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EINSTEIN SPACES WITH A CONFORMAL GROUP

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Abstract. The pseudo-Riemannian Einstein spaces with a conformal group of strictly positive dimension can be classified. In this article we give a straightforward and systematic proof. As a common generalization, this includes the global theorem by Yano and Nagano in the Riemannian case (1959 published in the Annals of Mathematics) and a pseudo-Riemannian analogue by Kerckhove in his 1988 thesis under Professor K.Nomizu. We extend and unify the previous results in the case of an indefinite metric by analogy with the case of a positive definite metric.

1. Introduction and notations

An interesting question in global Riemannian geometry is the following: Which spaces admit a global 1-parameter group of conformal transformations? One of the highlights in the theory is the theorem of Alekseevski and others on the classification of Riemannian manifolds admitting a complete and essential conformal vector field. It is still an open problem to find an appropriate analogue in the case of an indefinite metric. Under additional curvature conditions on the manifold like the Einstein condition or the constancy of the scalar curvature this is different. Here the theorem by Yano and Nagano [43] states that the standard sphere is the only complete Einstein space admitting a complete conformal vector field which is non-homothetic. The proof was based on a result by S.Kobayashi on Killing vector fields. It was the idea of K.Nomizu to carry that over to the pseudo-Riemannian case, mainly by considering the Lie algebra of conformal vector fields. In his thesis at Brown University M.Kerckhove worked on this problem and obtained an analogue for pseudo-Riemannian Einstein spaces of non-vanishing scalar curvature [23, Thm.3.1]. This was based on the study of closed conformal vector fields because the gradient of the divergence of any conformal vector field on an Einstein space is again conformal with the same conformal factor, up to a constant. The differential equation characterizing closed conformal gradient fields was essentially solved already already by Brinkmann in the 1920's and by Fialkow in the 1930's.

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Dedicated to the memory of Katsumi Nomizu.

In the sequel we give a unified and essentially self-contained approach to both results, also including the Ricci flat case. This is based on a local analysis of the differential equation leading to a normal form of the metric on the one hand and the discussion of singularities of the vector field or critical points of its divergence on the other hand. Lie-theoretic arguments are not used. The main theorem on complete conformal vector fields is Theorem 2.1. However, we also state the results for conformal vector fields which are not necessarily complete but which live on a complete space. This includes previous results by Kanai [21] and corrects a statement by Kerckhove [23, Thm.3.2].

We consider a pseudo-Riemannian manifold (M, g), which is defined as a smooth manifold M (here *smooth* means of class C^∞, at least C^3) together with a pseudo-Riemannian metric of arbitrary signature $(k, n-k), 0 \le k \le n$. All manifolds are asssumed to be connected. A conformal mapping between two pseudo-Riemannian manifolds $(M, g), (N, h)$ is a smooth mapping $F : (M, g) \to (N, h)$ with the property $F^*h = \alpha^2 g$ for a smooth positive function $\alpha : M \to \mathbb{R}^+$. In more detail this means that the equation

$$h_{F(x)}\left(dF_x(X), dF_x(Y)\right) = \alpha^2(x) g_x\left(X, Y\right)$$

holds for all tangent vectors $X, Y \in T_xM$. Particular cases are homotheties resp. dilatations, for which α is constant and isometries, for which $\alpha = 1$.

A (local) one-parameter group Φ_t of conformal mappings of a manifold into itself generates a conformal (Killing) vector field V, sometimes also called an infinitesimal conformal transformation, by $V = \frac{\partial}{\partial t}\Phi_t$. We need that V itself is of class at least C^3. Vice versa, any conformal vector field generates a local one-parameter group of conformal mappings. It is well known since [42] that a vector field V is conformal if and only if the Lie derivative $\mathcal{L}_V g$ of the metric g in direction of the vector field V satisfies the equation

$$\mathcal{L}_V g = 2\sigma g \tag{1}$$

for a certain smooth function $\sigma : M \to \mathbb{R}$. Necessarily this conformal factor σ coincides with the divergence of V, up to a constant $\sigma = \mathrm{div}V/n$. Particular cases of conformal vector fields are homothetic vector fields for which σ is constant, and isometric vector fields, also called Killing vector fields, for which $\sigma = 0$. On the (pseudo-)Euclidean space the divergence of a conformal vector field is always a linear function. This follow from Corollary 3.3 below.

Furthermore it is well known that the image of a lightlike geodesic under any conformal mapping is again a lightlike geodesic and that for any lightlike geodesic γ and any conformal vector field V the quantity $g(\gamma', V)$ is constant along γ. Conformal vector fields V with non-vanishing $g(V, V)$ can be made into Killing fields within the same conformal class of metrics, namely, for the metric. $\overline{g} = |g(V,V)|^{-1}\, g$. This is a special case of a so-called inessential conformal vector field.

A vector field V on a pseudo-Riemannian manifold is called closed if it is locally a gradient field, i.e., if locally there exists a function f such that $V = \mathrm{grad} f$. Consequently, from Equation 1 and $\mathcal{L}_V g = 2\nabla^2 f$ we see that a closed vector field V is conformal if and only

$$\nabla_X V = \sigma X \tag{2}$$

for all X or, equivalently $\nabla^2 f = \sigma g$. Here $\nabla^2 f(X,Y) = g\left(\nabla_X \mathrm{grad} f, Y\right)$ denotes the Hessian $(0,2)$-tensor and $n\sigma = \Delta f = \mathrm{div}\ (\mathrm{grad}\ f)$ is the Laplacian of f. If the symbol $(\quad)^\circ$ denotes the traceless part of a $(0,2)$-tensor, then grad f is conformal if and only if $(\nabla^2 f)^\circ \equiv 0$. This equation

$$(\nabla^2 f)^\circ = 0 \tag{3}$$

allows explicit solutions in many cases, for Riemannian as well as for pseudo-Riemannian manifolds, see the discussion in Section 3 below.

A vector field is called complete if the flow is globally defined as a 1-parameter group $(\Phi_t)_{t\in\mathbb{R}}$ of diffeomorphisms, i.e. a global smooth mapping $\Phi\colon \mathbb{R}\times M \to M$, $(t,x)\mapsto \Phi_t(x)$ satisfying $\Phi_{t+s} = \Phi_t \circ \Phi_s$. As usual,

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Y - \nabla_{[X,Y]} Z \tag{4}$$

denotes the (Riemann) curvature $(1,3)$-tensor. Then the Ricci tensor as a symmetric $(0,2)$-tensor is defined by the equation

$$\mathrm{Ric}(X,Y) = \mathrm{trace}\big(V \mapsto R(V,X)Y\big).$$

The associated $(1,1)$ tensor is denoted by ric where $\mathrm{Ric}(X,Y) = g\left(\mathrm{ric}(X),Y\right)$. Its trace $S = \mathrm{trace}\big(V \mapsto \mathrm{ric}(V)\big)$ is called the scalar curvature. A manifold is conformally flat, if every point has a neighborhood which is conformally equivalent to an open subset of pseudo-Euclidean space.

A pseudo-Riemannian manifold of dimension $n \geq 3$ is called an Einstein space if the equation

$$\mathrm{Ric} = \lambda g \tag{5}$$

holds with a factor $\lambda = S/n$ which is necessarily constant and which is called the Einstein constant. For convenience the normalized Einstein constant will be denoted by $k = \lambda/(n-1)$ so that we have $k = 1$ on the unit sphere of any dimension. For a survey on Einstein spaces in general we refer to [3], [10].

