REPRESENTATION AND APPROXIMATION OF POSITIVITY PRESERVERS

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Dedicated to the memory of Julius Borcea

Abstract. We consider a closed set $S \subseteq \mathbb{R}^n$ and a linear operator

$$\Phi\colon \mathbb{R}[X_1,\ldots,X_n] \to \mathbb{R}[X_1,\ldots,X_n]$$

that preserves nonnegative polynomials, in the following sense: if $f \geq 0$ on S, then $\Phi(f) \geq 0$ on S as well. We show that each such operator is given by integration with respect to a measure taking nonnegative functions as its values. This can be seen as a generalization of Haviland's Theorem, which concerns linear *functionals* on $\mathbb{R}[X_1,\ldots,X_n]$. For compact sets S we use the result to show that any nonnegativity preserving operator is a pointwise limit of very simple nonnegativity preservers with finite dimensional range.

1. Introduction

Linear operators that preserve *hyperbolic* polynomials have already been studied a hundred years ago by Pólya and Schur [PoSc]. A univariate real polynomial p is called hyperbolic, if all of its roots are real, and a linear map $\Phi\colon \mathbb{R}[t] \to \mathbb{R}[t]$ is called a *hyperbolicity preserver*, if for any hyperbolic $p \in \mathbb{R}[t]$, the image $\Phi(p)$ is again hyperbolic. For example, simple differentiation $p \mapsto \frac{\partial}{\partial t}p$ is a hyperbolicity preserver, which follows from Rolle's Theorem.

In [GS1], the authors study ellipticity-, positivity- and nonnegativity-preserving operators on polynomial algebras. Although there is a huge amount of literature on positive operators in Banach lattices (see for example [AlBu]), results specific to the polynomial case seem to be rare.

Let $\mathbb{R}[\underline{X}] = \mathbb{R}[X_1,\ldots,X_n]$ be the real polynomial algebra in n variables. Let

$$\mathcal{N}(\mathbb{R}^n) := \{p \in \mathbb{R}[\underline{X}] \mid p(x) \geq 0 \text{ for all } x \in \mathbb{R}^n\}$$

denote the set of *globally nonnegative polynomials*, let

$$\mathcal{P}(\mathbb{R}^n) := \{p \in \mathbb{R}[\underline{X}] \mid p(x) > 0 \text{ for all } x \in \mathbb{R}^n\}$$

be the set of *globally positive polynomials*, and let

$$\mathcal{E}(\mathbb{R}^n) = \{p \in \mathbb{R}[\underline{X}] \mid p(x) \neq 0 \text{ for all } x \in \mathbb{R}^n\}$$

be the set of polynomials without real zeros, also called *elliptic polynomials.*

A linear map $\Phi\colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ is called *nonnegativity-*, *positivity-* or *ellipticity-preserving*, if $\Phi\left(\mathcal{N}(\mathbb{R}^n)\right) \subseteq \mathcal{N}(\mathbb{R}^n)$, $\Phi\left(\mathcal{P}(\mathbb{R}^n)\right) \subseteq \mathcal{P}(\mathbb{R}^n)$ or $\Phi\left(\mathcal{E}(\mathbb{R}^n)\right) \subseteq \mathcal{E}(\mathbb{R}^n)$

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holds, respectively. One wants to characterize and describe these kinds of operators as well as possible. It turns out that this question is closely related to both real algebra as well as to functional analysis.

Fundamental work towards a characterization of all nonnegativity-, positivity- and ellipticity-preservers is done in [GS1, GS2], and [Bo] then contains a full characterization of such operators, in terms of differential operator representations. So the problem can be considered as completed, where of course other kinds of characterizations would still be of interest.

In this work we consider operators that preserve polynomials which are positive, nonnegative or elliptic on a certain *subset* S of $\mathbb{R}^n$. All the results in the spirit of [Bo, GS1, GS2] that one might expect turn out to be false in the general case.

We therefore choose a different approach to the problem. Our first main result is an integral representation of general nonnegativity preservers. Haviland's Theorem says that every linear functional on $\mathbb{R}[\underline{X}]$ that maps S-nonnegative polynomials to nonnegative reals is always given by integration on S. A similar statement is true for S-nonnegativity preserving linear operators on $\mathbb{R}[\underline{X}]$. The occuring measures can of course not be real-valued in general; they take certain nonnegative functions as their values instead. Our main integral representation result is Theorem 5.2 below. It applies to any closed set S and any S-nonnegativity preserver.

In the case of a compact set S, the integral representation can be strengthened (Theorem 6.3). Compared to the standard integral representation results for operators in $C(S)$ (as in [DuSc]), it is different in the sense that it does not assume compactness or weak compactness of the operator; it uses its nonnegativity instead. The occuring measures can be used to check whether an S-nonnegativity preserver is of finite dimensional range (Theorem 6.4), compact (Theorem 6.5) or weakly compact (Theorem 6.6). The last two results are standard from the representation theory of operators on $C(S)$, where the first one is specific to nonnegative operators on $\mathbb{R}[\underline{X}]$.

In the last section we propose another possible solution to the classification problem of nonnegativity preservers. The idea is to first provide a class of standard operators that preserve nonnegativity, and then check which operators can be approximated by these operators in a suitable sense. In the case of a compact set S, Theorem 7.1 below is such an approximation result.

2. Preliminaries and known results

Let $S \subseteq \mathbb{R}^n$ be a set. In analogy with the previous case, let

$$\begin{aligned}\mathcal{N}(S) &= \{p \in \mathbb{R}[\underline{X}] \mid p(x) \geq 0 \text{ for all } x \in S\}\\ \mathcal{P}(S) &= \{p \in \mathbb{R}[\underline{X}] \mid p(x) > 0 \text{ for all } x \in S\}\\ \mathcal{E}(S) &= \{p \in \mathbb{R}[\underline{X}] \mid p(x) \neq 0 \text{ for all } x \in S\}\end{aligned}$$

denote the set of polynomials that are nonnegative, positive and elliptic on S, respectively. We clearly have $\mathcal{P}(S) \subseteq \mathcal{N}(S)$ and $\pm\mathcal{P}(S) \subseteq \mathcal{E}(S)$. If S is connected, then $\mathcal{E}(S) = \mathcal{P}(S) \cup -\mathcal{P}(S)$.

