Essential conformal fields in pseudo–Riemannian geometry

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1 Introduction

Conformal symmetries, conformal transformations and conformal vector fields are of great importance in *general relativity*, as is well known since the early 1920's, see [PR], [Ha]. Brinkmann [Br] investigated in 1925 conformal transformations from one Einstein space to another, Riemannian or pseudo–Riemannian. Later conformal vector fields on Einstein spaces (arising from local 1–parameter groups of conformal diffeomorphisms) were reduced to the case of gradient fields, leading to a very fruitful theory of conformal gradient fields in general. Conformal vector fields also occur as associated vector fields of twistor spinors, see [Ra] and [KR1].

A conformal vector field is *essential* if it is not an isometric field for any conformally equivalent metric. In particular conformal gradient fields with a zero are essential. The existence of a conformal gradient field $V = \nabla\psi$ is equivalent to the existence of a solution of the equation

$$\nabla^2\psi = \lambda\, g \tag{1}$$

for some function λ which does not vanish identically. Here $\nabla^2\psi$ denotes the *hessian* of the function ψ and g denotes the metric. There are three types of complete Riemannian manifolds carrying a conformal gradient field according to the number $N \in \{0, 1, 2\}$ of zeros. If $N = 2$ then M is conformally equivalent to the *standard sphere*, if $N = 1$ it is conformally equivalent to the *Euclidean space* or to the *hyperbolic space*, if $N = 0$ it is conformally equivalent to a *product* of a real interval with an $(n-1)$–dimensional manifold M_*. These results are essentially due to Tashiro [Ta], Bourguignon [Bo], Kerbrat [Ke1], compare [Kue1], [Lf1] and [Ra]. If in addition the manifold is Einstein and $N \geq 1$ then the metric has constant sectional curvature, cf. [YN], [Ta], [Ka].

In the case of pseudo–Riemannian manifold with *indefinite* metric, the situation turns out to be fairly different. Local results in the case of conformal gradient fields are due to Brinkmann [Br] and Fialkow [Fi]. Corresponding global results are rare. Kerbrat [Ke2] studies the equation $\nabla^2\psi = \psi\, g$ on a complete manifold of signature $(k, n-k)$ and shows the following: If the conformal gradient field $\nabla\psi$ has a zero then M is isometric to the *pseudo–hyperbolic space* $S^n_k(-1) := \{x \in \mathbb{R}^{n+1}_k \mid \langle x, x\rangle = -1\}$ if $k \geq 2$, and it is an isometric covering of $S^n_1(-1)$ if $k = 1$.

In the present paper we study pseudo–Riemannian manifolds (M, g) with indefinite metric carrying a non–isometric conformal gradient field $V = \nabla\psi$ with a zero or, equivalently, a non–constant solution ψ with a critical point of Equation (1). In a rough formulation our main results are the following existence and classification results. We use the following completeness assumption: A pseudo–Riemannian manifold with a gradient field $V = \nabla\psi$ is *C–complete* if every point on the manifold can be joined with a critical point of ψ (i.e. a zero of V) by a geodesic and if every geodesic through a critical point of ψ is defined on $\mathbb{R}$.

Theorem A *1.) For any signature $(k, n-k)$ with $1 \leq k \leq n-1$ there exists a smooth pseudo–Riemannian manifold of dimension n carrying a complete conformal gradient field $V = \nabla\psi$ with an arbitrary prescribed number $N \geq 1$ of isolated zeros (including the case of infinitely many zeros in two different ways corresponding to $\mathbb{N}$ or $\mathbb{Z}$). These manifolds are C–complete.*

2.) For $N = \infty$ there exist analytic examples.

3.) For any N there are analytic examples where the vector field is complete but the metric is not C–complete.

4.) For any even number N there exists an analytic and C–complete example carrying a complete conformal field which is closed, i.e. locally a gradient field.

For a precise formulation see Theorems 4.3 and 5.5. The examples in Part 2.) can be described as the complete manifolds carrying a solution ψ of the *pendulum equation* $\nabla^2\psi + \omega^2 \sin\psi = 0$ for some positive constant $\omega > 0$ with at least one critical point, cf. [KR2].

Part 3.) follows from Theorem 5.5 since one can cut out zeros of the complete vector field. For $N = 1, 2$ the pseudo–Euclidean space resp. the pseudo–hyperbolic space $S(-1)$ provide analytic examples. On $S(-1)$ the vector field is not complete. Hence we obtain examples of manifolds carrying complete essential conformal fields with arbitrary number N of zeros. On a Riemannian manifold a complete essential conformal field has only 1 or 2 zeros and this occurs only on the sphere or Euclidean space with the standard conformal structure. This was shown by Alekseevskii [Al1], see also Ferrand [F1], [F2]. The compact case was proved by Obata [Ob] and Lelong–Ferrand [LF], see also [Lf2]. In the indefinite case there are also homothetic essential conformal fields on non–flat spaces, cf. [Al2].

Theorem B *Assume that M^n_k is a C–complete pseudo–Riemannian manifold of signature $(k, n-k)$ with $1 \leq k \leq n-1$ carrying a nontrivial conformal gradient field with at least one zero. Then M^n_k is (locally) conformally flat.*

Theorem B is a consequence of Theorem 6.3.

Theorem C *Let M^n_k be a geodesically complete pseudo–Riemannian manifold of signature (k, n) with $2 \leq k \leq n-2$ carrying a non–trivial conformal gradient field with at least one zero.*

1. *The diffeomorphism type of M^n_k is uniquely determined by the number N of zeros. Here in the case of infinitely many zeros we have to distinguish between $\mathbb{N}$ and $\mathbb{Z}$.*

2. *Every manifold is conformally equivalent to a standard manifold $M(J)(\alpha,\beta)$ defined at the end of section 4.*

3. *If in addition the vector field is complete then the conformal type is uniquely determined by the number N of zeros.*

Theorem C follows from Theorem 6.4 and Theorem 6.5. In the case $k=1$ the disconnectedness of the geodesic distance spheres opens up more possibilities for the global conformal types which can be described by the *gluing graph*, cf. Remark 6.6. In the Riemannian case part 3.) of Theorem C is given in [Bo]. We show in Corollary 6.9 that a complete manifold of constant scalar curvature with a conformal gradient field with a zero has constant sectional curvature. For related results in the Riemannian case see [Lf1].

The paper is organized in the following way: Section 2 presents basic results about closed conformal vector fields in the general context of pseudo–Riemannian geometry. Section 3 gives a discussion of the behaviour of the function resp. the vector field near a critical point resp. near a zero. This relies on a thorough study of geodesic polar coordinates in pseudo–Riemannian manifolds. The Taylor expansion of the function ψ in spacelike and timelike directions leads to a pair ψ_+,ψ_- of real functions satisfying certain compatibility conditions. In Section 4 examples are constructed, based on *building blocks*. Each building block contains exactly one zero of the vector field. Analytic examples with infinitely many critical points will be constructed via *elliptic functions* in Section 5. In Section 6 the local and global conformal types of pseudo–Riemannian manifolds with conformal gradient fields are investigated, and the classification Theorem C is obtained.

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2 Closed conformal vector fields

We consider an n–dimensional connected pseudo–Riemannian manifold (M,g) carrying a *closed conformal non–isometric vector field* V (We also use the symbol $g=\langle .,.\rangle$). Hence there is a smooth (C^∞–) function $\lambda\in C^\infty(M)$ which does not vanish identically, such that

$$\nabla_X V=\lambda X \tag{2}$$

for all vector fields X. Here ∇ denotes the Levi–Civita connection on M. Then one can find for every point $p\in M$ a neighborhood U and a function $\psi\in C^\infty(M)$ such that $V=\nabla\psi$ where $\nabla\psi$ denotes the *gradient* of ψ. It follows that the *Hessian* $\nabla^2{}_{X,Y}\psi:=\langle\nabla_X\nabla\psi,Y\rangle$ satisfies:

$$\nabla^2\psi=\lambda g\,. \tag{3}$$

If $(e_1,\ldots,e_n)$ is an orthonormal basis of the tangential space T_pM, i.e. $\langle e_i,e_j\rangle=\epsilon_i\delta_{ij}$, $\epsilon_i\in\{\pm1\}$ then we obtain for the *Laplacian* $\Delta\psi$ respectively the *divergence* div$V$:

$$\Delta\psi = \mathrm{div}V = \sum_{i=1}^{n} \langle \nabla_{e_i} V, e_i \rangle \epsilon_i = \lambda \cdot n \,. \tag{4}$$

From Equation (2) we obtain immediately the following *Ricci identity* for the *Riemannian curvature tensor* $\mathrm{R}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$:

$$\mathrm{R}(X,Y)V = X(\lambda)Y - Y(\lambda)X \,. \tag{5}$$

By contraction we obtain for the Ricci tensor:

$$\mathrm{Ric}(X,V) = (1-n)X(\lambda) \,. \tag{6}$$

An easy but useful observation is the following

Proposition 2.1 (cf. [Ke2, Prop.1]) *Let V be a closed conformal vector field and let $\gamma : I = [0,a) \longrightarrow M$ be a geodesic on M with $V(\gamma(0)) = k \cdot \gamma'(0)$ for some $k \in \mathbb{R}$. Denote by $\theta_\gamma(t) := \int_0^t \lambda(\gamma(s))ds$, then*

$$V(\gamma(t)) = (k + \theta_\gamma(t))\gamma'(t) \,. \tag{7}$$

Proof. Let $(e_1(t), \ldots, e_n(t))$ be a parallel orthonormal basis field along γ. Then $\gamma'(t) = \sum_{i=1}^n a_i e_i(t)$, $a_i \in \mathbb{R}$ and $V(\gamma(t)) = \sum_{i=1}^n \phi_i(t) e_i(t)$, $\phi_i \in C^\infty(I)$, $\phi_i(0) = ka_i$. Then

$$\frac{\nabla}{dt} V(\gamma(t)) = \sum_{i=1}^{n} \phi_i'(t) e_i(t) = \lambda(\gamma(t))\gamma'(t) \,,$$

hence $\phi_i' = a_i \lambda$ respectively $\phi_i(t) = (k + \theta_\gamma(t))a_i$ $\square$

Remark 2.2 a) If $V(\gamma(0)) = 0$ then $V(\gamma(t)) = \theta_\gamma(t)\gamma'(t)$.

b) If $V = \nabla\psi$, let $\psi(t) := \psi(\gamma(t))$, $\lambda(t) := \lambda(\gamma(t))$, then it follows from $\psi''(t) = {\nabla^2}_{\gamma',\gamma'}\psi$ that

$$\psi''(t) = \lambda(t)\langle \gamma', \gamma' \rangle \,. \tag{8}$$

Hence if $\langle \gamma', \gamma' \rangle \neq 0$, then

$$\nabla\psi(\gamma(t)) = \left(k + \frac{\psi'(t) - \psi'(0)}{\langle \gamma', \gamma' \rangle} \right) \gamma'(t) \,. \tag{9}$$

If in addition $\gamma(0)$ is a critical point of ψ, i.e. $k = 0, \nabla\psi(\gamma(0)) = 0$:

$$\nabla\psi(\gamma(t)) = \psi'(t) \frac{\gamma'(t)}{\langle \gamma', \gamma' \rangle} \,. \tag{10}$$

If γ is a null geodesic then $\psi'(t) = \langle \nabla\psi, \gamma' \rangle(t) = (k + \theta_\gamma(t))\langle \gamma', \gamma' \rangle = 0$, hence $\psi(t) = \psi(0)$ for all t.

c) The last statement is a particular case of the fact that for any conformal vector field V (not necessarily closed) and every null geodesic γ the product $\langle V, \gamma' \rangle$ is constant along the geodesic γ. If $V = \nabla\psi$ then this shows, that

$$\frac{d^2}{dt^2}\psi(\gamma(t)) = \frac{d}{dt}\langle \nabla\psi, \gamma' \rangle = 0$$

hence $\psi(\gamma(t)) = \langle \nabla\psi, \gamma' \rangle(\gamma(0)) \cdot t + \psi(\gamma(0))$.

Proposition 2.3 [Ke2, Prop.2] *Let V be a non–trivial closed conformal vector field on the n–dimensional pseudo–Riemannian manifold (M,g).*

1. *If $V(p) = 0$, then $\mathrm{div}V(p) = n \cdot \lambda(p) \neq 0$, in particular all zeros of V are isolated.*

2. *Denote by $\mathcal{C} = \mathcal{C}(M,g)$ the vector space of closed conformal vector fields, then $\dim \mathcal{C} \leq n+1$.*

Proof. 1.) Let $\gamma : I \longrightarrow M$ be a geodesic with $\gamma(0) = p$ and let $\theta_\gamma(t)$ be the function from Proposition 2.1. It follows from Equation (6) and Proposition 2.1 that the function θ_γ satisfies the following differential equation:

$$\begin{aligned} \theta_\gamma''(t) &= \frac{d}{dt}\lambda(\gamma(t)) = \gamma'(\lambda)(t) = \frac{1}{1-n}\mathrm{Ric}(\gamma',V) = && (11) \\ &= \frac{\theta_\gamma}{1-n}\mathrm{Ric}(\gamma',\gamma')\,. && (12) \end{aligned}$$

Since $V(\gamma(0)) = 0$ we have $\theta_\gamma(0) = 0$. If $\theta_\gamma'(0) = \lambda(p) = 0$ then θ_γ vanishes identically for all geodesics γ starting form p. Hence V vanishes in an open neigborhood of p where then also λ vanishes. Therefore the set

$$A := \{q \in M \mid V(q) = 0\} \cap \{q \in M \mid \lambda(q) = 0\}$$

is an open and closed subset of M. Hence $A = \emptyset$ since we assume V to be non–trivial.