2. Conformal vector fields on Einstein spaces: Results

Since the early 1920's conformal changes of Ricci flat and Einstein metrics were studied. One of the questions was: When is an Einstein metric still Einstein after a conformal change, locally or globally ? The local problem was essentially solved by Brinkmann [5], [6] but global results were investigated much later. The following statement is the main theorem of the present paper.

Theorem 2.1. (common extension of [43] and [23], initiated by Nomizu) *Assume that a geodesically complete pseudo-Riemannian Einstein space admits a complete and non-homothetic conformal vector field. Then it is isometric (or anti-isometric) with the standard sphere. In particular the metric must be definite.*

Kerckhove [23, Thm.3.1] stated that in the case of a non-vanishing Einstein constant the metric has to be definite. He used the completeness of the gradient of the divergence of any given complete conformal vector field (which seems to be true but which was not verified). We do not use the completeness of the gradient. The classification of complete Einstein spaces carrying a not necessarily complete conformal vector field is given below in Theorem 2.7 and Theorem 2.8. This includes more cases, also with an indefinite metric and also spaces of non-constant sectional curvature. Moreover, there is the following stronger version:

Theorem 2.2. (alternative version) *Assume that there is a global and non-homothetic conformal diffeomorphism $F : M_1 \to M_2$ between two complete pseudo-Riemannian Einstein spaces. Then each of them is isometric or anti-isometric with the Riemannian standard sphere. In particular the metric must be definite.*

Corollary 2.3. *On a complete pseudo-Riemannian Ricci flat manifold there is no complete and non-homothetic conformal vector field such that the gradient of its divergence is everywhere isotropic.*

Example 2.4. Let g_1 denote the standard metric on the unit $(n-1)$-sphere. The unit n-sphere with the warped product metric $g = dt^2 + (\sin t)^2 g_1$ carries the conformal vector field $V_1(t,x) = \sin t \cdot \partial_t$ with two antipodal zeros (north and south pole). It is the gradient of the function $f(t,x) = -\cos t$. The divergence is $\mathrm{div} V(t) = n \cos t$. The function f can be regarded as the height function in vertical direction with its minimum at the south pole. Another conformal vector field V_2 can be defined as the stereographic preimage of a translational vector field on $\mathbb{R}^n$. This is not closed and has precisely one zero in the stereographic pole. See Example 2.9 (with $\epsilon_i = +1$) for a stereographic projection from the antipodal pole back to $\mathbb{R}^n$. Each of them is a complete vector field since the sphere is compact.

The original version of Theorem 2.1 in Riemannian geometry is the following.

Theorem 2.5. (Yano and Nagano [43])
Assume that a complete Riemannian Einstein space admits a complete and non-homothetic conformal vector field. Then it is isometric with the standard sphere.

According to personal communication by Professor Nomizu, in 1959 this result was regarded as a great achievement. However, the proof in [43] does not refer to previous work in the 1920's by Brinkmann on conformal mappings between Einstein spaces (Riemannian or pseudo-Riemannian). With this reference the proof can be simplified. Moreover, there is the following more general version which implies Theorem 2.5:

Theorem 2.6. (alternative version, see [33, Thm.10.3] if $M_1 = M_2$ and [26, Thm.27])
Assume that there is a global and non-homothetic conformal diffeomorphism $F : M_1 \to M_2$ between two complete Riemannian Einstein spaces. Then both of them are isometric with the standard sphere.

An infinitesimal version of Theorem 2.6 is given in Kanai's theorem below, an infinitesimal version of Theorem 2.2 is Theorem 2.8. Under the assumption of compactness there is, in either case, a simpler proof using the differential equation $\nabla^2\varphi = c\varphi g$ with a constant c since any non-constant function must have a critical point. For Riemannian manifolds this equation was solved in [38] and, more generally, in [41].

Theorem 2.7. (Kanai [21, Thm.G])
Assume that (M, g) is a complete Riemannian Einstein space of dimension $n \geq 3$ admitting a non-homothetic conformal vector field V. Then the following hold:

1. *If there is a critical point of $\mathrm{div}V = n\sigma$ then (M, g) is isometric either with the standard sphere or with the hyperbolic space.*
2. *If there is no critical point of σ then (M, g) is isometric (up to scaling) with one of the following spaces:*
   - (a) *The Euclidean space $\mathbb{E}^n$ where σ is a linear coordinate function,*
   - (b) *the product $\mathbb{R} \times M_*$, equipped with the warped product metric $g = dt^2 + e^{2t}g_*$ where $\sigma(t) = e^t$ and where (M_*, g_*) is a complete Einstein space of dimension $n-1$ with an Einstein constant $k_* = 0$,*
   - (c) *the product $\mathbb{R} \times M_*$, equipped with the warped product metric $g = dt^2 + (\cosh t)^2 g_*$ where $\sigma(t) = \sinh(t)$ and where (M_*, g_*) is a complete Einstein space of dimension $n-1$ and with a normalized Einstein constant $k_* = -1$.*

The cases $(2b)$ and $(2c)$ contain also the hyperbolic space as special subcases. A pseudo-Riemannian analogue of Theorem 2.7 is slightly different as follows. It was initiated by Kerckhove [23] where the case $(2b)$ is discussed but not $(2a)$ and $(2c)$.

Theorem 2.8. (compare [23], [28]) *Assume that a (M, g) is a geodesically complete pseudo-Riemannian Einstein space of dimension $n \geq 3$ and signature $(j, n-j), 1 \leq j \leq n-1$ admitting a non-homothetic conformal vector field V. Then the following hold:*

(1) *If there is a critical point of* $\mathrm{div}V = n\sigma$ *then (M, g) is a space of constant sectional curvature $k \neq 0$. More precisely, it is isometric with S^n_k, H^n_k or with a covering of S^n_{n-1} or H^n_1 in the notation of* [40, p.108].
(2) *If there is no critical point of σ then (M, g) is isometric with one of the following cases:*
   (a) *(M, g) splits as a product $M \cong \mathbb{R} \times M_*$, equipped with a warped product metric $g = \pm dt^2 + (\cosh t)^2 g_*$ where (M_*, g_*) is a complete Einstein space of dimension $n-1$,*
   (b) *(M, g) is isometric with the pseudo-Euclidean space and σ is a linear coordinate function in a spacelike or timelike direction. The gradient of σ is a parallel and non-isotropic vector field.*
   (c) *(M, g) is Ricci flat and the gradient of σ is a parallel and isotropic vector field (in other words: (M, g) is a Ricci flat Brinkmann space).*

Each of these cases really occurs. There are closed conformal vector fields in the cases (1) and $(2a)$ where $k \neq 0$. On the flat pseudo-Euclidean space we have special conformal vector fields in Example 2.9 for Case $(2b)$ and in Example 2.13 for Case $(2c)$. A non-flat example for Case $(2c)$ is also given in Example 2.13. For 4-dimensional Lorentz manifolds the class of Ricci flat Brinkmann spaces is precisely the class of Ricci flat *pp*-waves, see Definition 2.12.