A linear map (also called an operator) $\Phi \colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ is called an *S-nonnegativity preserver*, if $\Phi\left(\mathcal{N}(S)\right) \subseteq \mathcal{N}(S)$. It is called an *$S$-positivity preserver* if $\Phi\left(\mathcal{P}(S)\right) \subseteq \mathcal{P}(S)$, and an *$S$-ellipticity preserver* if $\Phi\left(\mathcal{E}(S)\right) \subseteq \mathcal{E}(S)$ holds.

We say that S is *Zariski dense*, if it is not contained in the zero set of a polynomial from $\mathbb{R}[\underline{X}] \setminus \{0\}$. In that case, any two different polynomials define different functions on S. We will assume the Zariski denseness of S most of the time.

The following result is a generalized combination of Lemma 2.1, Theorem 2.3 and Theorem 2.5 from [GS1]. We include a short alternative proof.

Proposition 2.1. *(i) Each S-positivity preserver is an S-nonnegativity preserver.*

(ii) Let $\Phi\colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ be an S-nonnegativity preserver. Assume that $\Phi(1)(x) = 0$ for some $x \in S$. Then $\Phi(p)(x) = 0$ for all $p \in \mathbb{R}[\underline{X}]$. In particular, if S is Zariski dense, then $\Phi(1) = 0$ implies $\Phi = 0$.

(iii) If either S is compact, or $n = 1$ and $S \subseteq \mathbb{R}$ is closed, then for any S-nonnegativity preserver Φ and any $p \in \mathcal{P}(S)$, the zero locus of $\Phi(p)$ in S equals the zero locus of $\Phi(1)$ in S. In particular, Φ is an S-positivity preserver if and only if $\Phi(1) > 0$ on S in that case.

(iv) If S is connected, then Φ is an S-ellipticity preserver if and only if Φ or $-\Phi$ is an S-positivity preserver.

Proof. (i) is proven exactly as in [GS1], Theorem 2.1. Namely, if Φ is S-positivity preserving and $p \geq 0$ on S, then for all $\varepsilon > 0$,

$$\Phi(p) + \varepsilon\Phi(1) = \Phi(p + \varepsilon) > 0 \text{ on } S,$$

so $\Phi(p) \geq 0$ on S.

For (ii) let $p \in \mathbb{R}[\underline{X}]$ be arbitrary and note that

$$0 \leq \Phi\left((p + \lambda)^2\right)(x) = \Phi(p^2)(x) + 2\lambda\Phi(p)(x)$$

holds for any $\lambda \in \mathbb{R}$. So clearly $\Phi(p)(x) = 0$. Now if $\Phi(1) = 0$, then $\Phi(p)(x) = 0$ for all p and all $x \in S$. The Zariski denseness of S then implies $\Phi = 0$.

(iii) is again proven similar to [GS1], Theorem 2.5: If $p > 0$ on S, then $p \geq \varepsilon$ on S for some suitable $\varepsilon > 0$ (since S is compact or $S \subseteq \mathbb{R}$ closed). So

$$\Phi(p) - \varepsilon\Phi(1) = \Phi(p - \varepsilon) \geq 0 \text{ on } S,$$

for any S-nonnegativity preserver Φ. So the zero locus of $\Phi(p)$ in S is contained in the zero locus of $\Phi(1)$ in S. Equality follows from (ii).

(iv) Let S be connected and let Φ be S-ellipticity preserving. $\Phi(1)$ has no zeros in S, so suppose $\Phi(1) > 0$ on S (otherwise replace Φ by $-\Phi$). Now let $p \in \mathbb{R}[\underline{X}]$ be strictly positive on S. Then for any $\lambda \in [0, 1]$, the polynomial $\lambda p + (1 - \lambda)$ does not have zeros in S, so

$$\Phi(\lambda p + (1 - \lambda)) = \lambda\Phi(p) + (1 - \lambda)\Phi(1)$$

does not have zeros in S, for any $\lambda \in [0, 1]$. As $\Phi(1) > 0$ on S, this is clearly only possible if also $\Phi(p) > 0$ on S. So Φ is S-positivity preserving. The other direction follows immediately from $\mathcal{E}(S) = \mathcal{P}(S) \cup -\mathcal{P}(S)$. $\square$

So in view of Proposition 2.1, we can restrict ourself to examining S-nonnegativity preservers, at least in the case $S \subseteq \mathbb{R}$ closed or S compact.

Before we can describe the main results from [Bo, GS1, GS2], we introduce certain classes of linear operators on $\mathbb{R}[\underline{X}]$. Therefore let always $\mathbb{N} = \{0, 1, 2, \ldots\}$. For any $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, let $\alpha! := \alpha_1! \cdots \alpha_n!$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, write $\alpha \preceq \beta$ if $\alpha_i \leq \beta_i$ for all i, and define $\beta - \alpha := (\beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n)$. We also use the notation $\underline{X}^\alpha$ for $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

Here is a list of certain kinds of linear operators on $\mathbb{R}[\underline{X}]$:

Example 2.2.