It follows that $\lambda(p) = \theta_\gamma'(0) \neq 0$ for all geodesics starting from the zero p of V. Since $V(\gamma(t)) = \theta_\gamma(t)\gamma'(t)$ by Proposition 2.1 it follows that p is an isolated zero of V.

2.) For every $p \in M$ the linear mapping

$$V \in \mathcal{C}(M,g) \mapsto (V(p), \mathrm{div}V(p)) \in T_pM \oplus \mathbb{R}$$

is injective, since by 1.) it follows from $V(p) = 0$ and $\mathrm{div}V(p) = 0$ that V vanishes identically $\square$

In the sequel we will show that the metric in a neighborhood of a point where V is not null has the form of a *warped product* $I \times_f M_*$, i.e. the metric g on the product $I \times M_*$ has the form

$$g = \eta dt^2 + f^2(t) g_*\,,\ \eta = \pm 1\,.$$

Here (M_*, g_*) is a pseudo–Riemannian manifold and f is a nowhere vanishing C^∞–function on the interval I.

Remark 2.4 a) A consequence is this: For no conformally equivalent metric the vector field V becomes an isometric (Killing) field, i.e. V is an *essential conformal field.* This follows since the value $\mathrm{div}V(p)$ of a conformal vector field at a zero p is a conformal invariant.

b) If the dimension of the space of closed conformal fields is maximal, i.e. $\dim C(M,g) = n+1$, then the manifold has constant sectional curvature. This can be shown as follows: At first it follows from $\dim C(M,g) \geq 2$ that there is a $k \in \mathbb{R}$ such that for all $V \in C(M,g)$ the function $\psi = \mathrm{div} V$ satisfies $\nabla^2\psi = k\psi \cdot g$, cf. [Ke2, Prop.4]. Then one can use arguments as in the proof of [Ke2, Thm.6] resp. of Corollary 6.7 to show that the sectional curvature is constant.

Lemma 2.5 *Let ∂_t be the unit tangent vector in direction of the first factor of the product $I \times M_*$ and let X, Y, Z be lifts of vector fields on M_*. Here I denotes an open interval in $\mathbb{R}$. Denote by $\nabla^*, \mathrm{R}_*, \mathrm{Ric}_*, \rho_*$ the Levi–Civita covariant derivative, the Riemannian curvature tensor and the normalized scalar curvature of (M_*, g_*). (The* normalized scalar curvature *of the standard sphere with sectional curvature* 1 *is also* 1*). Then we have the following formulae for the corresponding geometric quantities $\nabla, \mathrm{R}, \mathrm{Ric}, \rho$ of the warped product metric*

$$(I, \eta dt^2) \times_f (M_*, g_*) = \Big((I \times M_*)\ ,\ \Big(g = \eta dt^2 + f^2(t) g_*\Big)\Big) :$$

1.

$$\begin{aligned} \nabla_{\partial_t}\partial_t &= 0 & (13)\\ \nabla_{\partial_t} X &= \nabla_X \partial_t = \frac{f'}{f} X & (14)\\ \nabla_X Y &= -\frac{g(X,Y)}{f}\eta f' \partial_t + \nabla^*_X Y & (15) \end{aligned}$$

2.

$$\begin{aligned} \mathrm{R}(X,Y)Z &= \mathrm{R}_*(X,Y)Z - \frac{f'^2}{f^2}\eta\{g(Y,Z)X - g(X,Z)Y\} & (16)\\ \mathrm{R}(X,Y)\partial_t &= 0 & (17)\\ \mathrm{R}(X,\partial_t)\partial_t &= -\frac{f''}{f} X & (18) \end{aligned}$$

3.

$$\begin{aligned} \mathrm{Ric}(Y,Z) &= \mathrm{Ric}_*(Y,Z) - \frac{\eta}{f^2}\{(n-2)f'^2 + f'' f\} g(Y,Z) & (19)\\ \mathrm{Ric}(Y,\partial_t) &= 0 & (20)\\ \mathrm{Ric}(\partial_t,\partial_t) &= -(n-1)\frac{f''}{f} & (21) \end{aligned}$$

4.

$$f^2\rho = \frac{n-2}{n}\rho_* - \frac{n-2}{n} f'^2 \eta - \frac{2}{n}\eta f'' f \qquad (22)$$

This follows from the formulae for warped products, cf. [ON, ch. 7](observe that the Riemannian curvature tensor in [ON] has the opposite sign) since

$$\nabla f = f'\eta\partial_t\,,\ \nabla^2{}_{\partial_t,\partial_t} f = g(\nabla_{\partial_t}\nabla f, \partial_t) = f''\,.$$

The formulae in the Riemannian case and the pseudo–Riemannian case coincide if we consider in the case $\eta = -1$ the warped product $\tilde{g} = dt^2 + f^2(t)\tilde{g}_*\,,\ \tilde{g}_* = -g_*$ which is anti–isometric to g (then $\tilde{\rho} = -\rho, \tilde{\rho}_* = -\rho_*, \ldots$). In particular we obtain as in the Riemannian case the

Corollary 2.6 *The warped product* $(I,\eta dt^2)\times_f (M_*,g_*)$ *is an Einstein metric (a metric of constant sectional curvature) if and only if* g_* *is an Einstein metric (a metric of constant sectional curvature) and* $f'^2+\rho\eta f^2=\eta\rho_*$.

Lemma 2.7 [Fi] [Kue1, Lemma 12] *Let* (M,g) *be a pseudo–Riemannian manifold. Then the following conditions are equivalent:*

1. *There is a non–constant solution* ψ *of* $\nabla^2\psi=\frac{\Delta\psi}{n}g$ *in a neighborhood of a point* $p\in M$ *with* $\langle\nabla\psi(p),\nabla\psi(p)\rangle\neq 0$.
2. *There is a neighborhood* U *of* p *, a* C^∞*–function* $\psi:(-\epsilon,\epsilon)\to\mathbb{R}$ *with* $\psi'(t)\neq 0$ *for all* $t\in(-\epsilon,\epsilon)$ *and a pseudo–Riemannian manifold* (M_*,g_*) *such that* (U,g) *is isometric to the warped product*
$$\Big((-\epsilon,\epsilon)\ ,\ \eta dt^2\Big)\times_{\psi'}(M_*,g_*))=\Big((-\epsilon,\epsilon)\times M_*\ ,\ \eta dt^2+\psi'(t)^2g_*\Big) \tag{23}$$
where $\eta:=\operatorname{sign}\langle\nabla\psi(p),\nabla\psi(p)\rangle\in\{\pm 1\}$.

Proof. 2.) $\Rightarrow$ 1.): Define the function $\psi:(-\epsilon,\epsilon)\times M_*\to\mathbb{R}$, $\psi(t,x)=\psi(t)$. Then $\nabla\psi(t,x)=\psi'(t)\cdot\eta\cdot\partial_t$ and $\nabla_{\partial_t}\nabla\psi=\psi''(t)\cdot\eta\cdot\partial_t$ by Equation (13). Let X be a lift of a vector field on M_*, then by Equation (14):
$$\nabla_X\nabla\psi=\psi''\cdot\eta\cdot X.$$

1.) $\Rightarrow$ 2.) Let U be a neigborhood of $p\in M$ with compact closure and with $\langle\nabla\psi(q),\nabla\psi(q)\rangle\neq 0$ for all $q\in U$. Hence $c=\psi(p)$ is a regular value, let M_* be the connected component of $\psi^{-1}(c)$ containing p. Then there is an $\epsilon>0$ such that the normal exponential map
$$\exp^\perp:(-\epsilon,\epsilon)\times M_*\to M,(t,x)\mapsto\exp(t\nabla\psi(x))$$
defines a diffeomorphism onto the image. Let $q\in U$, $g(X,\nabla\psi(q))=0$, then it follows immediately that
$$Xg(\nabla\psi,\nabla\psi)=2\frac{\Delta\psi}{n}g(\nabla\psi,X)=0\,. \tag{24}$$
Hence $\langle\nabla\psi,\nabla\psi\rangle$ is constant along the level hypersurfaces $\psi^{-1}(c')$ and the level hypersurfaces $\psi^{-1}(\psi(\exp(t,x_0)))$, $t\in(-\epsilon,\epsilon)$ are parallel. Therefore they coincide with the t–levels and ψ can be regarded as a function of t alone: $\psi(t,x)=\psi(t)$ and $\nabla\psi(t,x)=\psi'(t)\cdot\eta\partial_t$ as well as
$$\nabla^2\psi=2\psi''\eta g=\frac{\Delta\psi}{n}g\,.$$
$g(\partial_t,\partial_t)=\eta=\operatorname{sign}\langle\nabla\psi(p),\nabla\psi(p)\rangle$ follows since $t\mapsto\exp(t\nabla\psi(x))$ is a geodesic. Let X be a lift of a vector field on M_*, then $g(\partial_t,X)=0$ by the Gauß–Lemma. If X_1,X_2 are vectors tangential to M_* at x_0 and
$X_i(t)=d\exp(t,x_0)(X_i), i=1,2$ then
$$\begin{aligned}\frac{d}{dt}|_{t=s}g(X_1,X_2)(t) &= \mathcal{L}_{\partial_t}g(X_1,X_2)(s)=\frac{\eta}{\psi'(s)}\mathcal{L}_{\nabla\psi}g(X_1,X_2)(s)=\\ \frac{2\eta}{\psi'(s)}\nabla^2_{X_1(s),X_2(s)}\psi &= 2\frac{\psi''(s)}{\psi'(s)}g(X_1,X_2)(s)\,.\end{aligned}$$

Here $\mathcal{L}_Z g(X,Y) = g(\nabla_X Z, Y) + g(X, \nabla_Y Z)$ is the *Lie derivative* of the metric in direction of the vector field Z. Hence $t \mapsto (\psi'(t))^{-2} g(X_1, X_2)(t)$ satisfies the differential equation $((\psi')^{-2} g(X_1, X_2))'(t) = 0$. Hence if $g_*(X_1, X_2) = (\psi')^{-2}(0) g(X_1, X_2)$ the claim follows. The metric g_* is non–degenerate: If $g_*(X,Y) = 0$ for some X and all Y tangent to M_* then also $g(X, \partial_t) = 0$. Since g is non–degenerate it follows that $X = 0$ $\square$

Remark 2.8 If $\nabla\psi$ is a null vector on an open set of points in M and if $\nabla^2\psi$ is a multiple of the metric then it easily follows that $\nabla^2\psi = 0$, i.e. $\nabla\psi$ is parallel. Therefore this case does not have to be discussed if we assume that $\nabla\psi$ has at least one zero. In general relativity these metrics are called *plane gravitational waves*, see [Half] and [HE].

3 Geodesic polar coordinates on pseudo–Riemannian manifolds

We denote by $\mathbb{R}^n_k = (\mathbb{R}^n, \langle .,. \rangle)$ the pseudo–Euclidean space of signature $(k, n-k)$, i.e. $\langle x, x \rangle = -(x_1^2 + \cdots + x_k^2) + x_{k+1}^2 + \cdots + x_n^2$. For $n \geq 2, \eta \in \{\pm 1\}$ let $S(\eta) := \{x \in \mathbb{R}^n_k \,|\, \langle x, x \rangle = \eta\}$ and we denote by $|x| := \sqrt{|\langle x, x \rangle|} \geq 0$ the *pseudo–norm*.

Then $S(1)$ is the *pseudo–sphere* which is with the induced metric a pseudo–Riemannian manifold of signature $(k, n-1-k)$ and of constant sectional curvature $K \equiv 1$. $S(1)$ is diffeomorphic to $\mathbb{R}^k \times S^{n-1-k}$, we denote by $S^0(1)$ the connected component of $S(1)$ containing the point $(0, 0, \ldots, 0, 1)$. The pseudo–sphere $S(1)$ is connected for $1 \leq k < n-1$, otherwise it consists of two components.