Example 2.9. On the pseudo-Euclidean space $\mathbb{R}^n = \{(t, x_1, \ldots, x_{n-1})\}$ with the metric $g = dt^2 + \sum_i \epsilon_i dx_i^2$ we define the vector field

$$V(t, x_1, \ldots, x_{n-1}) = \left(\tfrac{1}{2}(t^2 - \textstyle\sum_i \epsilon_i x_i^2), tx_1, \ldots, tx_{n-1}\right).$$

This is conformal with divergence $\mathrm{div}V = t + (n-1)t = nt$, so the function σ is nothing but the coordinate function t. On the t-level M_* in the product decomposition $M = \mathbb{R} \times M_*$ the vector field appears as t times the homothetic radial vector field. The origin $t = x_1 = \cdots = x_{n-1} = 0$ is a zero of V and of $\mathrm{div}V$ simultaneously. In the Euclidean plane one can describe the vector field in terms of complex numbers $z = t + ix$ simply as $V(z) = \frac{1}{2}z^2$.

The classical case of dimension 4 (Riemannian or not) is contained in these results as follows.

Corollary 2.10. *Any* 4*-dimensional Einstein space admitting a non-homothetic conformal vector field V (or a non-trivial conformal mapping onto some other Einstein space) is one of the following:*

(1) *If* $\mathrm{grad}(\mathrm{div}V)$ *is not isotropic on any open subset then the space is of constant sectional curvature.*
(2) *If* $\mathrm{grad}(\mathrm{div}V)$ *is isotropic on an open subset then the space is Ricci flat and carries an isotropic and parallel vector field. If it is a Lorentzian manifold then this is also called a* vacuum *pp*-wave.

Obviously the intersection of these two cases (with two distinct vector fields on the same space) consists of flat spaces only.

This local result is due to Brinkmann [6], it holds also globally. In particular it follows that under the same assumption a 4-dimensional Einstein space which is not Ricci flat must have constant curvature, a fact which was also observed in [18].

Corollary 2.11. *Any vacuum spacetime admitting a non-homothetic conformal vector field is either locally flat or it is locally a pp-wave, defined as follows.*

Definition 2.12. *The class of pp*-waves *(or plane-fronted waves) in general is given by all Lorentzian metrics g on open parts of $\mathbb{R}^4 = \{(u, v, x, y)\}$ which are of the form*

$$g = -2H(u, x, y)du^2 - 2dudv + dx^2 + dy^2$$

with an arbitrary function H, the potential, which does not depend on v. The subclass of plane waves *is given by all potentials H of the form*

$$H(u, x, y) = a(u)x^2 + 2b(u)xy + c(u)y^2.$$

A *pp*-wave is Ricci flat if and only if $H_{xx} + H_{yy} = 0$. A particular class is the class of polarized exact plane waves with a potential $H(u, x, y) = h(u)(x^2 - y^2)$ [7]. Isometric, homothetic and conformal vector fields of *pp*-waves were classified in a kind of a recursive normal form in [37], starting from the possible Killing fields. Furthermore it is well known that the isometry group is of codimension at most one in the homothety group, and that in turn the homothety group is of codimension at most one in the conformal group, compare [19]. The dimension of the conformal group is at most 7, see [20]. Moreover, all vacuum spacetimes admitting a 7-dimensional conformal group (together with the vector fields themselves) can be explicitly determined in terms of elementary functions and a finite number of parameters [30]. Moreover there is one family admitting a non-homothetic conformal vector field.

Example 2.13. A typical example of a non-homothetic conformal vector field on a *pp*-wave is the *standard special conformal vector field* (SCKV in [37])

$$Z_1 = u^2\partial_u + \tfrac{1}{2}(x^2+y^2)\partial_v + ux\partial_x + uy\partial_y$$

for the Ricci flat metric $g = -2u^{-4}(x^2-y^2)du^2 - 2dudv + dx^2 + dy^2$ which is isotropic and which is also conformal on the flat Minkowski space with the metric $g_0 = -2dudv + dx^2 + dy^2$, see [30]. The flow of Z_1 is explicitly given by

$$\Phi_t(u,v,x,y) = \tfrac{1}{1-2tu}\big(u, v(1-2tu) + t(x^2+y^2), x, y\big).$$

Any fixed trajectory is a straight line. However, the vector field is not complete since there is always a pole along the u-lines. Note that u is the divergence of the field. An attempt to visualize the flow can be found in [32].

An example of a *complete and Ricci flat pp-wave carrying a non-homothetic conformal vector field* is the modified metric

$$g = -2(u^2+1)^{-2}(x^2-y^2)du^2 - 2dudv + dx^2 + dy^2$$

with the modified vector field

$$V = Z_1 + \partial_u = (u^2+1)\partial_u + \tfrac{1}{2}(x^2+y^2)\partial_v + ux\partial_x + uy\partial_y,$$

see [30, Thm.1]. This metric is geodesically complete by [7, Prop.3.5].

The case of pseudo-Riemannian spaces of constant scalar curvature carrying non-isometric local gradient fields can also be classified, see [28, Thm.4.3]. In particular there are generalizations of Ejiri's compact example at the end of this article, all as warped product metrics. The possible warping functions can be explicitly determined.

There are many more examples of non-complete Einstein spaces carrying complete non-homothetic vector fields. As an example, on the standard sphere without north and south pole with the metric $g = dt^2 + (\sin t)^2 g_1$ one can replace the equatorial spheres with metric g_1 by any Einstein space (M_*, g_*) of the same Einstein constant as the unit sphere. Then the warped product metric $dt^2 + (\sin t)^2 g_*$ admits the same complete gradient field $V_1 = \sin t \cdot \partial_t$ as in Example 2.4 above. Particular cases use a non-standard Einstein metric on the sphere M_*, see [29]. This leads to an Einstein sphere with two isolated metrical singularities.

3. Conformal vector fields on Einstein spaces: Proofs

The proof of the results in Section 2 involves several steps as follows:

(1) If there is a non-homothetic conformal vector field V then there is also a non-trivial conformal gradient field. This in turn is characterized by the equation $(\nabla^2\sigma)^\circ = 0$ where $\sigma = \mathrm{div}V$.

(2) There is a local normal form for metrics admitting a non-trivial solution of $(\nabla^2\sigma)^\circ = 0$.
(3) If the manifold if complete and if gradσ is not isotropic on an open subset then it has to contain a global warped product of the type $\eta dt^2 + \sigma'^2(t)\, g_*$ where g_* is a complete Einstein metric on the level. Moreover, the function σ' has to satisfy a standard ODE of type $\sigma''^2 + \eta k \sigma'^2 = \sigma k_*$, and the only possibly solutions (as functions of t) are sin, cos, sinh, cosh and linear functins.
(4) If in addition the function σ has a critical point then the level (M_*, g_*) is isometric with the standard "sphere" of constant curvature, and around the critical point we have polar coordinates with t as the radius. Moreover, the space (M, g) is of constant sectional curvature. Consequently, in the compact case we have always a space of constant curvature, for Riemannian and pseudo-Riemannian metrics.
(5) In the cases without a critical point (essentially only $\sigma(t) = e^t$ and $\sigma(t) = \sinh t$) one has to decide whether the space can be complete and whether there can be a conformal mapping onto another complete manifold.
(6) If gradσ is isotropic on an open subset then similarly the manifold has to contain globally a Brinkmann space carrying a parallel and isotropic vector field, and the manifold must be Ricci flat. One has to decide whether this can be complete and whether there can be a complete conformal vector field which is non-homothetic.

Step 1: The conformal gradient field. For Einstein spaces with a conformal vector field V we have the special situation that the divergence $\sigma = \mathrm{div}V$ satisfies the differential equation $(\nabla^2\sigma)^\circ = 0$. This means that the gradient of σ is again conformal. This can be used for further discussions whenever σ is not constant, i.e., whenever V is non-homothetic.

