - \epsilon$ or $\gamma \in U - \{c\}$. Hence this map defines a strong deformation retraction of $\Lambda^{a-\epsilon} \cup S^1.\Sigma_c$ onto $\Lambda^{a-\epsilon} \cup S^1.V_c$ which is not energy increasing. From Proposition 2.2 (b) the conclusion follows. □

Here the set $S^1.V_c$ is S^1-equivariantly hoemeomorphic to $S^1 \times_{\mathbb{Z}_m} V_c$ and $\Lambda^{a+\epsilon}$ is S^1-homotopy equivalent to the space obtained by adjoining $S^1.V_c$ to $\Lambda^{a-\epsilon}$.

Proposition 2.4. *Let c be a non-degenerate closed geodesic of multiplicity m, energy $a = E(c)$, index $i = \operatorname{ind}(c)$ and $\Lambda(k,b) \subset \Lambda$ a finite-dimensional approximation with $a < b$ such that m divides k and such that the critical orbit $S^1.c$ consists of all closed geodesics of energy in the interval $[a-\epsilon, a+\epsilon]$. Then there is an orthogonal representation of the group $\mathbb{Z}_m$ on an i-dimensional vector subspace $\mathbb{R}^i \subset T_c\Lambda(k, a+\epsilon)$ of the tangent space with corresponding disc $D^i = \{x \in \mathbb{R}^i; \|x\| \le \delta\}$ for some $\delta > 0$ and a diffeomorphism $\phi : D^i \to D^-(c) \subset \Lambda(k, a+\epsilon)$ such that the following holds: $E(D^-(c) - \{c\}) \subset (0,a)$ and for sufficiently small $\epsilon > 0$ the set $\Lambda^{a-\epsilon} \cup S^1 D^-(c) = \Lambda^{a-\epsilon} \cup_\phi (S^1 \times_{\mathbb{Z}_m} D^i)$ is a strong S^1-deformation retract of the sublevel set $\Lambda^{a+\epsilon}$.*

Remark 2.1. *The disc D^i with its orthogonal $\mathbb{Z}_m$-action resp. its image $\phi(D^i)$ under the diffeomorphism is also called a* negative disc *$D^-(c)$ of the closed geodesic c. It carries the structure of a finite $\mathbb{Z}_m$-CW complex with subcomplex S^{i-1}, cf.* [9, Prop.1.10]. *This cell decomposition allows the computation of the homology $H^*(D^i/\mathbb{Z}_m, S^{i-1}/\mathbb{Z}_m; R)$ of the quotient space $(D^i/\mathbb{Z}_m, S^{i-1}/\mathbb{Z}_m)$ for rings $R = \mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}$, cf.* [9, Prop.1.13]. *In particular there is at least one cell in D^i/S^{i-1} in any dimension $k \in \{\operatorname{ind}(c_0), \ldots, \operatorname{ind}(c)\}$ where $c = c_0^m$ with a prime closed geodesic c_0. This $\mathbb{Z}_m$-CW decomposition on D^i with subcomplex S^{i-1} induces in a canonical way a S^1-CW-structure on the twist product $S^1 \times_{\mathbb{Z}_m} D^i$ with subcomplex $S^1 \times_{\mathbb{Z}_m} S^{i-1}$.*