$S(-1)$ is the *pseudo–hyperbolic space* which is with the induced metric a pseudo–Riemannian manifold of signature $(k-1, n-k)$ and of constant sectional curvature $K \equiv -1$. It is diffeomorphic to $S^{k-1} \times \mathbb{R}^{n-k}$, cf. [ON, p.110]. We denote by $S^0(-1)$ the connected component of $S(-1)$ containing the point $(1, 0, \ldots, 0)$. $S(-1)$ is connected if $k \geq 2$, for $k = 1$ it consists of two components. In the Lorentzian case $k = 1, n = 4$ the pseudo–sphere is also called *de Sitter space–time*, the pseudo–hyperbolic space is called *anti–de Sitter space–time*. A region of de Sitter space served in general relativity as a model in the *steady state theory*, which was proposed in 1948 by Bondi–Gold and by Pirani, cf. [HE, ch. 5.2].

Let $\Sigma := S^0(1) \cup S^0(-1)$ and let $C := \{x \in \mathbb{R}^n_k \,|\, \langle x, x \rangle = 0\}$ be the *light cone*. Then we to introduce *polar coordinates* on $\mathbb{R}^n_k - C$. We construct as in the Riemannian case a map

$$y \in \mathbb{R}^n_k - C \mapsto \Phi(y) = (r(y), \phi(y)) \in \mathbb{R} \times \Sigma$$

where $r(y)$ is the radial part, i.e. the absolute value of $r(y)$ equals the pseudo–norm $|y|$. Here the image $G := \Phi(\mathbb{R}^n_k - C) \subset \mathbb{R} \times \Sigma$ of the polar coordinates depends on the signature k and the dimension n, resp. it depends on the number q of components of $S(1) \cup S(-1)$.

If $2 \leq k \leq n-2$ then $q = 2$ and $\Phi(y) = (\operatorname{sgn}(\langle y, y \rangle)|y|, |y|^{-1} y)$, i.e. $G = (\mathbb{R}^+ \times S(1)) \cup (\mathbb{R}^- \times S(-1))$. If $k = 1, n \geq 3$ then $q = 3$ and we set for $\langle y, y \rangle < 0 : \Phi(y) = (\eta|y|, \eta|y|^{-1} y) \in \mathbb{R} \times S^0(-1)$ where $\eta \in \{\pm 1\}$ and for $\langle y, y \rangle > 0 : \Phi(y) = (|y|, |y|^{-1} y)$, i.e. $G = \mathbb{R} \times S^0(-1) \cup \mathbb{R}^+ \times S(1)$. If $k = n-1, n \geq 3$ then $q = 3$

and we set for $\langle y,y\rangle > 0 : \Phi(y) = (\eta|y|, \eta|y|^{-1}y) \in \mathbb{R} \times S^0(1)$ where $\eta \in \{\pm 1\}$ and for $\langle y,y\rangle < 0 : \Phi(y) = (|y|, |y|^{-1}y)$, i.e. $G = \mathbb{R} \times S^0(1) \cup \mathbb{R}^+ \times S(-1)$. Finally let $k = 1, n = 2$. Then $q = 4$ and $\Phi(y) = (\eta|y|, \eta|y|^{-1}y) \in \mathbb{R} \times \Sigma$, i.e. $G = \mathbb{R} \times \Sigma$.

Then it is clear how to define *geodesic polar coordinates* around any point $p \in M$ of an arbitrary pseudo–Riemannian manifold via the exponential map. Let $C_p := \{X \in T_pM \mid \langle X,X\rangle = 0\}$ be the *light cone* at p. There is an open neigborhood $\tilde{U}$ of the zero vector in $T_pM \cong \mathbb{R}^n_k$ such that

$$\phi : \Phi(\tilde{U} - C_p) \subset G \longrightarrow U \subset M$$

$\phi(r,x) = \exp_p(\Phi^{-1}(r,x))$ defines geodesic polar coordinates.

In these coordinates we consider warped product metrics of the form $\eta dr^2 + f_\eta(r)^2 \cdot g_1(x)$, $\eta \in \{\pm 1\}, (r,x) \in G \subset \mathbb{R} \times \Sigma$, where g_1 is the standard metric on Σ. This implies that these metrics have around 0 an $\mathbb{O}(k, n-k)$–symmetry, here $\mathbb{O}(k,n-k) := \{A \in \mathrm{End}(\mathbb{R}^n_k) \mid \langle A(y), A(y)\rangle = \langle y,y\rangle\}$ is the *orthogonal group* of $\mathbb{R}^n_k$. First we study which pairs of functions $f_\pm$ define a smooth metric in a neighborhood of the origin, i.e. we give sufficient and necessary conditions for the functions $\psi_\pm$, such that the metric also extends onto the light cone on which $r = 0$:

Definition 3.1 1.) We define the following set $\mathcal{F}$ of two C^∞–functions $f = (f_+, f_-) : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions: $f_\pm^{(2m+1)}(0) = 0$, $f_+^{(2m)}(0) = (-1)^m f_-^{(2m)}(0)$ for all $m \geq 0$ and $f_+''(0) = -f_-''(0) \neq 0$.

2.) We define the set $A_f \subset \mathbb{R}^n_k - C$ in geodesic polar coordinates $(r,x) \in G \subset \mathbb{R} \times \Sigma$ as follows: $(r,x) \in A_f$ if and only if f'_η , $\eta = \langle x,x\rangle$ does not vanish between 0 and r.

Remark 3.2 Let $f_\pm : \mathbb{R} \to \mathbb{R}$ be two smooth (i.e. C^∞–) resp. analytic functions. Then the following conditions are equivalent:

1. $f_\pm^{(2m+1)}(0) = 0$, $f_+^{(2m)}(0) = (-1)^m f_-^{(2m)}(0)$ for all $m \geq 0$.
2. The function

$$F(t) := \begin{cases} f_+(\sqrt{t}) & ; \quad t \geq 0 \\ f_-(\sqrt{-t}) & ; \quad t \leq 0 \end{cases}$$

is smooth resp. analytic in a neighborhood of 0.

Remark 3.3 Let $f_\pm$ be two real analytic functions defined on the real line whose Taylor expansions around 0 are given by

$$f_\pm(x) = \sum_{j=0}^{\infty} (\pm 1)^j \frac{a_{2j}}{(2j)!} x^{2j}$$

with $a_2 \neq 0, a_{2j} \in \mathbb{R}$ for all $j \geq 2$. Hence $f_\pm \in \mathcal{F}$. Then one can form a holomorphic function $F : U \to \mathbb{C}$ whose expansion around 0 is given by

$$F(z) = \sum_{j=0}^{\infty} \frac{a_{2j}}{(2j)!} z^{2j} .$$

F is defined on an open neigborhood U of the union $\mathbb{R} \cup i\,\mathbb{R}$ of the real and the imaginary axis. Given such holomorphic function we can recover $f_\pm$ by $f_+(x) = F(x)$ and $f_-(x) = F(ix)$ for all $x \in \mathbb{R}, i = \sqrt{-1}$. On the other hand given a holomorphic function $F : U \to \mathbb{C}$ which satisfies $F(z) = F(-z)$ and $F''(0) \neq 0$ we can form $f_+(x) = \mathrm{Re}F(x)$, $f_-(x) = \mathrm{Re}F(ix)$ for all $x \in \mathbb{R}$ and obtain $f_\pm \in \mathcal{F}$.

Lemma 3.4 *Let a smooth pseudo–Riemannian metric g be given in geodesic polar coordinates $(r, x) \in G \subset \mathbb{R} \times \Sigma$ by*

$$g(r, x) = \eta dr^2 + \frac{f_\eta'(r)^2}{f_\eta''(0)^2}\, g_*$$

with a C^∞–metric g_* on Σ. Then g_* coincides with the standard metric g_1 on Σ of constant sectional curvature η.

Proof. Let σ be a plane spanned by the orthonormal vectors X, Y with $\epsilon = \langle X, X\rangle\langle Y, Y\rangle = \pm 1$ which are both orthogonal at $r = r_0$ to the radial geodesic $r \mapsto (r, x_0)$ for a fixed $x_0 \in \Sigma$. Let $K(\sigma)$ resp. $K_*(\sigma)$ be the sectional curvature of σ in (M, g) resp. in (Σ, g_*). Then it follows from the formulae (16) for the curvature of a warped product (see Lemma 2.5):

$$\begin{aligned} K(\sigma) &= \epsilon\, g(R(X,Y)Y,X) = \epsilon\, g(R_*(X,Y)Y,X) - \eta \left(\frac{f_\eta''(r_0)}{f_\eta'(r_0)}\right)^2 && (25)\\ &= \frac{\epsilon}{f_\eta'(r_0)^2}\left(K_*(\sigma) f_\eta''(0)^2 - \eta\, f_\eta''(r_0)^2\right) . && (26)\end{aligned}$$

Since $K_*(\sigma)$ is independent of r we obtain for $r_0 \to 0$ that $K_*(\sigma) = \eta$. Hence (Σ, g_*) has constant sectional curvature η, i.e. g_* is isometric to g_1 □

Proposition 3.5 *Let $\psi_\pm$ be two smooth real functions with $\psi_+''(0) = -\psi_-''(0) \neq 0$. Then we define the functions $\psi(r, x) = \psi_\eta(r)$, $\lambda(r, x) = \lambda_\eta(r)$, $\lambda_\eta = \eta\psi_\eta''$ on the complement $\mathbb{R}^n_k - C$ of the light cone C in the pseudo–Euclidean space. Here $(r, x) \in G \in \mathbb{R} \times \Sigma$ are geodesic polar coordinates of the pseudo–Euclidean space. We also define the metric*

$$g(r, x) := g_\psi(r, x) = \eta dr^2 + \frac{\psi_\eta'(r)^2}{\psi_\eta''(0)^2}\, g_1 \qquad (27)$$

where $\eta = \langle x, x\rangle \in \{\pm 1\}$ on the subset A_ψ, on which ψ_η' does not vanish, see Definition 3.1 2.). Then the following holds:

1. *The functions ψ, λ extend smoothly onto $B_\psi := A_\psi \cup C \subset \mathbb{R}^n_k$, i.e. onto the light cone, if and only if $\psi_\pm \in \mathcal{F}$, i.e.*

$$\psi_\pm^{(2m+1)}(0) = 0\,,\ \psi_+^{(2m)}(0) = (-1)^m \psi_-^{(2m)}(0) \qquad (28)$$

   for all $m \geq 0$.

2. *The metric g_ψ extends smoothly onto the light cone if and only if $\psi_\pm \in \mathcal{F}$ and g_ψ is conformally flat.*

3. If $\psi_\pm \in \mathcal{F}$ then the function $\psi : B_\psi \to \mathbb{R}$ has a critical point in 0 and satisfies

$$\nabla^2 \psi = \lambda\, g$$

for some function λ.

Proof. 1.) Let $x \in \mathbb{R}^n_k$ be the standard coordinates of $\mathbb{R}^n_k$. Then let $\gamma(t) := x_0 + y_0 t$ with null vectors $x_0, y_0 \neq 0$ satisfying $\langle x_0, y_0 \rangle = 1/2$, i.e. γ is a null geodesic intersecting the light cone at time $t = 0$ in x_0 and $\langle \gamma(t), \gamma(t) \rangle = t$. Hence

$$\psi(\gamma(t)) = \begin{cases} \psi_+(\sqrt{t}) & ; \ t > 0 \\ \psi_-(\sqrt{-t}) & ; \ t < 0 \\ \psi_+(0) = \psi_-(0) & ; \ t = 0 \end{cases} .$$

By Remark 3.2 $\psi \circ \gamma$ is smooth if and only if Equations (28) hold. If $\gamma(t) = x_0 + y_0 t$ for a null vector x_0 and $\langle x_0, y_0 \rangle = 1/2$ is an arbitrary geodesic intersecting the light cone at time $t = 0$ then $\langle \gamma(t), \gamma(t) \rangle = t + \langle y_0, y_0 \rangle t^2$. Then it follows that

$$\psi(\gamma(t)) = \begin{cases} \psi_+(\sqrt{t + \langle y_0, y_0 \rangle t^2}) & ; \ t > 0 \\ \psi_-(\sqrt{-t - \langle y_0, y_0 \rangle t^2}) & ; \ t < 0 \\ \psi_+(0) = \psi_-(0) & ; \ t = 0 \end{cases}$$

is smooth since $\psi_\pm$ satisfy equations (28).

2.) For the following computation we assume for simplicity that $2 \leq k \leq n - 2$. In case of signature $k = 1$ resp. $k = n - 1$ similar arguments work. Then we have $(r, x) \in (\mathbb{R}^+ \times S(1)) \cup (\mathbb{R}^- \times S(-1))$ as range of the geodesic polar coordinates. Instead of the functions $\psi_\pm$ we form a single function $\psi_* : \mathbb{R} \to \mathbb{R}$ with $\psi_*(t) := \psi_+(t), t > 0$ and $\psi_*(t) := \psi_-(-t), t < 0$.