1. For $\alpha \in \mathbb{N}^n$, let D^α denote the corresponding differential operator, i.e. the linear operator that sends $\underline{X}^\beta$ to $\frac{\beta!}{(\beta-\alpha)!} \cdot \underline{X}^{\beta-\alpha}$ for $\alpha \preceq \beta$, and to 0 otherwise.
2. Let $f \in \mathbb{R}[\underline{X}]$ be a fixed polynomial, then multiplication by f is a linear operator, denoted by M_f. Clearly, M_f is S-nonnegativity preserving if and only if $f \in \mathcal{N}(S)$, and S-positivity preserving if and only if $f \in \mathcal{P}(S)$.
3. Let $f_1, \ldots, f_n \in \mathbb{R}[\underline{X}]$ be fixed polynomials, and consider the operator $E_{\underline{f}} \colon p \mapsto p(f_1, \ldots, f_n)$. Such an operator is even multiplicative, i.e. it is an $\mathbb{R}$-algebra endomorphism of $\mathbb{R}[\underline{X}]$. Each algebra endomorphism is of that form. To see whether $E_{\underline{f}}$ is S-nonnegativity or S-positivity preserving, consider the corresponding polynomial map
$$\underline{f} \colon \mathbb{R}^n \to \mathbb{R}^n; x \mapsto (f_1(x), \ldots, f_n(x)).$$
One checks that $E_{\underline{f}}$ is S-nonnegativity preserving if and only if $\underline{f}\,(S) \subseteq \overline{S}$, and $E_{\underline{f}}$ is S-positivity preserving if and only if $\underline{f}\,(S) \subseteq S$.
4. A special class of operators is the following: for polynomials $f_1, \ldots, f_r$ and points $x_1, \ldots, x_r \in \mathbb{R}^n$ consider
$$\Phi_{\underline{f},\underline{x}} \colon p \mapsto f_1 \cdot p(x_1) + \cdots + f_r \cdot p(x_r).$$
This operator has a finite dimensional range; in case that the x_i are pairwise disjoint, its range equals the subspace of $\mathbb{R}[\underline{X}]$ spanned by $f_1, \ldots, f_r$. If all f_i are nonnegative on S and all $x_i \in S$, the operator is clearly S-nonnegativity preserving. These operators will be used in the approximation result below.
5. A generalization of (4) is the following. Let $L_1, \ldots, L_r$ be linear functionals on $\mathbb{R}[\underline{X}]$ and $f_1, \ldots, f_r \in \mathbb{R}[\underline{X}]$. Then
$$\Phi_{\underline{f},\underline{L}} \colon p \mapsto \sum_{i=1}^{r} f_i \cdot L_i(p)$$
is a linear operator on $\mathbb{R}[\underline{X}]$. It also has a finite dimensional range; in case that all the L_i are linearly independent, it is the subspace spanned by the f_i. Each linear operator with finite dimensional range is of this form (see for example in the proof of Theorem 6.4). If all $f_i \geq 0$ on S and all L_i map S-nonnegative polynomials to $[0, \infty)$, this operator is S-nonnegativity preserving; however, it should be noted that this is not necessary (see open problem (4) below).
6. Any two linear operators Φ, Ψ on $\mathbb{R}[\underline{X}]$ can be summed up and composed, i.e. one can consider $\Phi + \Psi \colon p \mapsto \Phi(p) + \Psi(p)$ and $\Phi \circ \Psi \colon p \mapsto \Phi\left(\Psi(p)\right)$. The sum and composition of two S-nonnegativity or S-positivity preservers is again an S-nonnegativity or S-positivity preserver, respectively.
7. Let $(\Phi_i)_{i \in I}$ be a family of linear operators on $\mathbb{R}[\underline{X}]$, such that for any polynomial $p \in \mathbb{R}[\underline{X}]$ only finitely many of the values $\Phi_i(p)$ are not zero. Then
$$\sum_{i \in I} \Phi_i \colon p \mapsto \sum_{i \in I} \Phi_i(p)$$
is a well defined linear operator. For example, with a (multi-)sequence $(q_\alpha)_{\alpha \in \mathbb{N}^n}$ of polynomials we can define the operator
$$\sum_{\alpha \in \mathbb{N}^n} q_\alpha D^\alpha \colon p \mapsto \sum_{\alpha} q_\alpha \cdot D^\alpha(p).$$

> Such an operator is called a *differential operator with polynomial coefficients*. If only finitely many of the polynomials q_α are non-zero, then the operator is called *of finite order*; otherwise it is called *of infinite order*. If the q_α are all real numbers, then the operator is called a *differential operator with constant coefficients*.

The following fact is folklore, we include a short proof for the sake of completeness:

Lemma 2.3. *Every linear operator on* $\mathbb{R}[\underline{X}]$ *is a differential operator with polynomial coefficients. The corresponding multisequence* $(q_\alpha)_{\alpha\in\mathbb{N}^n}$ *of polynomial coefficients is unique.*

Proof. Let Φ be a linear operator on $\mathbb{R}[\underline{X}]$ and set $p_\beta := \Phi(\underline{X}^\beta)$ for any $\beta \in \mathbb{N}^n$. We have to find polynomials q_α such that $p_\beta = \sum_{\alpha\preceq\beta} \frac{\beta!}{(\beta-\alpha)!} q_\alpha \underline{X}^{\beta-\alpha}$ for all β. This is done by induction on $|\alpha|$. We first choose $q_0 = p_0$. Then for any $\beta \neq 0$, from

$$p_\beta = \sum_{\alpha\preceq\beta} \frac{\beta!}{(\beta-\alpha)!} q_\alpha \underline{X}^{\beta-\alpha} = q_\beta \beta! + \sum_{\alpha\preceq\beta, |\alpha|<|\beta|} \frac{\beta!}{(\beta-\alpha)!} q_\alpha \underline{X}^{\beta-\alpha}$$

we deduce

$$q_\beta = \frac{1}{\beta!}\left(p_\beta - \sum_{\alpha\preceq\beta, |\alpha|<|\beta|} \frac{\beta!}{(\beta-\alpha)!} q_\alpha \underline{X}^{\beta-\alpha} \right).$$

The differential operator with coefficient sequence $(q_\alpha)_\alpha$, defined inductively by the above rule, then coincides with Φ, and there is clearly no other possible sequence of coefficients for Φ. □

Example 2.4. (i) The multiplication operator M_f defined above is already in the form of a differential operator, namely $M_f = f \cdot D^0$.