Lemma 3.1. *The following formula holds for any conformal change $g \mapsto \overline{g} = \varphi^{-2} g$:*

(6) $$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = \varphi^{-2}\Big((n-2)\cdot\varphi\cdot\nabla^2\varphi + \Big[\varphi\cdot\Delta\varphi - (n-1)\cdot\|\mathrm{grad}\varphi\|^2\Big]\cdot g\Big).$$

Moreover, if V is a conformal vector field with $\mathcal{L}_V g = 2\sigma g$ then the formula

(7) $$\mathcal{L}_V \mathrm{Ric} = -(n-2)\nabla^2\sigma - \Delta\sigma\cdot g$$

holds and the following conditions are eqivalent:
(i) $\mathcal{L}_V\mathrm{Ric} = \mu g$ *for a certain function* μ
(ii) grad(div$V$) *is conformal*
(iii) $(\nabla^2\sigma)^\circ = 0$

Equation 6 follows from the relationship between the two Levi-Civita connections $\nabla, \overline{\nabla}$ associated with g and $\overline{g}$:

$$\overline{\nabla}_X Y - \nabla_X Y = -X(\log\varphi)Y - Y(\log\varphi)X + g(X,Y)\mathrm{grad}(\log\varphi).$$

For a proof in the Riemannian case see [26, p.107]. Equation 7 can be found in [42, p.160].

Corollary 3.2. *The Einstein property of a metric is in general not preserved under conformal changes. If g is an Einstein metric then the conformally transformed metric $\overline{g} = \varphi^{-2}g$ is Einstein if and only if*

$$(\nabla^2\varphi)^\circ = 0,$$

that is, if the Hessian of φ is a scalar multiple of the metric tensor.

This follows directly from Equation 6 and from the assumption $n \geq 3$.

Corollary 3.3. *Assume that an Einstein space carries a conformal vector field V which is not homothetic or isometric. Then it carries also a conformal gradient field, namely, the gradient of $\sigma = \frac{1}{n}\mathrm{div}V$. This function satisfies the equation $\nabla^2\sigma = -k\sigma g$ where k is the normalized Einstein constant. This gradient field does not vanish identically but it can happen that it is a parallel vector field, hence isometric.*

The proof follows from Equation 7 in connection with $n \geq 3$ since for Einstein spaces with $\mathrm{Ric} = \lambda g = (n-1)kg$ it reads as

$$2\lambda\sigma \cdot g = \mathcal{L}_V((n-1)kg) = \mathcal{L}_V\mathrm{Ric} = -(n-2)\nabla^2\sigma - \Delta\sigma \cdot g.$$

It follows that $\nabla^2\sigma$ must be some scalar multiple of g. From the trace of this equation we obtain $\Delta\sigma = -nk\sigma$.

Step 2: The local normal form of the metric. For any given smooth function f the equation $(\nabla^2 f)^\circ = 0$ or, equivalently, $\nabla^2 f = \frac{\Delta f}{n}g$ was already analyzed by Brinkmann [6] in the 1920's. He was the first who proved that in the case $g(\mathrm{grad}f, \mathrm{grad}f) \neq 0$ the metric g is a warped product. Furthermore, he proved that in the case $g(\mathrm{grad}f, \mathrm{grad}f) = 0$ the metric has a specific form carrying a parallel isotropic vector field (now called a Brinkmann space) which in dimension four became later important in physics as a *pp*-wave.

Theorem 3.4. (Brinkmann [6])
Assume that (M, g) is an Einstein space of dimension $n \geq 3$ admitting a non-constant solution f of the equation $(\nabla^2 f)^\circ = 0$. Then the following hold:

(1) *Around any point p with $g(\mathrm{grad}f(p), \mathrm{grad}f(p)) \neq 0$ the metric tensor is a warped product $g = \eta dt^2 + (f'(t))^2 g_*$ where $\mathrm{grad}f = f'\eta\partial_t$, $\eta = \pm 1$ and where the $(n-1)$-dimensional Einstein metric g_* does not depend*

on t. Moreover, f satisfies the ODE $f''' + k\eta f' = 0$ where k denotes the normalized Einstein constant.

(2) *If in addition $g(\mathrm{grad} f, \mathrm{grad} f) = 0$ on an open subset then $\mathrm{grad} f$ is a parallel isotropic vector field on that subset, and the metric tensor can be brought into the form $g = dudv + g_*(u)$ where $\mathrm{grad} f = \partial_u = \mathrm{grad} v$ and where the $(n-2)$-dimensional metric $g_*(u)$ is Ricci flat for any fixed u and does not depend on v. Consequently g itself must be Ricci flat. These coordinates u, v, x_i; $i = 1, 2, \ldots, n-2$ are sometimes called* Rosen coordinates.

The proof follows from Lemma 3.6 in connection with Lemma 3.5, as far as Part (1) is concerned. The particular form of the ODE above follows from the one in Lemma 3.5 by differentiation. The ODE above is nothing but the equation of the harmonic oscillator with standard solutions for $f'(t)$ such as $\sin t, \cos t, \sinh t, \cosh t$ or linear functions $at+b$. For Part (2) see the discussion of Case 6 (the isotropic case). In a local classification Kerckhove [24] listed the warped product case also with a higher-dimensional basis under an extra assumption on a symmetric bilinear form on the space of conformal vector fields.

Lemma 3.5. *The warped product $(I, \eta dt^2) \times_f (M_*, g_*)$ is an Einstein metric (a metric of constant sectional curvature) if and only if g_* is an Einstein metric (a metric of constant sectional curvature) and $f'^2 + k\eta f^2 = \eta k_*$ where k, k_* are the normalized Einstein constants of g, g_*.*

This follows from the formulae for warped products in general, cf. [40, ch.7]. For the Riemannian case see [26].

It turns out that one can integrate Equation 3 (without any additional curvature assumption) by reducing it to an ODE whenever the gradient of f is not isotropic. In particular this leads to the warped product metric. This step of the proof can be done along the lines of Brinkmann's results [6]. The following lemma was stated by Fialkow [14, p.471].

Lemma 3.6. *Let (M, g) be a pseudo–Riemannian manifold. Then the following conditions are equivalent:*

(1) *There is a non–constant solution f of the equation $\nabla^2 f = \frac{\Delta f}{n} g$ in a neighborhood of a point $p \in M$ with $g(\mathrm{grad} f(p), \mathrm{grad} f(p)) \neq 0$.*

(2) *There is a neighborhood U of p , a C^∞–function $f : (-\epsilon, \epsilon) \to \mathbb{R}$ with $f'(t) \neq 0$ for all $t \in (-\epsilon, \epsilon)$ and a pseudo–Riemannian manifold (M_*, g_*) such that (U, g) is isometric to the warped product*

$$\big((-\epsilon, \epsilon)\ ,\ \eta dt^2\big) \times_{f'} (M_*, g_*)) = \big((-\epsilon, \epsilon) \times M_*\ ,\ \eta dt^2 + (f'(t))^2 g_*\big)$$

where $\eta := \mathrm{sign} g(\mathrm{grad} f(p), \mathrm{grad} f(p)) \in \{\pm 1\}$.

Proof. (2) $\Rightarrow$ (1): Define the function f by $f(t,x) = f(t)$. Then we have $\mathrm{grad} f = f'\eta\partial_t$ and $\nabla_{\partial_t}\mathrm{grad} f = f''\eta\partial_t$. Let X be a lift of a vector field on M_*, then we have $\nabla_X \mathrm{grad} f = f''\eta X$ by standard computations in warped products [40].