The function ψ_*' has an expansion around 0 of the form

$$\psi_*'(r) = (\mathrm{sgn} r)\, \psi_*''(0+)\, r + \mathcal{O}(r^2)\,,$$

therefore the function $h : (r_-, r_+) \to \mathbb{R}$,

$$h(r) := (\mathrm{sgn} r) \frac{\psi_*''(0+)}{\psi_*'(r)} - \frac{1}{r} \tag{29}$$

is continuous in 0. Then one verifies that

$$\rho = \rho(r) = \frac{\rho_0}{r_0}\, r\, \exp\left\{ \int_{r_0}^{r} h(\xi) d\xi \right\} \tag{30}$$

satisfies

$$g_\psi(r, x) = \eta dr^2 + \frac{\psi_*'(r)^2}{\psi_*''(0)^2}\, g_1(x) = F^2(r) \left\{ (\mathrm{sgn}\rho) d\rho^2 + \rho^2\, g_1(x) \right\} . \tag{31}$$

Here the conformal factor is given by

$$F = F(r) = \frac{r_0}{\rho_0} \frac{\psi_*'(r)}{\psi_*''(0)} \frac{1}{r} \exp\left\{ - \int_{r_0}^{r} h(\xi) d\xi \right\} . \tag{32}$$

Hence the transformation $(\rho, x) = (\rho(r), x)$, $\text{sgn}\langle x, x\rangle = \text{sgn}r$ is a conformal transformation. Let $F_\pm(r) := F(\pm r)$, $r \geq 0$, then one verifies that $F_\pm \in \mathcal{F}$ if $\psi_\pm \in \mathcal{F}$. Hence g_ψ extends onto the light cone since g_0 does and since $F(0) > 0$.

3.) It follows from Lemma 2.7 that outside the light cone $\nabla^2\psi = \lambda g$ with $\lambda(r, x) = \psi''_{\langle x,x\rangle}(r)\langle x, x\rangle$. Since $\psi_\pm \in \mathcal{F}$ the function λ is smooth on $A_\psi \cup C$. □

Proposition 3.6 *Let (k, n) be the signature with $2 \leq k \leq n-2$. On the building block $B(a, b) = \{y \in \mathbb{R}^n_k \mid -a^2 < \langle y, y\rangle < b^2\}$ where $a, b \in \mathbb{R} \cup \{\infty\}$ we consider the metric g_ψ as defined in Proposition 3.5. for a continuous function ψ_* on $\mathbb{R}$ which is smooth outside 0 and the pseudo–Euclidean metric g_0 and we let $\psi_\pm(t) = \psi_*(\pm t)$. We denote by r_+, r_- the first positive, resp. negative zero of ψ'_*, and we allow $r_+, -r_- = \infty$. The functions $\rho = \rho(r)$, $F = F(r)$ for an arbitrary $r_0 \in (0, r_+)$ and $\rho_0 > 0$ are defined as in Equations (30) and (32). We set $\rho_\pm := \rho(r_\pm)$, hence $\pm\rho_\pm \in \mathbb{R}^+ \cup \{\infty\}$.*

If $r_+ < \infty$ resp. $r_- > -\infty$ then $\rho_+ = \infty$ resp. $\rho_- = -\infty$. Then $(B(r_-, r_+), g_\psi)$ and $(B(\rho_-, \rho_+), g_0)$ are conformally equivalent with conformal factor F, i.e.

$$g_\psi(r, x) = \eta dr^2 + \frac{\psi'_*(r)^2}{\psi''_*(0)^2} g_1(x) = F^2(r)\left\{(\text{sgn}\rho)d\rho^2 + \rho^2 g_1(x)\right\} \qquad (33)$$

with $(\rho, x) = (\rho(r), x)$, $\text{sgn}\langle x, x\rangle = \text{sgn}r$.

Proof. It follows from Part 2.) of the proof of Proposition 3.5 that $(B(r_-, r_+), g_\psi)$ and $(B(\rho_-, \rho_+), g_0)$ are conformally equivalent with conformal factor F. It remains to prove that ρ_+ resp. ρ_- is infinite if r_+ resp. r_- is finite. Now we assume that $r_+ < \infty$. We use around the critical point q (which corresponds to $r = r_+$) geodesic polar coordinates $(\tilde{r}, \tilde{x}) \in \tilde{G}$. Then it follows that $\tilde{r} = r - r_+$ for $r \in (0, r_+)$ and $\tilde{x} = x$. The metric g_ψ around $\tilde{p}$ for $\tilde{r} < 0$ is of the form

$$d\tilde{r}^2 + \frac{\psi'_*(\tilde{r} + r_+)^2}{\psi''_*(r_+)^2} g_1(x)\,.$$

It follows from Lemma 3.4 that

$$\psi''_*(r_+) = -\psi''_*(0)\,.$$

Hence the function $l : (0, r_+) \to \mathbb{R}$

$$l(r) = \frac{\psi''_*(0)}{\psi'_*(r)} + \frac{1}{r - r_+} = h(r) + \frac{1}{r - r_+} + \frac{1}{r}$$

is continuous in r_+ and we obtain from Equation (30):

$$\rho(r) = \rho_0 \frac{r_0 - r_+}{r - r_+} \exp\left\{\int_{r_0}^{r} l(\xi)d\xi\right\}\,.$$

Hence

$$\lim_{\substack{r \to r_+ \\ r < r_+}} \rho(r) = \infty\,.$$

The same argument shows that $\rho_- = -\infty$ provided $r_- > -\infty$. □

Remark 3.7 Since the conformal field $\nabla\psi$ under the conformal transformation of Proposition 3.5 resp. Proposition 3.6 is mapped onto the radial field on $B(\rho_-,\rho_+)$ for every signature $1 \le k \le n-1$ we obtain that $\nabla\psi$ is complete if ρ_- and ρ_+ are infinite.

Remark 3.8 In the case $2 \le k \le n-2$ we can also write the metric g_ψ as follows: Instead of the coordinate r we choose $s = \langle y, y\rangle$, $y \in \mathbb{R}^n_k$, i.e. $s = \eta r^2$ resp. $ds/dr = 2\eta r = 2\eta\sqrt{\eta s}$. The functions $\psi_\pm$ satisfy Equation (28) if and only if $\phi(s) := \eta\psi''_\eta(0)^{-1}\psi_\eta(\sqrt{\eta s})$ is smooth, cf. Lemma 3.2. Then we can write

$$g_\psi(s,x) = \frac{ds^2}{4s} + 4|s|\phi'(s)^2 g_1 \,. \tag{34}$$

The pseudo–Euclidean metric in this coordinates is given by

$$\frac{ds^2}{4s} + |s| g_1 \,. \tag{35}$$

We also have $\phi'(0) = 1/2$. Then Proposition 3.5 shows that Equation (34) defines a metric which extends onto the light cone if and only if ϕ is smooth and $\phi'(0) = 1/2$.

Example 3.9 1.) We consider the *pseudo–Euclidean space* $\mathbb{R}^n_k$. Its metric in geodesic polar coordinates (r,x) around 0 is of the form:

$$\begin{array}{rcl} -dr^2 + r^2\, g_1(x) & , & \langle x, x\rangle < 0 \\ dr^2 + r^2\, g_1(x) & , & \langle x, x\rangle > 0 \,. \end{array}$$

Then the function

$$\begin{array}{rclcl} \psi(r,x) & = & r^2 & , & \langle x, x\rangle < 0 \\ \psi(r,x) & = & -r^2 & , & \langle x, x\rangle > 0 \end{array}$$

satisfies the equation $\nabla^2\psi = 2\, g$.

2.) We consider the *pseudo–hyperbolic space* $S(-1) = \{x \in \mathbb{R}^{n+1}_{k+1} \mid \langle x, x\rangle = -1\}$ of signature $(k, n-k)$ with its canonical embedding in the pseudo–Euclidean space $\mathbb{R}^{n+1}_{k+1}$. We can describe the conformal gradient fields as follows. Fix a vector $T \in \mathbb{R}^{n+1}_{k+1}$, then $\psi_T : S(-1) \to \mathbb{R}$, $\psi_T(p) = \langle T, p\rangle$ is the *height function* in direction T. Then one can easily show that

$$\nabla_X \psi_T = \psi_T X \tag{36}$$

for every vector field X on $S(-1)$. Hence $\nabla\psi_T$ is a conformal vector field, resp. ψ_T satisfies

$$\nabla^2\psi_T = \psi_T\, g \,, \tag{37}$$

cf. [Ker2, §2] or [Ke2, §3]. Proposition 2.3 b) shows that every closed conformal vector field can be written this way. Now assume that $n > 2$. Then $S(-1)$ is connected. The critical points of ψ_T are the intersection points of $S(-1)$ as subset of $\mathbb{R}^{n+1}_{k+1}$ with the line through 0 whose direction is T. Hence if T is timelike there are two critical points $p, -p$. It follows that in geodesic polar coordinates $(r, x) \in G$ we have around p resp. $-p$ the following expression for the metric:

$$\begin{array}{rcl} -dr^2 + \sin^2 r\, g_1(x) & , & \langle x, x\rangle < 0 \\ dr^2 + \sinh^2 r\, g_1(x) & , & \langle x, x\rangle > 0\,. \end{array}$$

The function ψ_T has the form

$$\begin{array}{rclcl} \psi_T(r, x) & = & \cos r & , & \langle x, x\rangle < 0 \\ \psi_T(r, x) & = & \cosh r & , & \langle x, x\rangle > 0 \end{array}$$

in geodesic polar coordinates around p and around $-p$:

$$\begin{array}{rclcl} \psi_T(r, x) & = & -\cos r & , & \langle x, x\rangle < 0 \\ \psi_T(r, x) & = & -\cosh r & , & \langle x, x\rangle > 0. \end{array}$$

The geodesics on $S(-1)$ are the intersections of planes in $\mathbb{R}^{n+1}_{k+1}$ through 0 with $S(-1)$. Hence all timelike geodesics emanating from p meet after time $r = \pi$ in the point $-p$ and close after time $r = 2\pi$. The spacelike and null geodesics from p resp. $-p$ go to infinity, more precisely the height ψ_T along these geodesics goes to $\pm\infty$. It also follows that the points on the null resp. spacelike geodesics starting from p do not have a geodesic connection to the point $-p$. This is an example of a pseudo–Riemannian manifold where all geodesics are defined on $\mathbb{R}$ i.e. the manifold is *geodesically complete* but the exponential map $\exp_p : T_pM \to M$ of the point p is *not* surjective. On the other hand every point on $S(-1)$ has a geodesic connection either to p or to $-p$, i.e. one says that $S(-1)$ is $\{p, -p\}$*–complete*, see Definition 4.2.

4 Existence results

We construct pseudo–Riemannian manifolds $M(J)$ with $J = \{1, \ldots, m\}$ for any $m \in \mathbb{N}$ or $J = \mathbb{N}$ or $J = \mathbb{Z}$ carrying two non–constant functions ψ, λ satisfying $\nabla^2\psi = \lambda g$ with the following property:

$\mathrm{Cr}(\psi) = \{p_j \in M(J) | j \in J\}$ is the set of critical points of ψ with $\psi(p_j) < \psi(p_{j+1})$ for all $j, j+1 \in J$. We use the following *building blocks*. For $a, b > 0$ let $B^+(a) := \{x \in \mathbb{R}^n_k | \langle x, x\rangle \leq a^2\}$, $B^-(b) := \{x \in \mathbb{R}^n_k | -b^2 \leq \langle x, x\rangle\}$ and $B(a, b) := B^+(a) \cap B^-(b) = \{x \in \mathbb{R}^n_k | -b^2 \leq \langle x, x\rangle \leq a^2\}$.

Lemma 4.1 *There is a smooth pseudo–Riemannian metric g of signature $(k, n-k)$ on every building block $B(a, b), B^+(a), B^-(b)$ with two non-constant functions ψ, λ satisfying $\nabla^2\psi = \lambda g$ such that 0 is the only critical point of ψ and $\lambda(0) = 1$. The metric g in a collar neighborhood of the boundary is the product metric on $[0, \epsilon] \times \partial B(a, b)$. resp. on $[0, \epsilon] \times \partial B^+(a)$ or on $[0, \epsilon] \times \partial B^-(b)$. In case $B^+(a)$ resp. $B^-(b)$ the metric is a product metric outside $\langle x, x\rangle > -1$ resp. $\langle x, x\rangle < 1$.*

Proof. We give the proof for the building block $B(a, b)$, the constructions for $B^\pm$ are analogous. On $B(a, b)$ we have polar coordinates $(r, x) \in G \cap ((-b, a) \times \Sigma)$ since $B(a, b)$ is a subset of $\mathbb{R}^n_k$. G is the range of polar coordinates which depends on the signature, cf. Section 3. Let $f_\pm : \mathbb{R} \to \mathbb{R}$ be smooth even functions with $f_- = -f_+$ satisfying the following assumptions, where $\epsilon > 0$ satisfies $10\epsilon < \min\{a, b\}$.

1. $f_+(t) = \frac{1}{2}t^2$ for $|x| < \epsilon$
2. 0 is the only critical value of $f_+|[-b, a]$
3. $f_+'(t) = 1$ for $t \in [a - \epsilon, a] \cup [-b, -b + \epsilon]$.