(ii) Consider the algebra endomorphism $E_{\underline{f}}$ as defined above. For any polynomial $p \in \mathbb{R}[\underline{X}]$ we have

$$p(f_1,\ldots,f_n) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} (f_1 - X_1)^{\alpha_1} \cdots (f_n - X_n)^{\alpha_n} \cdot D^\alpha(p). \tag{1}$$

This Taylor-formula is easily verified for monomials $p = \underline{X}^\beta$, and thus holds in general. Recall that the above sum is always finite, there is no convergence problem. So the sequence $(q_\alpha)_{\alpha\in\mathbb{N}^n}$ defined by

$$q_\alpha := \frac{1}{\alpha!} (f_1 - X_1)^{\alpha_1} \cdots (f_n - X_n)^{\alpha_n}$$

is the coefficient sequence for $E_{\underline{f}}$ in its representation as a differential operator.

Definition 2.5. A (multi-)sequence $(r_\alpha)_{\alpha\in\mathbb{N}^n}$ of real numbers is called a *moment sequence*, if there is a nonnegative Borel measure μ on $\mathbb{R}^n$ such that

$$r_\alpha = \int_{\mathbb{R}^n} \underline{X}^\alpha d\mu$$

holds for all $\alpha \in \mathbb{N}^n$.

The following Theorem sums up some of the most important results from [Bo, GS1, GS2]. It can be seen as a complete characterization of $\mathbb{R}^n$-nonnegativity preserving operators:

Theorem 2.6. *(i) A differential operator* $\Phi = \sum_{\alpha \in \mathbb{N}^n} r_\alpha D^\alpha$ *with constant coefficients is* $\mathbb{R}^n$*-nonnegativity preserving if and only if the sequence* $(\alpha! r_\alpha)_{\alpha \in \mathbb{N}^n}$ *is a moment sequence.*

(ii) A differential operator $\Phi = \sum_{\alpha \in \mathbb{N}^n} q_\alpha D^\alpha$ *with polynomial coefficients is* $\mathbb{R}^n$*-nonnegativity preserving if and only if for all* $a \in \mathbb{R}^n$ *the operator*

$$\Phi_a := \sum_{\alpha \in \mathbb{N}^n} q_\alpha(a) D^\alpha$$

is $\mathbb{R}^n$*-nonnegativity preserving (and (i) applies to each* Φ_a*).*

(iii) A differential operator of finite order (with constant or polynomial coefficients) can only be $\mathbb{R}^n$*-nonnegativity preserving if it is of order* 0*, i.e. if it is of the form* M_f *for some* $f \in \mathbb{R}[\underline{X}]$*.*

Part (i) is [Bo] Theorem 3.1, a special case is [GS1], Theorem 3.4 and also [GS2], Theorems A and B. Part (ii) is again [Bo], Theorem 3.1. Part (iii) follows from (i) and (ii), and was first proven in a constructive way in [GS1], Section 4.

So the problem of characterizing $\mathbb{R}^n$-nonnegativity preservers boils down to characterizing moment sequences. There is a huge amount of literature on the moment problem, we only refer to [Ak, Ham, Hav, Sm2] and the references therein. An important result is for example Hamburger's Theorem for the one dimensional case:

Theorem 2.7. *A sequence* $\mathbf{r} = (r_i)_{i \in \mathbb{N}}$ *is a moment sequence if and only if for all* $m \in \mathbb{N}$ *the matrix* $\mathcal{H}(\mathbf{r})_m := (r_{i+j})_{i,j=0}^m$ *is positive semidefinite.*

Note however that in the case of dimension $n \geq 2$, there is not such an easy characterization of moment sequences.

We will also need Haviland's Theorem in the following ([Hav], see also [Ma] for a proof):

Theorem 2.8 (Haviland). *Let* $S \subseteq \mathbb{R}^n$ *be a closed set and let* $L \colon \mathbb{R}[\underline{X}] \to \mathbb{R}$ *be a linear functional. Then there is a nonnegative Borel measure* μ *on* S *such that*

$$L(p) = \int_S p d\mu \text{ for all } p \in \mathbb{R}[\underline{X}]$$

if and only if $L(p) \geq 0$ *whenever* $p \geq 0$ *on* S*.*

Hamburger's Theorem follows for example from Haviland's Theorem and the fact that every nonnegative univariate polynomial is a sum of squares of polynomials.

3. Examples in the general case

In this section we consider possible generalizations of Theorem 2.6 to the case of arbitrary sets $S \subseteq \mathbb{R}^n$. So let $S \subseteq \mathbb{R}^n$ be arbitrary. An *S-moment sequence* is a sequence $(r_\alpha)_{\alpha \in \mathbb{N}^n}$ of real numbers, such that there exists a nonnegative Borel measure μ on S with

$$r_\alpha = \int_S \underline{X}^\alpha d\mu \quad \text{for all } \alpha \in \mathbb{N}^n.$$

The value $\int_S \underline{X}^\alpha d\mu$ is called the *(S-)moment of* μ *of order* α.

The natural guess how to generalize Theorem 2.6 would be to simply replace $\mathbb{R}^n$ by S each time. But only some parts remain true.