(1) $\Rightarrow$ (2) : Let U be a neigborhood of $p \in M$ with compact closure and with $g\,(\mathrm{grad} f(q), \mathrm{grad} f(q)) \neq 0$ for all $q \in U$. Hence $c = f(p)$ is a regular value, let M_* be the connected component of $f^{-1}(c)$ containing p. Then there is an $\epsilon > 0$ such that the normal exponential map $\exp^\perp : (-\epsilon, \epsilon) \times M_* \to M$ defines a diffeomorphism onto the image. Let $q \in U$, $g(X, \mathrm{grad} f(q)) = 0$, then it follows immediately that

$$Xg(\mathrm{grad} f, \mathrm{grad} f) = 2\tfrac{\Delta f}{n} g(\mathrm{grad} f, X) = 0\,. \tag{8}$$

Hence $g\,(\mathrm{grad} f, \mathrm{grad} f)$ is constant along the level hypersurfaces $f^{-1}(c')$ and the level hypersurfaces of f are parallel. Therefore they coincide with the t–levels and f can be regarded as a function only of t, written as $f(t,x) = f(t)$ by a slight abuse of notation and $\mathrm{grad} f(t,x) = f'(t)\cdot \eta\partial_t$ as well as

$$\nabla^2 f = 2f''\eta g = \tfrac{\Delta f}{n}\, g\,. \tag{9}$$

The equation $g(\partial_t, \partial_t) = \eta = \mathrm{sign} g\,(\mathrm{grad} f(p), \mathrm{grad} f(p))$ follows since each t-curve is a geodesic. Let X be a lift of a vector field on M_*, then $g(\partial_t, X) = 0$ by the Gauss Lemma. If X_1, X_2 are vectors tangential to M_* at x_0 and $X_i(t) = d\exp(t, x_0)(X_i), i = 1, 2$ then

$$\begin{aligned} \tfrac{d}{dt}|_{t=s} g(X_1, X_2)(t) &= \mathcal{L}_{\partial_t} g(X_1, X_2)(s) = \tfrac{\eta}{f'(s)}\mathcal{L}_{\mathrm{grad} f} g(X_1, X_2)(s) = \\ \tfrac{2\eta}{f'(s)} \nabla^2_{X_1(s), X_2(s)} f &= 2\tfrac{f''(s)}{f'(s)} g(X_1, X_2)(s)\,. \end{aligned}$$

It follows that the ODE $((f')^{-2} g(X_1, X_2))'(t) = 0$ is satisfied for any t. Hence we can define $g_*(X_1, X_2) = (f')^{-2} g(X_1, X_2)$ and use this metric g_* as a non–degenerate metric on the level hypersurface M_* since it is orthogonal to the time-like or space-like t-direction. $\square$

Step 3: The global warped product. If M is a complete Einstein manifold carrying a non-homothetic conformal vector field V then $\sigma = \frac{1}{n}\mathrm{div} V$ satisfies the equation $(\nabla^2\sigma)^\circ = 0$ by Step 1, and by Step 2 around any non-critical point of σ the manifold M contains a warped product part U of type $I \times_{\sigma'} M_*$ where M_* itself must be a complete Einstein space. Now we consider a maximal connected subset $U_0 = I_0 \times_{\sigma'} M_*$ of this type. There are two possible cases:

Case 1: $I_0 = \mathbb{R}$. Then we have a global warped product with a warping function σ' without a zero. However, it is not a trivial question whether this warped product metric on U_0 is complete. See Step 5 below.

Case 2: I_0 is bounded from above or from below. This can happen only if the function σ' runs into a zero at one of the ends of I_0 (or both). In this case the metric is not globally a warped product. Instead, there are exceptional points, namely, critical points of σ. However, these critical points are isolated according to the results of Step 4. Moreover in this case U_0 must be of constant sectional curvature since M_* must be of constant sectional curvature.

Step 4: Polar coordinates around a critical point. Near a regular point of a function f satisfying $\nabla^2 f = \lambda g$ the metric has the structure of a warped product, cf. Lemma 3.6. Around a critical point we can use geodesic polar coordinates and obtain the following.

Proposition 3.7. ([22], [27], in the Riemannian case [41])
Let (M, g) be a pseudo–Riemannian manifold with a non–constant solution f of the equation $\nabla^2 f = \lambda g$ for a function λ and with a critical point $p \in M$.

(1) (cf. [41], [26, Lemma 18] in the Riemannian case) *Then there are functions $f_\pm$ such that the metric in geodesic polar coordinates $(r, x) \subset \mathbb{R} \times \Sigma$ in a neighborhood U of p has the form*

$$g(r, x) = \eta dr^2 + \frac{f_\eta'(r)^2}{f_\eta''(0)^2}\, g_1(x)\,;\; \eta = g\,(x, x) \tag{10}$$

and $f(r, x) = f_\eta(r), \lambda(r, x) = \lambda_\eta(r)$ with $\lambda_\eta(r) = \eta f''(r)$ and with the standard metric g_1 on a hyperquadric Σ of constant sectional curvature. In particular the metric is conformally flat in a neighborhood of the critical point.

(2) *If all geodesics through p are defined on the whole real line $\mathbb{R}$ then the metric g is of the form above for all (r, x), as long as $f_\eta'(r)$ does not vanish.*

The proof follows by considering the two warped products near the critical point in timelike directions ($\eta = -1$) and in spacelike directions ($\eta = 1$). Along the lightlike directions they have to fit together smoothly. This implies the conditions above. For the details of the proof see [27].

Proposition 3.8. [22, Prop.2] [27]
Let V be a non–trivial closed conformal vector field on the n–dimensional pseudo–Riemannian manifold (M, g).

(1) *If $V(p) = 0$, then* $\mathrm{div} V(p) = n \cdot \lambda(p) \neq 0$, *in particular all zeros of V are isolated.*

(2) *Denote by $\mathcal{C} = \mathcal{C}(M, g)$ the vector space of closed conformal vector fields, then* $\dim \mathcal{C} \leq n + 1$.

Part (1) follows from 3.7 above. Part (2) is well known, as well as the following additional statement: If the dimension of the space of closed conformal vector fields is maximal, i.e., if $\dim C(M,g) = n+1$, then the manifold is of constant sectional curvature.

Corollary 3.9. *Any Einstein space with a warped product metric* $(I, \eta dt^2) \times_f (M_*, g_*)$ *is of constant sectional curvature if the function* f *has a zero on the interval* I.

This follows from 3.5 in connection with 3.7 since around the critical point the level M_* must be a sphere (or hyperquadric in pseudo-Euclidean space) of constant sectional curvature.

Step 5: Complete solutions without a critical point. Let us assume that there is a global solution of Equation 3 on an Einstein space. Up to scaling, the only cases without a critical point have to contain the warped product $\mathbb{R} \times M_*$ with the metric

$$\pm dt^2 + e^{2t} g_* \quad \text{or} \quad \pm dt^2 + (\cosh t)^2 g_*,$$

respectively. Here (M_*, g_*) has to be a complete Einstein space with an Einstein constant $k_* = 0$ or $k_* = \pm 1$, respectively. In the second case this warped product is always geodesically complete if M_* is. However, in the first case it is geodesically complete only if the metric is definite. Otherwise there is a null geodesic $\gamma(s) = (\log s, c(s))$ in $\mathbb{R} \times M_*$ whose natural parameter s cannot go beyond 0, see [40, p.209]. In this case it is the question whether the warped product $\mathbb{R} \times_{e^t} M_*$ can be part of a complete manifold M which contains a point $p = \lim_{s \to 0} \gamma(s)$.