(In the case $a = \infty$ resp. $b = \infty$ condition 3.) should be replaced by $f_+'(t) = t$ for $t \geq 1$). Then $f_\pm \in \mathcal{F}$. We choose the metric $\langle x, x\rangle dr^2 + f_{\langle x,x\rangle}'(r)^2\, g_1(x) = \langle x, x\rangle dr^2 + f_+'(r)^2\, g_1(x)$ on $B(a, b)$, where $\eta = \langle x, x\rangle \in \{\pm 1\}$. It follows from Proposition 3.5 that the functions ψ, λ on $B(a, b)$ with $\psi(r, x) = \langle x, x\rangle f_+(r)$ and $\lambda(r, x) = f_+''(r)$ satisfy $\nabla^2\psi = \lambda\, g$. By Property 2.) 0 is the only critical point of ψ and $\lambda(0) = 1$ by Property 1.). Property 3.) implies that the metric is a product metric in a collar neighborhood of the boundary. □

Definition 4.2 Let A be a subset of a pseudo–Riemannian manifold M. Then M is said to be *A–complete* if the following two conditions hold:

1. Every geodesic through A is defined on $\mathbb{R}$.
2. One can join every point on M with A by a geodesic.

In the Riemannian case it follows from the theorem of Hopf and Rinow that a metrically complete manifold M is geodesically complete, then in particular M is $\{p\}$–complete for every point $p \in M$. In the pseudo–Riemannian case this does not hold, as explained in Example 3.9 the pseudo–hyperbolic space $S(-1)$ is geodesically complete, but there are points p such that $S(-1)$ is not $\{p\}$–complete. On $\mathbb{R}^n_k$ one defines the *separation* $d(p, q) = |p - q| \geq 0$ as the pseudo–norm of the difference $p - q$, cf. [ON, ch.6]. Then we set for two points p, q on a pseudo–Riemannian manifold the *separation* $d(p, q)$ as the infimum of the absolute values of lengths of curves joining them.

Theorem 4.3 *Let $J = \{1, \ldots, m\}$ or $J = \mathbb{N}$ or $J = \mathbb{Z}$. Fix a set $D := \{d_j \,|\, j, j+1 \in J\} \subset \mathbb{R}^+$. Then there exists a smooth pseudo–Riemannian manifold $(M(J), g_D)$ of signature $(k, n-k)$ carrying smooth non–constant functions ψ, λ satisfying $\nabla^2\psi = \lambda\, g$ and such that $2d_j = d(p_j, p_{j+1})$ for all $j, j+1 \in J$. Here d denotes the separation induced by g_D. The set $\mathrm{Cr}(\psi) = \{p_j \,|\, j \in J\}$ of critical points of ψ is in natural bijection with J meaning that $\psi(p_j) < \psi(p_{j+1})$ for all $j, j+1 \in J$ and the manifold is $\mathrm{Cr}(\psi)$–complete (or C–complete for short) and the vector field $V = \nabla\psi$ is complete.*

Remark 4.4 In some sense the critical points together with their labeling form a linear graph.
$J = \{1, \ldots, m\}$ corresponds to •———•——— ⋯ ———•.
$J = \mathbb{N}$ corresponds to •———•——— ⋯ .
and $J = \mathbb{Z}$ corresponds to ⋯ ———•——— •——— ⋯ .

Proof. First we construct the underlying manifolds $M(J)$. If $J = \{1\}$ we take $M(J) = \mathbb{R}^n_k$. If $J = \{1, \ldots, m\}$, $m = 2l$, $l \geq 1$ we take

$$M(J) = B_1^+(d_1) \cup B_2(d_1, d_2) \cup \ldots \cup B_{m-1}(d_{m-2}, d_{m-1}) \cup B_m^+(d_{m-1})$$

Each $B_j^+(a)$ resp. $B_j(a,b)$ is a copy of $B^+(a)$ resp. $B(a,b)$ and here the gluing is as follows: The boundary $\partial B_1^+ = \{x \in \mathbb{R}_k^n \,|\, \langle x,x\rangle = d_1\}$ is identified with the component $\{x \in \mathbb{R}_k^n \,|\, \langle x,x\rangle = d_1\} \subset \partial B_2(d_1,d_2)$. Then the components $\{x \in \mathbb{R}_k^n \,|\, \langle x,x\rangle = -d_2\} \cap \partial B_2(d_1,d_2)$ and $\{x \in \mathbb{R}_k^n \,|\, \langle x,x\rangle = -d_2\} \cap \partial B_3(d_2,d_3)$ as well as $\{x \in \mathbb{R}_k^n \,|\, \langle x,x\rangle = d_3\} \cap \partial B_3(d_2,d_3)$ and $\{x \in \mathbb{R}_k^n \,|\, \langle x,x\rangle = d_3\} \cap \partial B_4(d_3,d_4)$ are identified. Proceeding this way we obtain $M(J)$.

If $J = \{1,\ldots,m\}$, $m = 2l-1$, $l \geq 2$ then

$$M(J) = B^+(d_1) \cup B_1(d_1,d_2) \cup \ldots \cup B_{m-2}(d_{m-2},d_{m-1}) \cup B^-(d_{m-1})\,.$$

with the analogous gluing. If $J = \mathbb{N}$ we let

$$M(J) = B^+(d_1) \cup B_1(d_1,d_2) \cup B_2(d_2,d_3) \cup \ldots$$

and if $J = \mathbb{Z}$ we let

$$M(J) = \ldots \cup B_{-1}(d_{-1},d_0) \cup B_0(d_0,d_1) \cup B_1(d_1,d_2) \cup \ldots$$

with the analogous gluing. On every building block we have by Lemma 4.1 a metric which is the product metric near the boundary. Therefore this defines a smooth metric g_D on the manifold $M(J)$ which is a product metric near the boundary of the building blocks. By construction g_D has the form $\langle x,x\rangle dr^2 + f_+'(r)^2\, g_1(x)$ with an even function f_+ satisfying the properties 1.),2.),3.) listed in the Proof of Lemma 4.1. Then for every $c \in \mathbb{R}$, $\epsilon \in \{\pm 1\}$ the function $\psi_{c,\epsilon}$ defined on the building block with

$$\psi_{c,\epsilon} = \epsilon \langle x,x\rangle f_+(r) + c$$

satisfies $\nabla^2 \psi_{c,\epsilon} = \lambda_\epsilon\, g$ where $\lambda_\epsilon(r,x) = \langle x,x\rangle \cdot f_+''(r) \cdot \epsilon$. By choosing the constants c,ϵ appropriately on every building block we obtain globally defined smooth functions ψ, λ satisfying $\nabla^2\psi = \lambda\, g$ with precisely one critical point on every building block. Here the value of ϵ has different signs on neighboring building blocks. V is complete, cf. Remark 3.7. □

Remark 4.5 The function ψ and hence λ as well as the metric can be described also by the following piecewise smooth function $\psi_* : \mathbb{R} \to \mathbb{R}$ of one variable. Let $t_1 = 0$ and $t_{j+1} = t_j + d_j$ for all $j, j+1 \in J$. Choose a broken geodesic $\gamma : \mathbb{R} \to M(J)$ with break points $p_j = \gamma(t_j)$, $j \in J$ at the critical points p_j of ψ. Hence $\gamma\,|\,[t_j,t_{j+1}]$ is a geodesic joining p_j and p_{j+1} whenever j and $j+1$ belong to J. The causal type of γ changes at every $t_j, j \in J$ between timelike and spacelike. If $\psi : M(J) \to \mathbb{R}$ is given we set $\psi_*(t) := \psi(\gamma(t))$. It follows that the smooth resp. analytic function $\psi : M(J) \to \mathbb{R}$ is completely determined by a piecewise smooth resp. analytic function $\psi_* : \mathbb{R} \to \mathbb{R}$ with the following properties: ψ_* is smooth resp. analytic outside the set $T := \{t_j | j \in J\} \subset \mathbb{R}$,

$$\psi_*^{(2m+1)}(t_j+) = \psi_*^{(2m+1)}(t_j-) = 0\,,\ \psi_*^{(2m)}(t_j+) = (-1)^m \psi_*^{(2m)}(t_j-) \qquad (38)$$

and

$$\psi_*''(t_j+) \in \{\pm 1\} \qquad (39)$$

for all $j \in J$ and all $m \geq 0$. Then the function ψ on $M(J)$ constructed in the preceding proof is defined by a function ψ_* of the following type:

1. $\psi_*(t_j+h)-\psi_*(t_j)=(-1)^j(\mathrm{sgn}h)h^2$ for sufficiently small h.

2. Outside $T\subset \mathbb{R}$ ψ_* has no critical value

3. $\psi_*'(t)=(-1)^j$ for t nearby d_j if $j,j+1\in J$.

On the other hand the proof of Theorem 4.3 shows also the following result:

Proposition 4.6 *Given a set $D=\{d_j\,|\,j,j+1\in J\}\subset \mathbb{R}^+$ together with the set $T:=\{t_j|j,j+1\in J\}$ where $t_1=0$ and $t_{j+1}=t_j+d_j$ for all $j,j+1\in J$ and given a continuous function $\psi_*:\mathbb{R}\to\mathbb{R}$ which is smooth outside T and satisfies Equations (38) and (39). Then there are smooth functions ψ,λ and a smooth metric g_ψ on $M(J)$ such that the following holds: The function ψ satisfies $\nabla^2=\lambda g_\psi$, the critical points of ψ are the points $p_j, j\in J$ and $2d_j=d(p_j,p_{j+1})$.*

We define on a manifold of the diffeomorphism type $M(J)$ a conformally flat structure $\mathrm{M(J)}(\alpha,\beta)$ as follows:

Fix numbers $\alpha,\beta\in\mathbb{R}^+\cup\{\infty\}$ as follows: In case $J=\mathbb{Z}$ we set $\alpha=\beta=\infty$, if $J=\mathbb{N}$ then $\alpha\in\{1,\infty\},\beta=\infty$. If $J=\{1,\ldots,m\}$ let either $\alpha=1$ and $\beta\in\mathbb{R}^+\cup\{\infty\}$ or $\alpha=\beta=\infty$.

If $J=\{1,\ldots,m\}$ let

$$\mathrm{M(J)}(\alpha,\beta):=B_1(\alpha,\infty)\cup B_2(\infty,\infty)\cup\ldots B_{m-1}(\infty,\infty)\cup B_m(\infty,\beta)\,.$$

Here each $B_j(a,b)$ is a copy of $B(a,b):=\{y\in\mathbb{R}^n_k\,|\,-a^2\le\langle y,y\rangle\le b^2\}$.

If $J=\mathbb{N}$ we let

$$M(J)(\alpha,\infty):=B_1(\alpha,\infty)\cup B_2(\infty,\infty)\cup B_3(\infty,\infty)\cup\ldots$$

and if $J=\mathbb{Z}$ we let

$$M(J)(\infty,\infty):=\ldots\cup B_{-1}(\infty,\infty)\cup B_0(\infty,\infty)\cup B_1(\infty,\infty)\cup\ldots$$

In all cases we use the following gluing (identification). We describe how to glue $B_j=B_j(a_j,b_j)$ with $B_{j+1}=B_{j+1}(a_{j+1},b_{j+1})$, resp. B_{j-1}. Let U_j be the interior of B_j, then we have on U_j the pseudo–Euclidean metric g_j which in polar coordinates (ρ_j,x_j) has the form

$$\mathrm{sgn}\langle x_j,x_j\rangle d\rho_j^2+\rho_j^2\,g_1(x_j)\,.$$

We identify the point (ρ_j,x_j) on U_j with $(\rho_{j+1},x_{j+1})=(1/\rho_j,x_j)$ on U_{j+1} whenever $(-1)^j\rho_j>0$ resp. with $(\rho_{j-1},x_{j-1})=(1/\rho_j,x_j)$ on U_{j-1} if $(-1)^j\rho_j<0$. On the overlaps $U_j\cap U_{j+1}$ we have

$$\mathrm{sgn}\langle x_j,x_j\rangle d\rho_j^2+\rho_j^2\,g_1(x_j)=\frac{1}{\rho_j^4}\left\{\mathrm{sgn}\langle x_{j+1},x_{j+1}\rangle d\rho_{j+1}^2+\rho_{j+1}^2\,g_1(x_{j+1})\right\}$$

hence the metrics $\{(U_j,g_j)_{j\in J}\}$ define a conformally flat structure.

The conformal factor is the same as the conformal factor of the *inversion*: $I(y)=y/\langle y,y\rangle$ for $\langle y,y\rangle\neq 0$. In geodesic polar coordinates the inversion has the form $I(r,x)=(1/r,x)$ and satisfies $I^*g_0=r^{-4}g_0$.

In general relativity conformally flat manifolds occur in the *Nordström theory*, cf. [HE, ch.3.4].

5 Analytic Examples

Now we are going to construct explicit analytic examples of metrics on $M(J)$ which satisfy the hypotheses of Theorem A in the Introduction, in particular these examples are $\mathrm{Cr}(\psi)$–complete. For the sake of simplicity we assume here, that $d(p_j, p_{j+1}) = 1$ for all $j, j+1 \in J$.