Lemma 3.1. *If* $0 \in S$ *and a differential operator* $\Phi = \sum_\alpha r_\alpha D^\alpha$ *with constant coefficients is* S*-nonnegativity preserving, then the sequence* $(\alpha! r_\alpha)_{\alpha\in\mathbb{N}^n}$ *is an* $\overline{S}$*-moment sequence.*

Proof. The proof is the same as in [Bo], Theorem 3.1. We consider the linear functional $p \mapsto \Phi(p)(0)$, which maps $\mathcal{N}(S)$ to $[0,\infty)$. So by Haviland's Theorem, there exists a measure μ on $\overline{S}$ such that

$$\Phi(p)(0) = \int_{\overline{S}} p d\mu$$

for all $p \in \mathbb{R}[X]$. In particular

$$\int \underline{X}^\beta d\mu = \Phi(\underline{X}^\beta)(0) = \sum_{\alpha\preceq\beta} r_\alpha \frac{\beta!}{(\beta-\alpha)!}\underline{X}^{\beta-\alpha}(0) = \beta! r_\beta$$

for all $\beta \in \mathbb{N}^n$. □

Of course, one can always ensure $0 \in S$ by switching to

$$\Phi' \colon p \mapsto \Phi\left(p(\underline{X}+a)\right)(\underline{X}-a)$$

for some suitable $a \in \mathbb{R}^n$, if $S \neq \emptyset$. However, without the assumption $0 \in S$, Lemma 3.1 fails:

Example 3.2. Let $n = 1$ and $S := [2,\infty) \subseteq \mathbb{R}$. The operator

$$E_{X+1} \colon p \mapsto p(X+1) = \sum_{i=0}^{\infty} \frac{1}{i!} D^i(p)$$

is even S-positivity preserving. But there is no Borel measure μ on $[2,\infty)$ such that

$$\int_2^\infty X^i d\mu = 1 \text{ for all } i \in \mathbb{N}.$$

Indeed from $1 = \int_2^\infty X^0 d\mu = \mu\left([2,\infty)\right)$ we would obtain

$$1 = \int_2^\infty X d\mu \geq 2 \cdot \mu\left([2,\infty)\right) = 2,$$

a contradiction.

The converse of Lemma 3.1 fails even if $0 \in S$:

Example 3.3. Let $S = [-1,0] \subseteq \mathbb{R}$ and let μ be the Lebesque-measure restricted to S. Define

$$r_i := \frac{1}{i!}\int_{-1}^0 X^i d\mu$$

for all $i \in \mathbb{N}$. Then for the linear operator Φ defined by $\Phi := \sum_{i=0}^\infty r_i D^i$ we have

$$\Phi(X+1)(-1) = \sum_{i=0}^{\infty} r_i D^i(X+1)(-1) = r_1 = \int_{-1}^0 X d\mu = -\frac{1}{2} < 0.$$

As $X + 1 \in \mathcal{N}(S)$, Φ is not S-nonnegativity preserving.

Now we turn to possible generalizations of (ii) in Theorem 2.6.

Lemma 3.4. *Let* $S \subseteq \mathbb{R}^n$ *and let* $\Phi = \sum_{\alpha\in\mathbb{N}^n} q_\alpha D^\alpha$ *be a differential operator with polynomial coefficients. Suppose* $\Phi_a = \sum_{\alpha\in\mathbb{N}^n} q_\alpha(a) D^\alpha$ *is an* S*-nonnegativity preserver for all* $a \in S$*. Then* Φ *is an* S*-nonnegativity preserver.*

Proof. As in the proof of [Bo], Theorem 3.1, the result is clear from $\Phi(p)(a) = \Phi_a(p)(a)$. $\square$

Again, the converse of this Lemma fails in general:

Example 3.5. Let $S := [-1,1] \subseteq \mathbb{R}$ and let

$$\Phi = E_{\frac{X}{2}} : p \mapsto p\left(\frac{X}{2}\right) = \sum_{i=0}^{\infty} \frac{1}{i!}\left(-\frac{X}{2}\right)^i D^i(p),$$

which is an S-positivity preserver. We have $1 \in S$ and

$$\Phi_1 = \sum_{i=0}^{\infty} \frac{1}{i!}\left(-\frac{1}{2}\right)^i D^i,$$

so $\Phi_1(p) = p\left(X - \frac{1}{2}\right)$ for all p, again using the Taylor-formula. The polynomial $p = X + 1$ belongs to $\mathcal{N}(S)$, but $\Phi_1(p) = X + \frac{1}{2}$ does not. So Φ_1 is not S-nonnegativity preserving.

So in the case of arbitrary sets $S \subseteq \mathbb{R}^n$, one has to look for different kinds of characterizations of S-nonnegativity preservers.

4. The adjoint map

We introduce the *adjoint map* to a nonnegativity preserver. This adjoint map, defined via Haviland's Theorem, was implicitly already used in [Bo, GS1, GS2].

Let $S \subseteq \mathbb{R}^n$ be closed and $\Phi \colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ be an S-nonnegativity preserver. For every nonnegative Borel measure μ on S with all finite moments consider the map

$$L_\mu \colon \mathbb{R}[\underline{X}] \to \mathbb{R};\ p \mapsto \int_S \Phi(p) d\mu.$$

L_μ is a linear functional and satisfies the assumption from Haviland's Theorem, since Φ is S-nonnegativity preserving. So there is another nonnegative Borel measure ν on S with all finite moments, such that

$$\int_S \Phi(p) d\mu = \int_S p d\nu \quad \forall p \in \mathbb{R}[\underline{X}].$$

Let $\mathcal{M}_+(S)$ denote the set of all nonnegative Borel measures on S with all finite moments. S can be considered as a subset of $\mathcal{M}_+(S)$, by identifying each point $x \in S$ with the Dirac measure δ_x centered at x.

Definition 4.1. A map $T \colon \mathcal{M}_+(S) \to \mathcal{M}_+(S)$ is called an *adjoint map* to Φ, if

$$\int_S \Phi(p) d\mu = \int_S p dT(\mu)$$

holds for all $p \in \mathbb{R}[\underline{X}]$ and all $\mu \in \mathcal{M}_+(S)$.

Proposition 4.2. *Let $S \subseteq \mathbb{R}^n$ be closed. Then*

(i) Every S-nonnegativity preserver Φ has an adjoint map.

(ii) If T and U are adjoint maps to Φ and Ψ respectively, then $T+U$ is an adjoint map to $\Phi + \Psi$ and $U \circ T$ is an adjoint map to $\Phi \circ \Psi$.

(iii) If S is Zariski dense and T is adjoint to both Φ and Ψ, then $\Phi = \Psi$.