Let us assume that a non-homothetic conformal vector field V and its divergence are defined on M. In particular they are defined in an open neighborhood of the limit point p. From the flow of $\mathrm{grad}(\frac{1}{n}\mathrm{div}V) = e^t \partial_t$ we see that the flow runs into a fixed point for $t \to -\infty$. This follows form the explicit formula

$$\Phi_\tau(t,x) = \big(-\log(e^{-t} - \tau), x \big)$$

for the flow Φ_τ of $e^t\partial_t$ with the property $\Phi_{\tau+\sigma} = \Phi_\tau \circ \Phi_\sigma$ whenever this is defined. This is independent of the metric. For the specific null geodesic $\gamma(s)$ above we obtain

$$\Phi_\tau(\gamma(s)) = \big(-\log\big(\tfrac{1}{s} - \tau\big), \tfrac{1}{s} \big)$$

and see that $\Phi_\tau(p) = \Phi_\tau(\lim_{s \to 0} \gamma(s)) = \lim_{s \to 0} \Phi_\tau(\gamma(s)) = p$ for an open interval in τ. It follows that $\mathrm{grad}(\mathrm{div}V)(p) = 0$ and $\mathrm{div}V(p) = \lim_{s\to 0} \mathrm{div}V(\gamma(s)) = \lim_{s \to 0} n e^{\log s} = 0$. This means that p is a critical point of a conformal gradient field, and moreover $\mathrm{div}(\mathrm{grad}\sigma)(p) = \lim_{t \to -\infty} n e^t = 0$, in contradiction with the results in Step 4 above. By Proposition 3.8 the divergence at a critical point cannot vanish. Therefore such a conformal vector field V on M cannot exist. Consequently, this case of a warped product $\mathbb{R} \times_{e^t} M_*$ can occur only in the Riemannian case if the manifold is assumed to be complete.

Step 6: The isotropic case. There remains a discussion of the case of a conformal gradient field which is isotropic or null on an open set. It was called the *improper case* in [6]. Here we have the following:

Theorem 3.10. (Brinkmann [6], Catalano [8])
Assume that (M, g) is a pseudo-Riemannian manifold of dimension $n \geq 3$ admitting a non-vanishing and isotropic conformal gradient field, i.e., a non-constant solution f of Equation 3 such that $\mathrm{grad} f$ *is isotropic on an open subset. Then* $\mathrm{grad} f$ *is in addition parallel, and the metric tensor can be brought into the form $g = 2dudv + g_*(u)$ where* $\mathrm{grad} f = \partial_v = \mathrm{grad} u$ *and where the $(n-2)$-dimensional metric $g_*(u)$ does not depend on v. If in addition (M, g) is Einstein then it is Ricci flat.*

Such spaces carrying a parallel isotropic vector field are often called Brinkmann spaces. The transition from a non-isotropic gradient to an isotropic one is further explained in [8]. It corresponds to passing to the limit $\alpha \to 0$ in the metric $g = -\alpha(u)du^2 + 2dudv + g_*(u)$.

Sketch of proof: By assumption we have $\nabla^2 f = \lambda g$ and $g(\mathrm{grad} f, \mathrm{grad} f) = 0$, hence $0 = \nabla_X(g(\mathrm{grad} f, \mathrm{grad} f)) = 2g(\nabla_X \mathrm{grad} f, \mathrm{grad} f) = 2\lambda g(X, \mathrm{grad} f)$ for any X. This implies $\lambda = 0$ and therefore $\nabla_X \mathrm{grad} f = 0$ for any X, so $\mathrm{grad} f$ is parallel. If we use the function f as a coordinate u then the metric can be brought into the form above, see [8]. By $\partial_v = \mathrm{grad} u$ the metric does not depend on v since $\mathrm{grad} u$ is parallel. From $\nabla_X \partial_v = 0$ one gets $R(X, Y)\partial_v = 0$ for any X, Y. It follows that $\lambda = \mathrm{Ric}(\partial_u, \partial_v) = 0$ if $\mathrm{Ric} = \lambda g$. □

In the isotropic case a generalized Liouville theorem was obtained in [31]. This concerns the possible conformal mappings preserving the Ricci tensor. In particular this assumption is satisfied for any conformal transformation of a Ricci flat space.

Proof of the results in Section 2.
Proof of Theorem 2.7: From Step 1 we obtain that $\sigma = \frac{1}{n}\mathrm{div} V$ is a non-constant function such that $\mathrm{grad}\sigma$ is a conformal gradient field. From Step 2 we see the local normal form of the metric as $g = dt^2 + (\sigma'(t))^2 g_*$ with a warping function $\sigma' = \frac{d\sigma}{dt}$ where the parameter t is the arc length along the trajectories of $\mathrm{grad}\sigma$. Moreover, from Lemma 3.5 we obtain the ODE $\sigma''^2 + k\sigma'^2 = k_*$ where k, k_* denote the normalized Einstein constants of M and M_*, respectively.

For $k \neq 0$ the solutions are $\sigma'(t) = a\cos t + b\sin t$ if $k = 1$ and $\sigma'(t) = a\cosh t + b\sinh t$ if $k = -1$, up to scaling of the metric. By Step 3 the warped product is global unless the function σ' has a zero (a critical point of $\mathrm{div}V$) which implies that the space is of constant sectional curvature. The global cases without a zero are the cases $a^2 \geq b^2$ for $k = -1$. Up to a change of the parameter we obtain only the cases $\sigma'(t) = e^t$ and $\sigma'(t) = \cosh t$. Here (M_*, g_*) must be complete and Einstein with $k_* = 0$ in the first case and $k_* = -1$ in the second case. In all other cases σ' has a zero, and by Step 4 the given space (M, g) is of constant sectional curvature $k \neq 0$. Moreover, starting from the zero the warped product implies that the space is simply connected and hence isometric with the sphere or with the hyperbolic space.

It remains to discuss the Ricci flat case $k = 0$ with $\sigma''^2 = k_*$, hence $\sigma'(t) = at+b$ is linear. From Corollary 3.3 we get $\nabla^2\sigma = 0$, hence $a = 0$. Therefore $\sigma' = b$ is constant with $b \neq 0$, and σ has no critical point. Up to scaling, we can assume $b = 1$ and $\sigma(t) = t$. Consequently, the gradient of σ is ∂_t which is parallel since the t-lines are geodesics. It follows that (M, g) splits as a Riemannian product $(\mathbb{R} \times M_*, dt^2 + g_*)$ where (M_*, g_*) is complete. Then V splits as an orthogonal sum $V = \alpha\partial_t + V_*$ with a vector field V_* on (M_*, g_*) for any fixed t which is not identically zero for some fixed t_0 and with a function α defined on M. In a first step from $2t = (\mathcal{L}_V g)(\partial_t, \partial_t)$ one obtains the equation $t = \frac{\partial\alpha}{\partial t}$, hence $\beta := \alpha - \frac{t^2}{2}$ is a certain function on M_* which is independent of t. In a second step from $0 = (\mathcal{L}_V g)(\partial_t, X) = g(\nabla_t V_*, X) + \nabla_X\alpha = g_*(\nabla_t V_*, X) + g_*(\mathrm{grad}\beta, X)$ for arbitrary X tangential to M_* it follows that $\nabla_t V_*$ is a gradient field. By $g(\nabla_t V_*, \partial_t) = 0$ this is tangential to M_* as well. Finally by a straightforward calculation it follows that $\mathcal{L}_{\nabla_t V_*} g_* = 2g_*$, so $\nabla_t V_*$ is homothetic on (M_*, g_*). On the other hand it is well known [41, Thm.2] that a nontrivial homothetic gradient field on a complete Riemannian manifold has precisely one zero, and the manifold is isometric with the Euclidean space. Compare [25, p.242] for the flatness of the space even if the homothetic vector field is not a gradient. This implies that M_* is isometric with the Euclidean space. Hence $M = \mathbb{R} \times M_*$ is Euclidean as well. Compare Example 2.9 where $\nabla_t V_*$ is nothing but the position vector on $\mathbb{R}^{n-1}$. □