Remark 5.1 One can use the *stereographic projection* $\mathbb{R}^n_k \to Q$,(cf. [Kui] and [CK]) into the *standard quadric* Q in the $(n+1)$–dimensional real projective space $P^{n+1}\mathbb{R}$ to construct a locally symmetric metric in the conformal class of $M(J)(\alpha, \beta)$. The standard quadric Q can be defined as follows: Let $C^{n+1} := \{x \in \mathbb{R}^{n+2}_{k+1} \mid \langle x, x\rangle = 0\}$ be the light cone in the $(n+2)$–dimensional pseudo–Euclidean space $\mathbb{R}^{n+2}_{k+1}$ of signature $(k+1, n-k+1)$ and $\pi : \mathbb{R}^{n+2} - \{0\} \to P^{n+1}\mathbb{R}$ be the canonical projection. Then $Q = \pi(C^{n+1})$. On $S^k \times S^{n-k}$ we have the product metric $(-g_k) \oplus g_{n-k}$ of signature $(k, n-k)$, where g_l is the standard Riemannian metric on S^l, this induces the symmetric metric g_Q on Q. For the topology of Q see also [Kue2]. Then one can check that the pull–back metrics $\sigma^* g_Q$ on the building blocks $B_i \cong \mathbb{R}^n_k$ define an analytic metric on $M(J)(\alpha, \beta)$, which then is locally symmetric, since it is induced by a symmetric metric. Hence the scalar curvature is constant but the sectional curvature is not. The proof of Corollary 6.9 implies that for this metric there is no solution ψ of ψ of $\nabla^2\psi = \lambda\, g$ with critical points.

Remark 5.2 Let $M_m := M(\{1, \ldots, m\})$, we use the description of ψ and the metric g_ψ by the function $\psi_* : \mathbb{R} \to \mathbb{R}$ as described in the above Remark 4.5.

m=1: We can choose the pseudo–Euclidean space $\mathbb{R}^n_k$ as M_1, cf. Example 3.9 1.). In the description of Remark 4.5 the function ψ resp. the metric g_ψ is given by the function $\psi_*(t) = t^2$ for $t > 0$ and $\psi_*(t) = -t^2$ for $t < 0$.

m=2: We can choose the pseudo–hyperbolic space $S(-1)$ as M_2, see Example 3.9 2.). The function ψ on M_2 as well as the metric g_ψ can be described by the function ψ :

$$\psi_*(t) = \begin{cases} \frac{2}{\pi^2}\cosh \pi(t-1) & ; \quad t \le 1 \\ \frac{2}{\pi^2}\cos \pi(t-1) & ; \quad t \in [1,2] \\ -\frac{2}{\pi^2}\cosh \pi(t-2) & ; \quad t \ge 2 \end{cases} \tag{40}$$

Then one computes from Lemma 2.5 that the sectional curvature is constant -1.

m=∞ The rest of the section is devoted to the construction of an analytic solution ψ on $M(\mathbb{Z})$ of the equation $\nabla^2\psi = \lambda g$ and with critical points $\{p_i | i \in \mathbb{Z}\}$. Then the functions $\psi_\pm$ on a building block B have to satisfy the following hypotheses:

Choose $\gamma_\pm : \mathbb{R} \to M(\mathbb{Z})$ a geodesic with $\gamma_\pm(0) = p_j, \gamma_\pm(1) = p_{j+1}, \langle \gamma'_\pm, \gamma'_\pm\rangle = \pm 1$ then $\gamma_\pm$ is a closed geodesic with $\gamma_\pm(t+2) = \gamma_\pm(t)$ for all $t \in \mathbb{R}$. Hence $\psi_{j,\pm}(t) := \psi(\gamma_\pm(t))$ are analytic periodic functions satisfying $\psi_{j,\pm} \in \mathcal{F}$.

Therefore we use for the construction of an analytic metric on $M(\mathbb{Z})$ with an analytic solution ψ of $\nabla^2\psi = \lambda g$ elliptic functions, cf. Remark 3.3:

Lemma 5.3 *There is a real analytic function $f : \mathbb{R} \to \mathbb{R}$ which is periodic and even satisfying the following properties:*

1. $f(t+2) = f(t)$ *for all* $t \in \mathbb{R}$.
2. $f^{(4m)}(0) = 0$ *for all* $m \geq 1$ *and* $f''(0) = 1$.
3. $f'(t+1) = -f'(t)$ *for all* $t \in \mathbb{R}$.
4. *If t is a critical point of f then* $t \in \mathbb{Z}$.

Proof. We use *Jacobi's elliptic functions* $u \in \mathbb{C} \mapsto \mathrm{sn}(u) = \mathrm{sn}(u,k)$, resp. $\mathrm{cn}(u) = \mathrm{cn}(u,k)$ for the *modulus* $k = \sqrt{2}/2$ and the *complementary modulus* $k' = k = \sqrt{2}/2$. A reference is [La, ch.2]. Then there is positive real number $K = K(\sqrt{2}/2) = 1.85407...$ such that the elliptic function

$$F(z) = -\frac{1}{\sqrt{2}K}\mathrm{cn}(2Kz+K) \tag{41}$$

has periods 2 and 2i, where $i = \sqrt{-1}$. Let $f'(t) = F(t)$ for $t \in \mathbb{R}$.From the identity $\mathrm{cn}(iu+K) = i\,\mathrm{cn}(u+K)$ for $k = k' = \sqrt{2}/2$ it follows that f is even and satisfies $f^{(4m)}(0) = 0$ for all $m \geq 1$. One also computes that $F'(0) = 1$. F can also be described as the unique elliptic function with periods 2 and $2i$ and $F'(0) = 1$ whose poles and zeros are simple and are given as follows: The zeros occur at $z = m + in\,;\, m,n \in \mathbb{Z}$ and the poles at $z = m + 1/2 + i(n+1/2)\,;\, m,n \in \mathbb{Z}$. Hence property 1. and 4. follow. Property 3.) follows since $\mathrm{cn}(u) = -\mathrm{cn}(u+2K)$. □

Remark 5.4 Hence we can choose

$$f(t) = 2\arcsin\left(\frac{\sqrt{2}}{2}\mathrm{sn}(2Kt+K)\right) \tag{42}$$

for $k = \sqrt{2}/2$. If we set $f_+(t) = f(t)\,,\, f_-(t) = \pi - f(t)$, then the pair $f_\pm$ satisfies the hypotheses of Proposition 3.5 1.) i.e. $f_\pm \in \mathcal{F}$. $f_\pm$ are the solutions of the *pendulum equation* $f_\pm'' \pm \sin(4K^2 f_\pm) = 0$ with $f_\pm(0) = \pi/2\,,\, f_\pm'(0) = 0$. Hence the function $\psi : B(2,2) \to \mathbb{R}$ defined by $\psi(r,x) := f_\eta(r)\,,\, \eta = \langle x, x\rangle$ satisfies the *pendulum equation*

$$\nabla^2\,\psi = -\sin(4K^2\psi)\,g\,. \tag{43}$$

Theorem 5.5 *There exists a complete analytic pseudo–Riemannian manifold $M(\mathbb{Z})$ of signature $(k, n-k)$ carrying an analytic function $\psi : M(\mathbb{Z}) \to \mathbb{R}$ satisfying $\nabla^2\psi = -\sin(C\psi)\,g$ for some positive constant C with the following property: The critical set $\mathrm{Cr}(\psi)$ of ψ is in natural bijection with $\mathbb{Z}$ in the sense of Theorem 4.3 and the manifold is $\mathrm{Cr}(\psi)$–complete. Furthermore the vector field $\nabla\psi$ is complete.*

Proof. Let f be the real analytic function given by equation (42) which is periodic and even and satisfies the hypotheses of the preceding Lemma 5.3. As explained in Remark 4.5 we can define the function ψ on $M(\mathbb{Z})$ as well as the corresponding metric g_ψ by the following continuous and piecewise analytic function $\psi_* : \mathbb{R} \to \mathbb{R}$:

$$\psi_*'(t) = \begin{cases} (-1)^j f'(t) & ; \quad t \in [2j, 2j+1] \\ & ; \\ (-1)^{j+1} f'(t) & ; \quad t \in [2j-1, 2j] \end{cases} . \tag{44}$$

Hence up to a sign and an additive constant on an interval $[j, j+1], j \in J$ ψ_* is given by f resp. $-f$. It follows from Lemma 5.3 2.),3.) that Equation (38) of Remark 4.5 holds. Since f solves the pendulum equation we obtain for the function $\psi : M(\mathbb{Z}) \to \mathbb{R}$ the equation $\nabla^2\psi + \sin(4K^2\psi)\, g = 0$. The metric in a neighborhood of p_j is given by $g(r, x) = \eta dr^2 + f'(r)^2 g_1(x)$. The metric is $\mathrm{Cr}(\psi)$–complete since by construction every non–null geodesic through a critical point is closed. For the completeness of $\nabla\psi$ compare Remark 3.7. One can use the pendulum equation (43) resp. (42) and the differential equations for geodesics in warped products (cf. [ON, p. 208]) to show that the metric is geodesically complete. □

Corollary 5.6 *For any even number $2m$ there is an analytic pseudo–Riemannian manifold $M(\mathbb{Z})$ of signature $(k, n-k)$ carrying a closed and complete conformal vector field with exactly $2m$ zeros.*

Proof. It follows from Theorem 5.5 that we we can find neighborhoods U_j of p_j and an isometry $\Phi : M(\mathbb{Z}) \to M(\mathbb{Z})$ with $\Phi U_j = U_{j+2}$, i.e. there is an infinite cyclic subgroup in the isometry group of $M(\mathbb{Z})$ acting freely. It follows that on the quotient $M(\mathbb{Z})/m\mathbb{Z}$ there is a closed conformal vector field with exactly $2m$ zeros. □

We will see in the following section that in case of signature $(k, n-k), 2 \leq k \leq n-2$ under a suitable completeness assumption all manifolds carrying a non–constant solution ψ of $\nabla^2\psi = \lambda g$ with a critical point are diffeomorphic to $M(J)$. In the case of signature $(1, n-1), n \geq 3$ resp. $(1, 1)$ there are a lot of further examples due to the fact that the manifold $\Sigma := \{x \in \mathbb{R}^n_k | \langle x, x\rangle = \pm 1\} = S(1) \cup S(-1)$ then has three resp. four components. Hence the gluing process can be more complicated.

6 Conformal classification

Near a regular point of a function ψ satisfying $\nabla^2\psi = \lambda g$ the metric has the structure of a warped product, cf. Lemma 2.7. In this section we study the case that ψ has critical points, which are isolated by Proposition 2.3. We show that in geodesic polar coordinates with origin at a critical point the level sets of ψ with the induced metric have constant sectional curvature. In particular the metric is conformally flat.

Proposition 6.1 *Let (M, g) be a pseudo–Riemannian manifold with a non–constant solution ψ of the equation $\nabla^2\psi = \lambda g$ for a function λ and with a critical point $p \in M$.*

1. (cf. [Ta], [Kue1, lemma 18] *in the Riemannian case) Then there are functions* $\psi_\pm \in \mathcal{F}$ *such that the metric in geodesic polar coordinates* $(r,x) \in A_\psi \subset \mathbb{R}\times\Sigma$ *in a neighborhood* U *of* p *has the form*

$$g(r,x) = g_\psi(r,x) = \eta dr^2 + \frac{\psi_\eta'(r)^2}{\psi_\eta''(0)^2}\, g_1(x)\,;\, \eta = \langle x,x\rangle \tag{45}$$

and $\psi(r,x) = \psi_\eta(r), \lambda(r,x) = \lambda_\eta(r)$ *with* $\lambda_\eta(r) = \eta\psi''(r)$.

2. *If all geodesics through* p *are defined on the whole real line* $\mathbb{R}$. *Then the metric* g *is of the form (45) for all* $(r,x) \in A_\psi$ *, i.e. as long as* $\psi_\eta'(r)$ *does not vanish.*

Proof. 1) Let $\gamma : [0,r_0) \to M$ be a geodesic with $\gamma(0) = p, \langle\gamma',\gamma'\rangle = \eta \in \{\pm 1\}$. Then it follows from Proposition 2.1 that

$$\nabla\psi(\gamma(r)) = \left\{\int_0^r \lambda(\gamma(s))ds\right\}\gamma'(s) \tag{46}$$

and from Proposition 2.3 we know that there is an open neighborhood U of p where p is the only critical point of ψ. It follows from Equation (46) that the normal vectors of the level hypersurfaces $\psi^{-1}(\psi(r_0))$ and of the distance spheres (the sets $\{r = r_0\}$) are proportional. Hence the connected components of $\{r = r_0\} \cap U$ and of $\psi^{-1}(\psi(\gamma(r_0)))$ containing $\gamma(r_0)$ coincide. Therefore there are two smooth real functions $\psi_\pm$ with $\psi(r,x) = \psi_{\langle x,x\rangle}(r)$. From Remark 3.3 it follows that $\psi_\eta^{(2m+1)}(0) = 0$ and $\psi_+^{(2m)}(0) = (-1)^m\psi_-^{(2m)}(0)$ for all $m \geq 0$, cf. the proof of Proposition 3.5.