(iv) If $f \geq 0$ on S, then an adjoint map for the multiplication operator M_f is the map

$$T_f \colon \mu \mapsto f \cdot \mu,$$

where $(f \cdot \mu)(A) := \int_A f d\mu$ *for Borel sets* $A \subseteq \mathbb{R}^n$.

(v) Let $f_1, \ldots, f_n \in \mathbb{R}[\underline{X}]$ *be such that* $E_{\underline{f}}$ *is an* S*-nonnegativity preserver. Then the map*

$$T_{\underline{f}} \colon \mu \mapsto \mu \circ \underline{f}^{-1},$$

(where $\left(\mu \circ \underline{f}^{-1}\right)(A) := \mu\left(\underline{f}^{-1}(A)\right)$ *for Borel sets* A*) is an adjoint map to* $E_{\underline{f}}$.

Proof. (i) is clear from Haviland's Theorem, as explained above, (ii), (iv) and (v) are standard results from intergration theory.

Towards (iii) let δ_x denote the Dirac measure centered at x, for every point $x \in S$. We have

$$\Phi(p)(x) = \int_S \Phi(p) d\delta_x = \int_S p dT(\delta_x) = \int_S \Psi(p) d\delta_x = \Psi(p)(x)$$

for every $p \in \mathbb{R}[\underline{X}]$ and $x \in S$. So $\Phi = \Psi$, by the Zariski denseness of S. □

Note that the adjoint map is not unique in general. The question whether the measure obtained in Haviland's Theorem is unique is also known as the "determinate moment problem" (see for example [PuSc, PuSm, PuVa]).

Remark 4.3. If S is compact, then the measures in Haviland's Theorem are unique, as for example pointed out in [Ma], Section 3.3. So the adjoint map for an S-nonnegativity preserver Φ is also unique in that case, and we denote it by Φ^*. In view of Proposition 4.2 we have $(\Phi + \Psi)^* = \Phi^* + \Psi^*$, $(\Phi \circ \Psi)^* = \Psi^* \circ \Phi^*$ and if S is Zariski dense, one has

$$\Phi^* = \Psi^* \Longleftrightarrow \Phi = \Psi.$$

For $x \in S$ we also write μ_x instead of $\Phi^*(\delta_x)$, if no confusion can arise (i.e. if only one operator is considered). With this notation we have

$$\Phi(p)(x) = \int_S p d\mu_x \tag{2}$$

for all $p \in \mathbb{R}[\underline{X}], x \in S$.

5. Integral representations of S-nonnegativity preservers

In this section we construct a general integral representation of S-nonnegativity preservers. It can be seen as a generalization of Haviland's Theorem, and does not assume compactness of S or any continuity of operators.

Let $S \subseteq \mathbb{R}^n$ be closed and Φ an S-nonnegativity preserver. Suppose T is an adjoint map to Φ. Then for each Borel set $A \subseteq S$ we can consider the map

$$\begin{aligned} T_A \colon S &\to \mathbb{R} \\ x &\mapsto T(\delta_x)(A). \end{aligned}$$

Write $\mathcal{B}(S)$ for the σ-algebra of Borel subsets of S. For any $A \in \mathcal{B}(S)$ and any $x \in S$ we have

$$0 \leq T_A(x) \leq T_S(x) = \int_S 1 dT(\delta_x) = \Phi(1)(x).$$

So all T_A are nonnegative functions on S that are bounded by $\Phi(1)$, pointwise on S.

Example 5.1. (i) If Φ is the identity operator, then the identity map T is an adjoint map to Φ. We have $T_A = \mathbb{1}_A$, the characteristic function of A. More general, consider a multiplication operator M_f with adjoint map T_f as defined in Proposition 4.2 (iv). Then $T_{f\,A} = f \cdot \mathbb{1}_A$. We see that the maps T_A can not be expected to be continuous in general.

(ii) For an S-nonnegativity preserving algebra homomorphism $E_{\underline{f}}$ with adjoint map $T_{\underline{f}}$ as above we have

$$T_{\underline{f}\,A} = \mathbb{1}_{\underline{f}^{-1}(A)}.$$

(iii) For finitely many polynomials $f_1, \ldots, f_r \geq 0$ on S and points $x_1, \ldots, x_r \in S$ consider the S-nonnegativity preserver

$$\Phi_{\overline{f},\overline{x}} \colon p \mapsto f_1 \cdot p(x_1) + \cdots + f_r \cdot p(x_r).$$

An adjoint map is

$$T \colon \mu \mapsto \sum_{i=1}^{r} \left(\int f_i d\mu \right) \cdot \delta_{x_i},$$

so we have

$$T_A = \sum_{\{i \mid x_i \in A\}} f_i,$$

a polynomial map that is in particular continuous.

We always have $T_\emptyset = 0$ and for a countable family $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint elements from $\mathcal{B}(S)$

$$T_{\bigcup_i A_i} = \sum_i T_{A_i}.$$

Write $\mathcal{F}(S)$ for the vector space of real valued and polynomially bounded functions on S. So the mapping

$$\begin{aligned} \mathfrak{m}_T \colon \mathcal{B}(S) &\to \mathcal{F}(S) \\ A &\mapsto T_A \end{aligned}$$

is a measure taking nonnegative functions in $\mathcal{F}(S)$ as its values.