Proof of Theorem 2.8: Again from Step 1 we obtain that $\sigma = \frac{1}{n}\mathrm{div}V$ is a non-constant function such that $\mathrm{grad}\sigma$ is a conformal gradient field. From Step 2 we see the local normal form of the metric as $g = \eta dt^2 + (\sigma'(t))^2\, g_*$ with a warping function $\sigma' = \frac{d\sigma}{dt}$. Moreover, from Lemma 3.5 we obtain the ODE $\sigma''^2 + \eta k \sigma'^2 = \eta k_*$ where k, k_* denote the normalized Einstein constants of M and M_*, respectively.

For $k \neq 0$ the solutions are $\sigma'(t) = a\cos t + b\sin t$ if $\eta k = 1$ and $\sigma'(t) = a\cosh t + b\sinh t$ if $\eta k = -1$, up to scaling ot the metric. By Step 3 the warped product is global unless the function σ' has a zero (a critical point of div$V$). The global cases without a zero are the cases $a^2 \geq b^2$ for $\eta k = -1$. Again this can be reduced to the cases $\sigma'(t) = e^t$ and $\sigma'(t) = \cosh t$. The first case $\sigma'(t) = e^t$ can be excluded by Step 5 above (this part was also claimed in [28, p.242] but the proof was rather short). There remains only the second case $\sigma'(t) = \cosh t$ with a complete Einstein space (M_*, g_*) with $k_* = -1$. In all other cases σ' has a zero, and by Step 4 the given space (M, g) is of constant sectional curvature around the zero where one of the functions λ_η in Proposition 3.7 is $\lambda_\eta(t) = \sin t$, the other one is $\lambda_\eta(t) = \sinh t$. This implies that there are only two critical points and the function σ is a height function on a standard hyperquadric in some pseudo-Euclidean space [40, p.108], hence (M, g) is isometric with S^n_j or H^n_j or, if $j = 1$ or $j = n-1$, to a covering of it.

It remains to discuss the Ricci flat case $k = 0$ with $\sigma''^2 = k_*$, hence $\sigma'(t) = at+b$ is linear. From Corollary 3.3 we get $\nabla^2\sigma = 0$, hence $a = 0$. It follows that the function $\|\mathrm{grad}\sigma\|^2$ is constant and that σ has no critical point. Here an extra case distinction comes in, called "proper" and "improper" by Brinkmann [6].

<u>The proper case:</u> $\|\mathrm{grad}\sigma\|^2 = \pm b^2 \neq 0$. By scaling we can assume that $b = 1$ and $\sigma(t) = t$. It follows that gradσ is a parallel vector field $\pm\partial_t$. Hence (M, g) splits as a pseudo-Riemannian product $(\mathbb{R} \times M_*, \pm dt^2 + g_*)$ where (M_*, g_*) is complete. Then V splits as an orthogonal sum $V = \alpha\partial_t + V_*$ with a vector field V_* on (M_*, g_*) which is not identically zero for some fixed t_0. As above in the proof of Theorem 2.7 we obtain that $\nabla_t V_*$ is a homothetic gradient field on M_*. Then Theorem 3 in [22] implies that (M_*, g_*) is isometric with the pseudo-Euclidean space. Consequently the same holds for (M, g). See Example 2.9 for an example in this particular case.

<u>The improper case:</u> $\|\mathrm{grad}\sigma\|^2 = 0$ but grad$\sigma \neq 0$. It follows that gradσ is a parallel isotropic vector field. Hence (M, g) is a Ricci flat Brinkmann space. □

Proof of Theorem 2.5 and Theorem 2.6:

We prove only the stronger version in Theorem 2.6. By assumption there is a conformal diffeomorphism $F : (M_1, g_1) \to (M_2, g_2)$. We may consider two complete metrics g_1 and F^*g_2 on one manifold M_1 such that the conformal factor φ in the equation $F^*g_2 = \varphi^{-2}g_1$ is a global and non-constant function which, therefore, does not have a zero. By Corollary 3.2 the equation $(\nabla^2\varphi)^\circ = 0$ is satisfied. From Theorem 3.4 we obtain the local form of the metric g_1 as the warped product $g = dt^2 + (\varphi'(t))^2 g_*$ with a complete Einstein space (M_*, g_*) and with $\varphi''' + k\varphi' = 0$. Up to scaling there are only the cases $k = \pm 1$ and $k = 0$, so the function is $\varphi'(t) = a\sinh t + b\cosh t$ if $k = -1$ or $\varphi'(t) = a\sin t + b\cos t$ if $k = 1$ or $\varphi'(t) = at + b$ if $k = 0$.

If φ' has a zero then by Corollary 3.9 (M, g_1) is a space of constant sectional curvature since (M_*, g_*) is isometric with the standard sphere. Moreover, in this case we have $\varphi'(t) = \sinh t$ or $\varphi'(t) = \sin t$ or $\varphi'(t) = t$ if the critical point corresponds to the parameter $t = 0$. In the first case we have the metric of the hyperbolic space in polar coordinates (t, x) with $t \in [0, \infty)$. This metric g_1 is complete. However, the conformally transformed metric $\varphi^{-2} g_1$ would not be complete in this case, a contradiction. In the last case we have the metric of the Euclidean space in polar coordinates. However, the conformally transformed metric $\varphi^{-2} g_1$ would not be complete since the t-geodesics would not be complete if $\varphi(t) = \frac{1}{2}t^2 + c$ for a constant $c > 0$. In the second case we obtain the standard sphere in polar coordinates. This case really occurs with a conformal factor $\varphi(t) = -\cos t + c$ for any constant c with $|c| > 1$. The metrics g_1 and $g_2 = \varphi^{-2} g_1$ have constant sectional curvatures $k_1 = 1$ and $k_2 = c^2 - 1$, respectively. By scaling of g_2 one obtains a 1-parameter family of conformal transformation $g_1 \mapsto \frac{c^2-1}{(c-\cos t)^2} g_1$ of the unit sphere onto itself, depending on $c \in (1, \infty)$.