In geodesic polar coordinates (r,x) around p it follows from $\nabla^2\psi = \lambda\, g$ that

$$\nabla_{\partial_r}\nabla\psi(r,x) = \lambda(r,x)\,\partial_r$$

and that for X tangential to $\{r = r_0\}$:

$$X(\langle\nabla\psi,\nabla\psi\rangle) = \nabla^2{}_{X,\nabla\psi}\psi = 0\,.$$

Hence $\langle\nabla\psi,\nabla\psi\rangle$ is constant along the levels $\{r = r_0\}$. It follows that there are two smooth real functions $\lambda_\pm$ with $\lambda(r,x) = \lambda_{\langle x,x\rangle}(r)$ and $\psi_\eta'' = \nabla^2{}_{\gamma',\gamma'}\psi = \lambda_\eta\cdot\eta$. By Proposition 2.3 $\lambda(0) \neq 0$ hence $\psi_+''(0) = \psi_-''(0) = \lambda_+(0) = -\lambda_-(0) \neq 0$. Therefore 0 is an isolated critical point of $\psi_\pm$. It follows from Lemma 2.7 that in geodesic polar coordinates (r,x) the metric in U is of the form

$$g(r,x) = \eta dr^2 + \frac{\psi_\eta'(r)^2}{\psi_\eta''(0)^2} g_*(x)$$

for a C^∞–metric g_* on $\Sigma = S^0(1) \cup S^0(-1)$. We obtain from Lemma 3.4 that g_* coincides with the standard metric g_1 of constant sectional curvature ± 1 on Σ, hence we obtain Equation (45). Since the metric g extends to a neighborhood U of p Proposition 3.5 implies that $\psi_\pm \in \mathcal{F}$. It follows also that $\lambda_\pm \in \mathcal{F}$.

2) Assume $(r_0,x_0) \in A_\psi$, i.e. $\psi_\eta'(r) \neq 0$ for all $r \in (0,r_0]$. Then there is $r_1 \in (0,r_0]$ such that $(r_1,x_0) \in U$. Let r_* be the supremum of the numbers $r > 0$ such

that for a neighborhood of the radial geodesic segment $t \in [0,r) \mapsto \phi(t,x_0) \in M$ the metric has the form (45). It follows that $\psi'_\eta(r^*) = 0$ by Lemma 2.7. Now we show that $\phi : A_\psi \to M$ is injective, assume $\phi(r_1,x_1) = \phi(r_2,x_2)$, $(r_j,x_j) \in A_\psi$. If x_1, x_2 lie in the same component of Σ it follows that $r_1 = r_2$ since $\psi(r_1) = \psi(r_2)$ and since ψ is strictly monoton. If x_1, x_2 lie in different components, let $\gamma_1(r) = \phi(r,x_1)$, $\gamma_2(r) = \phi(r,x_2)$ be the two geodesics emanating from p with $\gamma_1(r_1) = \gamma_2(r_2) = \phi(r_1,x_1) = \phi(r_2,x_2) = q$, $r_1 < 0 < r_2$. Since $\nabla\psi(q) = -\psi'_{\eta_1}(r_1)\gamma'(r_1) = \psi'_{\eta_2}(r_2)\gamma'_{r_2}$ (where $\eta_j = \langle \gamma'_j, \gamma'_j \rangle$) it follows that $\gamma'(r_1) = -\gamma'(r_2)$ (since $\gamma'(r_1) \neq \gamma'(r_2)$) and $\psi'_{\eta_1}(r_1) = \psi'_{\eta_2}(r_2)$ $(\eta_1 = \eta_2)$. But ψ'_η changes sign at 0 since $\psi''_\eta(0) = 1$ $\Box$

We give a sufficient condition for a pseudo–Riemannian manifold with a conformal gradient field $\nabla\psi$ to be $\mathrm{Cr}(\psi)$–complete. Here $\mathrm{Cr}(\psi)$ is the set of critical points of ψ.

Lemma 6.2 *Let (M,g) be a* null complete *pseudo–Riemannian manifold with a non–constant solution ψ of the equation $\nabla^2\psi = \lambda\, g$ for some function λ and with a critical point. Let all geodesics through critical points be defined on $\mathbb{R}$. Then every point on M can be joined with a critical point by a geodesic, i.e. the manifold is* $\mathrm{Cr}(\psi)$*–complete.*

Proof. For a critical point p of ψ we denote by $\mathcal{A}_p$ the set of all points on M for which there is a geodesic joining them with p. Now suppose that the union $\mathcal{A}$ of all sets $\mathcal{A}_p$ for a critical point p is *not* the whole of M. Then there is a point $y \in \partial\mathcal{A}_p - \mathcal{A}_p$ for some critical point p. Hence there are points $y_j \in A_p$ with $y = \lim_{j\to\infty} y_j$ and geodesics $\gamma_j : \mathbb{R} \to M$, $\gamma_j(0) = p$, $\gamma_j(t_j) = y_j$, $\langle \gamma'_j, \gamma'_j \rangle = \eta \in \{\pm\}$. In the component of $\mathcal{A}_p$ containing y_j we have the warped product structure $\eta dt^2 + \psi'_\eta(t)^2\, g_1$ of the metric. Here we can assume without loss of generality that $\psi''_+(0) = 1$. By continuity $\psi(\gamma_j(t_j)) = \psi(t_j) \to \psi(y)$. Since $y \notin \mathcal{A}_p$ it follows that $\langle \nabla\psi(y), \nabla\psi(y) \rangle = 0$ and we can assume $\nabla\psi(y) \neq 0$. Since $\nabla\psi(y_j) = \psi'_\eta(t_j)\eta\partial_{t_j}$ and $\langle \partial_{t_j}, \partial_{t_j} \rangle = \eta$ it follows that $\lim_{j\to\infty} \psi_\eta(t_j) = 0$. Now choose a sequence σ_j of non–degenerate planes in $T_{y_j}M = T_{\phi(t_j,x_j)}M$ orthogonal to $\gamma'(t_j)$ and converging to a non–degenerate plane σ in T_yM. Then we use the formula for warped products for the sectional curvature $K(\sigma_j)$ of σ_j and obtain (cf. Lemma 2.5)

$$K(\sigma_j) = \frac{1}{\psi'_\eta(t_j)^2}\left(\epsilon - \psi''_\eta(t_j)\eta\right)$$

where $\epsilon \in \{\pm 1\}$ equals $\langle x_j, x_j \rangle$. Since $\psi'_\eta(t_j) \to 0$ for $j \to \infty$ we obtain $\psi''(t_j) \to \epsilon$. Then it follows that $\lambda(y_j) = \lambda_\eta(t_j) = \eta\,\psi''_\eta(t_j) \to \eta\epsilon$. Now denote by τ the null geodesic with $\tau'(0) = \nabla\psi(y)$, then $\psi(\tau(t)) = \psi(y)$ for all t, cf. Remark 2.2. Since $\psi^{-1}(\psi(y))$ is pointwise the limit of the sublevel sets $\psi^{-1}(\psi(y_j)) \cap \mathcal{A}_p$ and since $\psi^{-1}(\psi(y_j))$ coincides with $\lambda^{-1}(\lambda(y_j))$ (more precisely: the corresponding connected components) it follows that λ is constant along $\psi^{-1}(\psi(y))$. Hence $\nabla\psi(\tau(t)) = (\eta\epsilon t + 1)\tau'(0)$ therefore τ connects y and the critical point $\tau(-\eta\epsilon)$ of ψ $\Box$

Theorem 6.3 *Let (M,g) be a pseudo–Riemannian manifold carrying a non–constant solution ψ of the equation $\nabla^2\psi = \lambda\, g$ having critical points. We assume either that* *all geodesics through critical points are defined on* $\mathbb{R}$ *and that* (M,g) *is null complete or that* (M,g) *is* $\mathrm{Cr}(\psi)$*–complete.*

Then the manifold (M,g) *is conformally flat. One can define neighborhoods* $\tilde{A}_j$ *for every critical point* p_j *on which the metric has the warped product structure*

$$\eta dr^2 + \frac{\psi'_{j,\eta}(r)^2}{\psi''_\eta(0)^2}\, g_1 \tag{47}$$

in geodesic polar coordinates $(r,x) \in G \subset \mathbb{R} \times \Sigma$ *around the critical point* p_j*. The function* ψ *on* $\tilde{A}_j$ *has the form* $\psi(r,x) = \psi_{j,\langle x,x\rangle}(r)$*. These neighborhoods cover* M*.*

Proof. For every critical point p_j we define in terms of geodesic polar coordinates (r,x) around p_j the functions $\psi_{j,\eta}(t) := \psi(t,x)\,,\ \eta = \langle x,x\rangle$. Then let $\tilde{A}_j := A_\psi \cup C_j$, where C_j is the light cone at p_j. I.e. $(r_0,x) \in \tilde{A}_j$ if and only if $\psi'_{j,\eta}(r)$ does not vanish between 0 and r_0 (cf. Definition 3.1). The form of the metric in $\tilde{A}_j$ is given by Equation (47), cf. Proposition 6.1. If $r_0 > 0$ is the first positive zero of $\psi'_{j,\eta}$ then all radial geodesics $\gamma : t \mapsto (t,x)\,,\ \langle x,x\rangle = \eta = \langle\gamma',\gamma'\rangle$ intersect at time r_0 in a critical point. Together with Lemma 6.2 it follows that the sets $\tilde{A}_j$ cover M $\square$

Hence it remains to study the global conformal type of M. Here the results depend on the signature $(k, n-k)$. Let (M,g) be a pseudo–Riemannian manifold satisfying the assumptions of the preceding theorem with critical points $\mathrm{Cr}(\psi) = \{p_j \,|\, j \in J\}$ and with signature $2 \le k \le n-2$. The set $\mathrm{Cr}(\psi)$ is non–empty and discrete and we can assume that either $J = \{1,2,\ldots,j\}$ for some $j \in \mathbb{N}$ or $J = \mathbb{N}$ or $J = \mathbb{Z}$ and that $\psi(p_j) < \psi(p_{j+1})$ for all $j \in J$. We will show in the following theorem that under suitable completness assumptions M is diffeomorphic to $M(J)$. Hence for signature $(k, n-k)$, $2 \le k \le n-2$ the diffeomorphism type is determined by the index set J.

Theorem 6.4 *Let* (M,g) *be a pseudo–Riemannian manifold with signature* $(k, n-k)\,,\ 2 \le k \le n-2$ *carrying a non-constant solution* ψ *of* $\nabla^2\psi = \lambda\, g$ *with critical points for some function* λ*. Assume either that all geodesics through critical points are defined on* $\mathbb{R}$ *and that* (M,g) *is null complete or that* (M,g) *is* $\mathrm{Cr}(\psi)$*–complete. If the set of critical points is* J *where* J *has to be interpreted as a linear graph (see Theorem 4.3 resp. Remark 4.4) and if* $D := \{d_j = d(p_j, p_{j+1}) \,|\, j, j+1 \in J\}$ *then* M *is diffeomorphic to the manifold* $M(J)$ *and there is a function* $\psi_* : \mathbb{R} \to \mathbb{R}$ *as in Proposition 4.6 which completely determines the metric* $g = g_D$*.*

Proof. Let $\tilde{A}_j$ be the neighborhoods of p_j constructed in the proof of Theorem 6.3, i.e. there are two positive numbers $r_{j,\eta} \in \mathbb{R} \cup \{\infty\}$ such that $\tilde{A}_j$ is of the form

$$\tilde{A}_j = \{y \in \mathbb{R}^n_k \,|\, -r^2_{j,-} < \langle y,y\rangle < r^2_{j,+}\}\,,$$

i.e in polar coordinates

$$\tilde{A}_j = \{(r,x) \in G \subset \mathbb{R} \times \Sigma \,|\, r^2 < r^2_{j,\eta} \text{ if } \langle x,x\rangle = \eta\}\,.$$

The metric in $\tilde{A}_j$ is the warped product metric

$$\eta dr^2 + \frac{\psi'_{j,\eta}(r)^2}{\psi''_\eta(0)^2}\, g_1\,.$$

Since $\nabla\psi(r,x) = \eta\psi'_{j,\eta}(r)\partial_r$ it follows that all geodesics $r \mapsto \gamma(r) = (r,x)$, $\langle x,x\rangle = \eta$ starting from the critical point $p_j = \gamma(0)$ meet at the critical point $\gamma(r_{j,\eta}) = p_{j+\eta}$. Then $d(p_j, p_{j+1}) = r_{j,\eta}$. If $r_{j,\eta} = \infty$ then $\gamma(r)$ is a regular point of ψ for all $r > 0$. We define $A_j \subset \tilde{A}_j$ in geodesic polar coordinates by $(r,x) \in A_j$ if and only if $2|r| \leq r_{j,\eta}$, $\langle x,x\rangle = \eta$.