For pairwise disjoint sets $A_1, \ldots, A_t \in \mathcal{B}(S)$ and a real valued step function $s = \sum_{i=1}^{t} r_i \cdot \mathbb{1}_{A_i}$ we define

$$\int_S s d\mathfrak{m}_T := \sum_{i=1}^{t} r_i \cdot \mathfrak{m}_T(A_i) = \sum_{i=1}^{r} r_i \cdot T_{A_i},$$

which is a well defined polynomially bounded real valued function on S. For $x \in S$ we have

$$\left(\int_S s d\mathfrak{m}_T \right)(x) = \sum_i r_i \cdot T_{A_i}(x) = \sum_i r_i \cdot T(\delta_x)(A_i) = \int s dT(\delta_x).$$

The following is our first main result. It is a general integral representation of S-nonnegativity preservers and can be seen as an generalization of Haviland's Theorem to operators:

Theorem 5.2. *Let $S \subseteq \mathbb{R}^n$ be closed and $\Phi \colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ a linear map. Then the following are equivalent:*

(i) *Φ is S-nonnegativity preserving*

(ii) *There is a measure* $\mathfrak{m}\colon \mathcal{B}(S) \to \mathcal{F}(S)$, *taking only nonnegative functions as its values, with*

$$\Phi(p) = \int_S p d\mathfrak{m} \text{ for all } p \in \mathbb{R}[\underline{X}],$$

in the following sense: For every sequence of real valued step functions $(s_j)_j$ *such that* $|s_j| \leq q$ *on* S *for a polynomial* q *and all* j, *if* $(s_j)_j$ *converges pointwise on* S *to* p, *then the sequence* $\left(\int_S s_j d\mathfrak{m}\right)_j$ *converges pointwise on* S *to* $\Phi(p)$.

Proof. (i)⇒(ii): Let T be an adjoint map to Φ and let $\mathfrak{m} := \mathfrak{m}_T$ be as described above. Let $p \in \mathbb{R}[\underline{X}]$ and let $(s_j)_j$ be a sequence of step functions converging pointwise on S to p, with $|s_j| \leq q$ on S for some polynomial q. For any $x \in S$ we have

$$\begin{aligned}\left|\left(\int s_j d\mathfrak{m}\right)(x) - \Phi(p)(x)\right| &= \left|\int_S s_j dT(\delta_x) - \int_S p dT(\delta_x)\right| \\ &\leq \int |s_j - p| dT(\delta_x),\end{aligned}$$

which converges to zero for $j \to \infty$, by the Theorem of Majorized Convergence, since $\int_S q dT(\delta_x) = \Phi(q)(x) < \infty$. (ii)⇒(i) is clear from the fact that for every nonnegative polynomial p there exists such an approximating sequence of step functions that are nonnegative themselves. □

6. Compact sets

During this whole section let $S \subseteq \mathbb{R}^n$ be *compact* and Zariski dense, and let Φ be an S-nonnegativity preserver. Let Φ^* denote the (unique) adjoint map to Φ, as defined above. Again write μ_x instead of $\Phi^*(\delta_x)$, for $x \in S$. We show that the integral representation from the last section holds in a stronger sense.

Clearly,

$$||p||_\infty := \max_{x \in S} |p(x)|$$

defines a norm on $\mathbb{R}[\underline{X}]$. Any nonnegativity preserver is continuous with respect to that norm:

Lemma 6.1. *Let* $\Phi\colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ *be an* S*-nonnegativity preserver. Then* Φ *is a continuous operator with operator norm* $||\Phi|| = ||\Phi(1)||_\infty$.

Proof. For $x \in S$ and $p \in \mathbb{R}[\underline{X}]$ we have

$$\begin{aligned}|\Phi(p)(x)| &= \left|\int_S p d\mu_x\right| \\ &\leq \int_S |p| d\mu_x \\ &\leq ||p||_\infty \cdot \mu_x(S) \\ &= ||p||_\infty \cdot \int_S 1 d\mu_x \\ &= ||p||_\infty \cdot \Phi(1)(x) \\ &\leq ||p||_\infty \cdot ||\Phi(1)||_\infty.\end{aligned}$$

So $||\Phi(p)||_\infty \leq ||p||_\infty \cdot ||\Phi(1)||_\infty$, thus Φ is continuous with $||\Phi|| \leq ||\Phi(1)||_\infty$. Equality follows from $||\Phi(1)||_\infty = ||1||_\infty \cdot ||\Phi(1)||_\infty$. $\square$

As $\mathbb{R}[\underline{X}]$ embeds densely into $C(S)$, the Banach space of continuous real valued functions on S, each S-nonnegativity preserver Φ has a unique continuous extension $\widetilde{\Phi}$ to $C(S)$. In the case of a compact set, the vector measures $\mathfrak{m}$ defined above have nicer properties:

Lemma 6.2. *For any Borel set $A \subseteq S$, the mapping*

$$\begin{aligned} \Phi_A^* \colon S &\to [0,\infty) \\ x &\mapsto \mu_x(A) \end{aligned}$$

is Borel measurable and we have $||\Phi_A^*||_\infty \leq ||\Phi(1)||_\infty$.

Proof. $||\Phi_A^*||_\infty \leq ||\Phi(1)||_\infty$ is clear for any A by what we have shown above. Now let first $A \subseteq S$ be closed. By Urysohn's Lemma and the Stone-Weierstraß approximation, choose a sequence of polynomials $(p_j)_j$ with $|p_j| \leq 2$ on S, that converge pointwise on S to $\mathbb{1}_A$, the characteristic function of A. We have $\Phi(p_j)(x) = \int p_j d\mu_x \overset{j}{\to} \mu_x(A) = \Phi_A^*(x)$ for all $x \in S$, by the Theorem of Majorized Convergence. So Φ_A^* is the pointwise limit of the polynomial functions $\Phi(p_j)$, and so clearly measurable.