Proof. (of Proposition 2.4) This is a standard argument using the equivariant Morse Lemma applied to the $\mathbb{Z}_m$-invariant and smooth restriction $E : V_c \to \mathbb{R}$. One can choose an arbitrary Riemannian metric g on the manifold M which induces a $\mathbb{Z}_m$-invariant metric on $V_c \subset \Lambda(k,a)$ where $\Lambda(k,a)$ is identified with a subspace of $\underbrace{M \times \ldots \times M}_{k\ times}$ endowed with the product metric $h = g \oplus \cdots \oplus g$. Since the closed geodesic is non-degenerate there is an orthogonal decomposition $T_cV_c = V_+ \oplus V_-$ of the tangent space at c into the sum V_+ resp. V_- of eigenspaces of positive resp. negative eigenvalues of the endomorphism associated to the hessian $d^2E(c)$ via the inner product h_c. By $\|.\|$ we denote the associated norm of h_c. Since $i = \operatorname{ind} c$ the space V_- has dimension i. There is a disc $D = \{x \in T_cV_c\,;\, h_c(x,x) \le \delta\} \subset T_cV_c$ for some $\delta > 0$ and a $\mathbb{Z}_m$-equivariant diffeomorphism $\psi : D \to \psi(D) \subset V_c$ such that $\psi(0,0,0) = c$ and $E\left(\psi(x_+,x_-)\right) = \|x_+\|^2 - \|x_-\|^2$. Then let $D^i := V_- \cap D$. We call the $\mathbb{Z}_m$-invariant subset $D^-(c) = \psi(D^i)$ a *negative disc,* it is a local i-dimensional submanifold of the slice Σ_c with $E\left(D^-(c) - \{c\}\right) \subset (0,a)$ and $c \in D^-(c)$. By standard arguments in (equivariant) Morse theory it follows that $(V_c \cap \Lambda^{a-\epsilon}) \cup D^-(c)$ is a strong $\mathbb{Z}_m$-deformation retract of $V_c \cap \Lambda^{a+\epsilon}$ for sufficiently small ϵ, cf. for example [13, §4]. By equivariant extension one obtains that the set $\left(S^1.V_c \cap \Lambda^{a-\epsilon}\right) \cup S^1.D^-(c)$ is a strong S^1-deformation retract of $S^1.V_c \cap \Lambda^{a+\epsilon}$. Then the conclusion follows from Proposition 2.3 □

This Proposition is the essential step in the proof of Theorem 1.1 which we now present and which is analogous to the proof of [9, Thm.4.2]:

Proof. (of Theorem 1.1). We show with induction by j that there is a relative S^1-CW complex (X,A) with a filtration by subcomplexes $(X_j,A)_{j\ge 0}$ with a S^1-homotopy equivalence $F_j : \Lambda^{a_j} \to X_j$. Let b_j be the strictly monotone increasing sequence of critical values of E with $a_j < b_j < a_{j+1}$ and $c_0 = 0$. Let $A = \Lambda^0$ which one can identify with the manifold M. We assume that the claim is proved for $j-1$. Since the metric is bumpy there are for any $a > 0$ only finitely many critical S^1-orbits of closed geodesics with energy $\le a$. Let $S^1.c_{j,l}\,;\, l = 1,2,\ldots,N_j$ be the S^1-orbits of closed geodesics $c_{j,l}$ with energy a_j. Let $i_{j,l} = \operatorname{ind}(c_{j,l})\,, m_{j,l} = \operatorname{mul}(c_{j,l})\,.$ We choose m_j as a multiple of m_{j-1} and $m_{j,1},\ldots,m_{j,N_j}$ such that $m_j > 2a_j/\eta^2$, here η is the injectivity radius. Hence we conclude from Proposition 2.4: Λ^{a_j} is S^1-homotopy equivalent to

$$\Lambda^{a_{j-1}} \cup \bigcup_{g_{j,l}, l=1,2,\ldots,N_j} S^1 \times_{\mathbb{Z}_{m_{j,l}}} D^{i_{j,l}}$$

where $g_{j,l} : S^{i_{j,l}-1} \to \Lambda^{a_{j-1}}$ are $\mathbb{Z}_{m_{j,l}}$-equivariant attaching maps. It follows from Remark 2.1 that $S^1 \times_{\mathbb{Z}_{m_{j,l}}} D^{i_{j,l}}$ carries the structure of a finite S^1-CW complex with subcomplex $S^1 \times_{\mathbb{Z}_{m_{j,l}}} S^{i_{j,l}-1}$. Then the equivariant cellular approximation theorem [12, II.2.1] implies that for every map $F_{j-1}\circ g_{j,l} : S^1 \times_{\mathbb{Z}_{m_{j,l}}} S^{i_{j,l}-1} \to X_{j-1}$ there is a S^1-homotopic map $\overline{g_{j,l}} : S^1 \times_{\mathbb{Z}_{m_{j,l}}} S^{i_{j,l}-1} \to X_{j-1}$ which is *cellular,* i.e. for any $r \ge 0$ the image of the r-skeleton of $S^1 \times_{\mathbb{Z}_{m_{j,l}}} S^{i_{j,l}-1}$ under $\overline{g_{j,l}}$ lies in the r-skeleton of the subcomplex $(X_{i-1},A)\,.$ Then we obtain the finite S^1-CW complex (X_j,A) by attaching cells to the complex (X_{j-1},A) via the attaching maps $\overline{g_{j,l}}, l = 1,2,\ldots,N_j$:

$$X_j = X_{j-1} \cup \bigcup_{\overline{g_{j,l}}, l=1,2,\ldots,N_j} S^1 \times_{\mathbb{Z}_{m_{j,l}}} D^{i_{j,l}}$$

By standard arguments for equivariant CW-complexes (cf. [12, Section II.1]) we conclude that there is a S^1-equivariant homotopy $F_j : \Lambda^{a_j} \to X_j$ extending F_{j-1}. □

Remark 2.2. *(a) Caponio et al. introduce in* [4, Section 2] *a localization procedure for the energy functional on the infinite-dimensional Hilbert space based on ideas of K.-C. Chang.*

(b) The statement of Theorem 1.1 can be extended to manifolds with a Morse metric. *For these metrics the critical set of the energy functional is the disjoint union of non-degenerate critical submanifolds, i.e. the energy functional is a Morse-Bott function.*

(c) One can use the Morse chain complex of the S^1-CW complex resp. of the $\left(\mathbb{Z}_{m_i}, S^1\right)$-CW complexes as in the author's work [9]. *Applications of equivariant Morse chain complexes can be found for example in Hingston's paper* [5].

References

[1] A.Abbondandolo & M.Schwarz: *A smooth pseudo-gradient for the Lagrangian action functional.* `arXiv:0812.4364` to appear in: Adv. Nonlin. Studies

[2] V.Bangert & Y.Long: *The existence of two closed geodesics on every Finsler 2-sphere.* Math. Ann. (publ. online August, 2009) `DOI 10.1007/s00208-009-0401-1`

[3] E.Caponio, M.Javaloyes & A.Masiello: *On the energy functional on Finsler manifolds and applications to stationary spacetimes.* `arXiv:math/0702323v3`

[4] E.Caponio, M.Javaloyes & A.Masiello: *Morse theory of causal geodesics in a stationary spacetime via Morse theory of geodesics of a Finsler metric.* `arXiv:0903.3519`

[5] N.Hingston: *On the equivariant Morse complex of the free loop space of a surface.* Trans. Amer. Math. Soc. **350** (1998) 1129–1141

[6] W.Klingenberg: *Lectures on closed geodesics.* Grundlehren der math.Wiss. **230** Springer–Verlag Berlin Heidelberg New York 1978

[7] J.Milnor: *Morse theory.* Annals of Math.Studies **51** Princeton Univ. Press, 3rd ed., Princeton 1969

[8] H.B.Rademacher: *On the average indices of closed geodesics.* J.Differential Geom. **29** (1989) 65–83

[9] H.B.Rademacher: *On the equivariant Morse chain complex of the space of closed curves.* Math. Zeitschr. **201** (1989) 279–302

[10] H.B.Rademacher: *Morse-Theorie und Geschlossene Geodätische.* Bonner Math. Schr. **229** (1992)

[11] H.B.Rademacher: *The second closed geodesic on Finsler spheres of dimension $n > 2$.* Trans. Amer. Math. Soc., online first: `DOI:10.1090/S0002-9947-09-04745-X`

[12] T. tom Dieck: *Transformation groups.* de Gruyter studies in Math. **8**, Berlin, New York 1987

[13] A.G.Wasserman: *Equivariant differential toplogoy.* Topology **8** (1969) 127-150

Mathematisches Institut, Universität Leipzig, D–04081 Leipzig, Germany, E-Mail:, `rademacher@math.uni-leipzig.de`; `www.math.uni-leipzig.de/MI/rademacher`