The building blocks A_j are of the form $B^+(r_{j,+}), B^-(r_{j,-})$ or $B(r_{j,+}, r_{j,-})$ as defined in the beginning of Section 4. Since the sets $\partial A_j \cap \{x \in M \,|\, \psi(x) < \psi(p_j)\}$ resp. $\partial A_j \cap \{x \in M \,|\, \psi(x) > \psi(p_j)\}$ have only one component the manifold M is of the diffeomorphism type $M(J)$. Here A_j is of the type B if j is not an extremal value of J. If j is an extremum of J then A_j is either of the form B^+ or B^-. Then $A_j = B^+$ if and only if there is a geodesic γ joining p_j with p_{j+1} resp. p_{j-1} and $\langle\gamma', \gamma'\rangle = 1$. Hence it follows that M is diffeomorphic to $M(J)$. It follows from Proposition 4.6 that the metric is determined by the function ψ_*. □

It follows from the cohomology rings of $M(J)$ that $M(J)$ is diffeomorphic to $M(J')$ only if $J = J'$. In the case of signature $2 \leq k \leq n-2$ we can classify the *global* conformal types of manifolds with solutions of $\nabla^2\psi = \lambda\, g$. It follows from Theorem 6.4 that there is a set $T = \{t_j \,|\, j \in J\}$ with $t_1 = 0$ and $t_{j+1} = t_j + d_j$ for all $j, j+1 \in J$ such that the following holds: The functions $\psi_{*,j} = \psi_*(r - t_j)$ satisfy $\psi_{*,j} \in \mathcal{F}$ for all $j \in J$ and such that there are neighborhoods $\tilde{A}_j = \mathrm{Interior} B(d_{j-1}, d_j)$ (here $d_j = \infty$ if $j \notin J$) and where the metric is of the form

$$\mathrm{sgn}\langle x,x\rangle dr_j^2 + \frac{\psi'_*(r_j - d_j)^2}{\psi''_*(d_j+)^2}\, g_1(r_j, x_j) \tag{48}$$

Now we define the numbers $\alpha, \beta \in \mathbb{R}^+ \cup \{\infty\}$: If $J = \mathbb{Z}$ then $\alpha = \beta = \infty$. For $j \in J$ let $\psi_{*,j}(r) = \psi_*(r - d_j)$ and

$$h_j(r) = (\mathrm{sgn} r)\frac{\psi''_j(0+)}{\psi'_j(r)} - \frac{1}{r}\,.$$

If $J = \{1, \ldots, m\}$ resp. $J = \mathbb{N}$ let

$$a_1 := -\lim_{r \to -\infty}\left(r \exp \int_0^r h_1(\xi) d\xi\right)$$

and if $J = \{1, \ldots, m\}$ let

$$a_m := (-1)^{m+1} \lim_{r \to (-1)^{m+1}\infty}\left(r \exp \int_0^r h_m(\xi) d\xi\right)\,.$$

If $J = \mathbb{N}$ then let $\alpha = 1$ if $a_1 < \infty$ and $\alpha = \infty$ if $a_1 = \infty$. If $J = \{1, \ldots, m\}$ we assume without loss of generality that $a_1 \leq a_m$. If $a_1 < \infty$ then let $\alpha = 1, \beta = a_m/a_1$. If $a_1 = \infty$ then $\alpha = \beta = \infty$.

Theorem 6.5 *Let (M,g) be a pseudo–Riemannian manifold with signature $(k, n-k)$, where $2 \leq k \leq n-2$ carrying a non–constant solution ψ of $\nabla^2\psi = \lambda\, g$ with at least one critical point. Assume that (M,g) is $\mathrm{Cr}(\psi)$–complete. If J is in natural bijection with the critical set $\mathrm{Cr}(\psi)$ and if α, β are defined as above then (M,g) is conformally equivalent to the conformally flat manifold $M(J)(\alpha,\beta)$ constructed in the end of section 4. $V = \nabla\psi$ is a complete vector field if and only if $\alpha = \beta = \infty$.*

Proof. For every neighborhood $\tilde{A}_j$ as defined in Theorem 6.3 we have that the metric is in polar coordinates (r_j, x_j) around the critical point p_j of the form

$$g_j(r_j, x_j) = \eta dr^2 + \psi_j'^2(r_j)\, g_1 \,.$$

Here by a linear change we assume that $\lambda(p_j) \in \{\pm 1\}$ for all $j \in J$. Hence with the notation of Proposition 3.6 we have

$$(\tilde{A}_j, g) = (B(d_{j-1}, d_j), g_j)$$

where $d_{j-1} = \infty$ if $j - 1 \notin J$. Then we define a map

$$\phi : M \longrightarrow M(J)(\alpha, \beta)$$

as follows. Here we use that $M(J)(\alpha, \beta)$ is covered by open sets $U_j, j \in J$ which are of the type $U_j = \mathrm{Int} B_j(a_j, b_j)$ together with the pseudo–Euclidean metric

$$\mathrm{sgn} \rho_j d\rho_j^2 + \rho_j^2\, g_1(x_j) \,.$$

Now we use Proposition 3.6. First we fix $r_0 \in (0, d_1)$ and choose $\rho_0 > 0$ arbitrary. Then we define $\tilde{A}_1 \longrightarrow U_1$ by $\phi((r_1, x_1)) = (\rho_1(r), x_1)$ with $\rho = \rho(r)$ given by Equation (30) in the Proof of Proposition 3.5. If $J = \mathbb{N}$ or $J = \{1, \ldots, m\}$ and $\alpha = 1$ then we have to choose ρ_0 such that $\rho(-\infty) = -1$.

Hence ϕ is fixed on $\tilde{A}_1 \cap \tilde{A}_2 = \{(r_1, x_1) | r_1 > 0\} = \{(r_2, x_2) | r_2 > 0\}$ since $r_1 = 1/r_2$ for $r_1, r_2 > 0$. Hence ϕ is uniquely determined by $\phi : A_j \to U_j$, $\phi((r_j, x_j)) = (\rho_j(r), x_j)$ with $\rho_j(r)$ given again by Equation (30). It follows from Proposition 3.6 that ϕ is a conformal transformation. Under the map $\phi : \tilde{A}_j \to U_j$ the gradient field $\nabla\psi$ is mapped onto the radial vector field $\rho_j \frac{\partial}{\partial \rho_j}$ in U_j. This is complete only if $\alpha = \beta = \infty$. □

Remark 6.6 If $k = 1$ or $k = n - 1$ then the diffeomorphism type of M can be classified by the *gluing graph* whose vertices are the critical points and where two vertices are joined by an edge if and only if there is a direct geodesic connection between them in M. By the ψ–levels of the critical points this graph is a directed graph in a canonical way.

We sketch the proof in the case $k = 1$ or $k = n - 1$. From the above definition of the gluing graph it follows that if (M, g) and (M_*, g_*) are globally conformally equivalent then the two gluing graphs are isomorphic.

A more detailed definition of the gluing graph involving spacelike or timelike edges and a more careful labeling implies that the assignement of this graph is injective on conformal classes, i.e. two graphs are non–isomorphic if the two manifolds are conformally inequivalent. So in this case the gluing graph really classifies the global conformal types. For $n = 2$ at any vertex at most four edges meet, for $n > 2$ at most three edges meet at any vertex because of the number of components of $S(\pm 1)$. In the case $2 \le k \le n - 2$ the gluing graph is just a linear graph, see Remark 4.4. In the cases $k = 1, n - 1$ the gluing graph can have cycles.

As an application of Theorem 6.3 we describe manifolds with solutions of $\nabla^2\psi = \lambda \cdot g$ for λ being a constant or $\lambda = \psi$ and manifolds of constant scalar curvature with a closed conformal gradient field with a zero.

Corollary 6.7 (Kerbrat, [Ke2, Thm.3 and Thm.6]) *Let (M, g) be a pseudo–Riemannian manifold of signature $(k, n-k)$.*

1. *If M admits a solution ψ of $\nabla^2\psi = c \cdot g$ for a constant $c \neq 0$ and if (M, g) is* $\mathrm{Cr}(\psi)$*–complete then (M, g) is isometric with the pseudo–Euclidean space.*
2. *If M admits a non–constant solution ψ of $\nabla^2\psi = \psi\, g$ with at least one critical point, if (M, g) is* $\mathrm{Cr}(\psi)$*–complete and if the signature satisfies $k \geq 2$ then (M, g) is isometric to the pseudo–hyperbolic space $S(-1)$ of constant sectional curvature -1. If $k = 1$ then (M, g) is a pseudo–hyperbolic space up to a covering map.*

Proof. 1.) Along the trajectories of $\nabla\psi$ ψ is a quadratic polynomial. Therefore ψ has exactly one critical point on M. Theorem 6.3 implies that the metric has the form $g(r, x) = \eta dr^2 + r^2 g_1(x)$ for all $(r, x) \in G$. This is the pseudo–Euclidean metric in polar coordinates.

2.) By Theorem 6.3 the metric in a neighborhood $\tilde{A}_j$ of the critical point p_j has the form

$$g(r, x) = \eta dr^2 + \psi_\eta'(r)^2 g_1(x) \tag{49}$$

where

$$\psi_+'' = \psi_+ \,,\; \psi_-'' = -\psi_- \,,\; \psi_+(0) = \psi_-(0) = 0 \,,$$

i.e.

$$\psi_+(r) = \cosh r \,,\; \psi_-(r) = \cos r$$

and these neighborhoods cover M. This implies that ψ has exactly two critical points.

But this is the metric of the pseudo–hyperbolic space $S(-1)$ in geodesic polar coordinates, see Example 3.9 2.) □

Remark 6.8 In Kerbrat's notation the quadratic form

$$\Phi(\psi) := \langle\nabla\psi, \nabla\psi\rangle - \psi^2$$

on the space of all solutions of $\nabla^2\psi = \psi\, g$ is assumed to be negative for some ψ: $\Phi(\psi) < 0$. Then it follows that ψ has a critical point on M and that the condition 2.) above holds [Ke2, Thm.6]. Moreover, if $\Phi(\psi) = 0$ for some nontrivial solution ψ then $\nabla\psi$ is not a null vector everywhere (otherwise $\nabla^2\psi = 0$), hence $\psi' = \pm\psi$ along a trajectory of $\nabla\psi$. Then Lemma 2.7 implies that locally the metric is a warped product $g = dt^2 + \exp(\pm 2t)\, g_*$ where g_* is not positive definite. (M, g) contains this warped product defined for $t \in \mathbb{R}$. However, by the argument given in [ON, p.209] certain null geodesics in this warped product $\mathbb{R} \times_{\exp \pm t} M_*$ are not complete for $t \to \infty$ or $t \to -\infty$. If g is complete then M must contain a limit point for $t \to \pm\infty$. At such a point ψ would have a zero and moreover a critical point. But this contradicts Proposition 2.3. This argument shows that Kerbrat's Theorem 6 in [Ke2] remains valid under the weaker assumption that $\Phi(\psi) \leq 0$ for at least one nontrivial solution ψ of $\nabla^2\psi = \psi\, g$. It is certainly not true if Φ is positive definite. The warped product $\mathbb{R} \times_{\cosh} M_*$ provides a counterexample because M_* may be chosen arbitrarily.

Corollary 6.9 *Let (M,g) be a pseudo–Riemannian manifold of constant scalar curvature. If M admits a non–constant solution ψ of $\nabla^2\psi = \lambda\, g$ for some function λ, if ψ has at least one critical point and if (M,g) is* $\mathrm{Cr}(\psi)$*–complete then (M,g) is a space of constant sectional curvature.*

Proof. By Proposition 6.1 we have in a neighborhood of the critical point $g(r,x) = \eta dr^2 + \frac{\psi_\eta'(r)^2}{\psi_\eta''(0)^2} g_1(x)$. By Equation (22) of Lemma 2.5 the functions ψ_η satisfy the following differential equations for constants ρ, ρ_*:

$$\psi_\eta'^2 \rho = \frac{n-2}{n}\rho_* - \frac{n-2}{n}\psi_\eta''^2\eta - \frac{2}{n}\eta\psi_\eta'''\psi_\eta'$$

and

$$\psi_\eta'(0) = 0\,.$$

We regard this as a differential equation for

$$y_\eta := \psi_\eta'$$

and obtain

$$\eta y_\eta y_\eta'' + \frac{n-2}{2}\eta y_\eta'^2 + \frac{n}{2}\rho y_\eta^2 - \frac{n-2}{2}\rho_* = 0$$

or, equivalently,

$$y_\eta^{n-2}\left(\eta y_\eta'^2 + \rho y_\eta^2 - \rho_*\right) = \text{ constant } = 0\,.$$

The latter holds by the initial condition $y_\eta(0) = 0$. Then Corollary 2.6 implies that the sectional curvature is constant. By Theorem 6.3 the neighborhoods $\tilde{A}_j$ cover M, hence the sectional curvature is constant everywhere. □

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