If there is no zero of φ' along the entire t-axis then we have either $k = -1$ and $\varphi'(t) = e^t$ or $\varphi'(t) = \cosh t$ (up to shift of the parameter) or we have $k = 0$ and φ' is constant. If φ' is constant or if $\varphi'(t) = \cosh t$ then the function φ would have a zero, a contradiction. In the last case we obtain the global warped products $\mathbb{R} \times M_*$ with the metric $g_1 = dt^2 + e^{2t} g_*$. This is complete if M_* is complete. However, the conformally transformed metric $\varphi^{-2} g_1$ would not be complete, a contradiction. This last case shows that without the assumption of the completeness of $\varphi^{-2} g_1$ the theorem would not be true. □

Remark: A verification of Equation 6 for $\varphi(t) = c - \cos t$, defined on the unit sphere, leads to the following expressions: From the equation $\varphi = c - \varphi''$ one obtains $\|\mathrm{grad}\varphi\|^2 = \varphi'^2$, $\Delta\varphi = n\varphi'' = n(c - \varphi)$ and $\nabla^2\varphi = (c - \varphi) g_1$ and

$$(n-2)\varphi\nabla^2\varphi + (\varphi\Delta\varphi - (n-1)\varphi'^2) g_1$$
$$= (n-1)(2\varphi''(c-\varphi'') - \varphi'^2) g_1 = -(n-1)(1 + \varphi''^2 - 2c\varphi'') g_1.$$

After multiplication with φ^{-2} we obtain

$$(n-1)((c^2-1) g_2 - g_1) = \mathrm{Ric}_2 - \mathrm{Ric}_1,$$

as expected. For $c^2 > 1$ the transformed metric g_2 is that of a round sphere, for $c^2 = 1$ a part of Euclidean space and for $c^2 < 1$ a part of hyperbolic space in polar coordinates. These cases may be called *elliptic, parabolic* and *hyperbolic*, respectively.

Proof of Theorem 2.1 and Theorem 2.2:
Again we prove here the stronger version Theorem 2.2. The Riemannian case was treated above. So we can assume that there is a conformal diffeomorphism $F : (M_1, g_1) \to (M_2, g_2)$ between two pseudo-Riemannian manifolds with an indefinite metric such that the conformal factor φ in the equation $F^* g_2 = \varphi^{-2} g_1$

is a global and non-constant function which does not have a zero. Again the equation $(\nabla^2\varphi)^\circ = 0$ is satisfied, and around any point with $\|\text{grad}\varphi\|^2 \neq 0$ we obtain the local form of the metric g_1 as the warped product $g = \eta dt^2 + (\varphi'(t))^2 g_*$ with an Einstein space (M_*, g_*) and with $\varphi''' + \eta k\varphi' = 0$. Up to scaling there are only the cases $k = \pm 1$ and $k = 0$, so the function is $\varphi'(t) = a\sinh t + b\cosh t$ if $\eta k = -1$ or $\varphi'(t) = a\sin t + b\cos t$ if $\eta k = 1$ or $\varphi'(t) = at + b$ if $k = 0$.

If φ' has a zero then by Corollary 3.9 (M, g_1) is a space of constant sectional curvature since (M_*, g_*) is isometric with a hyperquadric of constant sectional curvature (a "sphere"). Moreover, in this case we have $\varphi'(t) = \sinh t$ or $\varphi'(t) = \sin t$ or $\varphi'(t) = t$ if the critical point corresponds to the parameter $t = 0$. More precisely for $k \neq 0$ we have $\varphi'(t) = \sinh t$ in timelike [or spacelike] directions and $\varphi'(t) = \sin t$ in spacelike [or timelike] directions around the critical point. In any case there is a geodesic emanating from the critical point such that $\varphi'(t) = \sinh t$ along the geodesic where t is the arc length parameter. This implies that in the conformally transformed metric $g_2 = \varphi^{-2} g_1$ this geodesic is no longer complete, a contradiction. Hence the case of constant sectional curvature $k \neq 0$ cannot occur with a critical point of φ. In the case $k = 0$ we have the metric of the pseudo-Euclidean space in polar coordinates. However, the conformally transformed metric $\varphi^{-2} g_1$ would not be complete since the t-geodesics would not be complete if $\varphi(t) = \frac{1}{2}t^2 + c$ for a constant $c > 0$.

If there is no zero of φ' along the entire t-axis then we have either $\eta k = -1$ and $\varphi'(t) = e^t$ or $\varphi'(t) = \cosh t$ (up to shift of the parameter) or we have $k = 0$ and φ' is constant. If φ' is constant or if $\varphi'(t) = \cosh t$ then the function φ would have a zero, a contradiction. In the remaining case $\varphi'(t) = e^t$ we obtain the global warped products $\mathbb{R} \times M_*$ with the metric $g_1 = \eta dt^2 + e^{2t} g_*$. However, this case can be excluded by the same reasoning as in Step 5 above since in a limit point along the special null geodesic γ would have to be a critical point of φ.

It remains to discuss the improper case of an isotropic gradient with $\|\text{grad}\varphi\|^2 = 0$ but $\text{grad}\varphi \neq 0$ on an open subset. Then by Theorem 3.10 we have a Ricci flat Brinkmann space with a metric $g_1 = 2dudv + g_*(u)$. Moreover, the function φ coincides with u, up to shift of the parameter. The u-lines are null geodesics with their natural parameter since $\nabla_{\partial_u}\partial_u = 0$. However, the conformal factor φ has a zero along any u-line, a contradiction. Therefore this improper case cannot occur if g_1 is geodesically complete. □

Final remarks. In the case of a Riemannian manifold any conformal vector field without zeros can be made into an isometric vector field by a conformal change. Such a field is called inessential, otherwise it is essential. Since much is known about the isometry groups and Killing fields, it is here more interesting to study essential conformal vector fields, that is, conformal vector fields which never become isometric under a global conformal change of the metric. In the compact case it has been known as the *Lichnerowicz conjecture* [36]. Now the theorem states that *a Riemannian manifold of dimension* n *admitting a complete and essential conformal vector field is conformally diffeomorphic with either the standard sphere* S^n *or with the Euclidean space* $\mathbb{E}^n$*.* This was proved by Alekseevskii [1], Ferrand [12],[13] and Yoshimatsu [44], in the compact case also by Obata [39], Lelong-Ferrand [36], Lafontaine [35]. Several steps in the proof were made more precise in various papers, so the result cannot really be attributed to a single person. For a more recent and alternative proof see [17]. The case of a complete manifold carrying a complete and closed essential conformal vector field was solved by Bourguignon [4].

No analogous result seems to be known yet in the case of a pseudo-Riemannian manifold with an indefinite metric. It is the other part of the same *Lichnerowicz conjecture* that a compact and pseudo-Riemannian manifold carrying an essential conformal vector field is conformally flat [15]. So far it seems that in the case of an indefinite metric there is no example of a non-homothetic conformal vector field wih a zero.

The situation with respect to inessential conformal vector fields is totally different, even in the compact case and even under additional curvature restrictions.

Example 3.11. For any n there is a compact Riemannian n-manifold of constant scalar curvature admitting a conformal vector field without zeros. The simplest example of this kind for $n = 4$ is the product $S^1 \times S^3$ with the warped product metric $g = dt^2 + (2 + \cos t)g_1$ where g_1 is the standard metric on the unit sphere. In this case the closed vector field $V = \sqrt{2 + \cos t}\ \partial_t$ is conformal (and inessential), see [9, p.277]. There are similar examples $g = dt^2 + (f(t))^2 g_*$ in any dimension, with a periodic warping function f which can be explicitly given. It has to satisfy the ODE $nkf^2 + (n-2)f'^2 + 2ff'' = (n-2)k_*$ where k, k_* are the constant (normalized) scalar curvatures of g, g_*, respectively. These examples can be extended to the case of a pseudo-Riemannian metric, see [28].

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