For any Borel set A we have

$$\Phi_{S\setminus A}^* = \Phi(1) - \Phi_A^*$$

and for Borel sets $A_1 \subseteq A_2 \subseteq \dots$

$$\Phi_{\bigcup_i A_i}^* = \lim_i \Phi_{A_i}^* \quad \text{pointwise on } S,$$

so the general result follows by transfinite induction. $\square$

So the measure $\mathfrak{m}_{\Phi^*}$ takes its values in a bounded subset of $\mathbb{B}(S)$, the Banach space of bounded measurable real valued functions on S, equipped with the sup-norm. A whole theory of integration with respect to a measure with values in an arbitrary Banach space is for example developed in [DuSc], IV.10. In our case we obtain the following strong integral representation of Φ:

Theorem 6.3. *Let $S \subseteq \mathbb{R}^n$ be compact and Zariski dense. Let $\Phi\colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ be an S-nonnegativity preserver. Then the measure $\mathfrak{m} := \mathfrak{m}_{\Phi^*}$ takes only nonnegative measurable functions as its values, and we have*

$$\Phi(p) = \int_S p d\mathfrak{m} \textit{ for all } p \in \mathbb{R}[\underline{X}],$$

in the sense of Theorem 5.2 (ii). Furthermore, if a sequence of real valued step functions $(s_j)_j$ converges uniformly on S to p (and such sequences exist for every p), then the sequence $\int_S s_j d\mathfrak{m}$ converges uniformly on S to $\Phi(p)$.

Proof. Clear from the above results and the proof of Theorem 5.2. $\square$

Note that the result does only assume that Φ is nonnegativity preserving, in contrast to standard representation results as in [DuSc], VI.7, where the operators have to be *weakly compact* (see there for the notion of compact or weakly compact operator). We can further investigate nonnegativity preservers and their vector measures $\mathfrak{m}_{\Phi^*}$:

Theorem 6.4. *Let $S \subseteq \mathbb{R}^n$ be compact and Zariski dense. Let $\Phi\colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ be an S-nonnegativity preserver. Then the following are equivalent:*

(i) *There is some $d \in \mathbb{N}$ such that all Φ^*_A are polynomials of degree $\leq d$.*
(ii) *Φ has finite dimensional range.*
(iii) *$\widetilde{\Phi}$ has finite dimensional range.*

Proof. (ii)⇔(iii) is clear. (i)⇒(ii) is clear from Theorem 6.3. For (ii)⇒(i) let $q_1, \dots, q_r \in \mathbb{R}[\underline{X}]$ be such that $f_1 := \Phi(q_1), \dots, f_r := \Phi(q_r)$ form a basis of $\Phi(\mathbb{R}[\underline{X}])$. Using the fact that each polynomial is a difference of two squares of polynomials, we can assume that all q_i and therefore all f_i are nonnegative on S.

For each $p \in \mathbb{R}[\underline{X}]$ exist uniquely determined real numbers $L_1(p), \dots, L_r(p)$ such that

$$\Phi(p) = \sum_{i=1}^{r} L_i(p) f_i.$$

The mappings $L_i\colon \mathbb{R}[\underline{X}] \to \mathbb{R}$ are linear functionals. In other words, we have $\Phi = \Phi_{\underline{f},\underline{L}}$ (see Example 2.2 (5)).

By the Riesz-Representation Theorem ([DuSc] IV.6.3), there are finite signed Borel measures ν_i on S such that $L_i(p) = \int_S p d\nu_i$ for all p and i. By the determinateness of the Moment Problem for compact sets, we have

$$\mu_x = \sum_{i=1}^{r} f_i(x) \cdot \nu_i \text{ for all } x \in S.$$

So for each Borel set $A \subseteq S$ we have

$$\Phi^*_A = \sum_{i=1}^{r} \nu_i(A) \cdot f_i,$$

which proves the claim. □

The following two results are standard results from the theory of compact and weakly compact operators:

Theorem 6.5. *Let $S \subseteq \mathbb{R}^n$ be compact and Zariski dense. Let $\Phi\colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ be an S-nonnegativity preserver. Then the following are equivalent:*

(i) *The familiy $(\Phi^*_A)_{A \in \mathcal{B}(S)}$ is equicontinuous.*
(ii) *$\widetilde{\Phi}$ is a compact operator on $C(S)$.*

Proof. Use [DuSc], VI.7.7 Theorem 7 and the fact that a subset of $C(S)$ is contained in a compact set if and only if it is equicontinuous and pointwise bounded (this is the Arzela-Ascoli Theorem). □

Theorem 6.6. *Let $S \subseteq \mathbb{R}^n$ be compact and Zariski dense. Let $\Phi\colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}]$ be an S-nonnegativity preserver. Then the following are equivalent:*

(i) *All the functions Φ^*_A are continuous.*
(ii) *$\widetilde{\Phi}$ is a weakly compact operator on $C(S)$.*

Proof. This is [DuSc], VI.7.3 Theorem 3. □

Remark 6.7. In case that there is a finite positive Borel measure μ on S such that

$$||\Phi(p)||_\infty \leq \int |p| d\mu$$

holds for all $p \in \mathbb{R}[\underline{X}]$, it follows from [BaBe] Theorem 8 that $\widetilde{\Phi}$ is weakly compact. So all the functions Φ_A^* are continuous in that case.

7. Approximation of nonnegativity preservers

In this section we show that all S-nonnegativity preservers can be approximated by very simple ones, at least in the case of a compact set S. The setup is the following. Let $\mathcal{W}$ denote the set of all operators of the form

$$\Phi_{\underline{f},\underline{x}}\colon p \mapsto \sum_{i=1}^{r} p(x_i) \cdot f_i,$$

with $r \in \mathbb{N}$, $x_1, \ldots, x_r \in S$ and polynomials $f_1, \ldots, f_r$ which are nonnegative on S; see also Example 2.2 (4). $\mathcal{W}$ is a convex cone contained in the cone of all S-nonnegativity preservers. All elements from $\mathcal{W}$ have a finite dimensional range. Our goal is to prove that each S-nonnegativity preserver can be approximated by a sequence of elements from $\mathcal{W}$, pointwise on $\mathbb{R}[\underline{X}]$ with respect to $|| \; ||_\infty$. This convergence is also often called *convergence in the strong operator topology.*

Therefore let S be compact, Φ an S-nonnegativity preserver and $\mathfrak{m} := \mathfrak{m}_{\Phi^*}$ the function valued measure constructed from Φ as above. Integration of bounded measurable functions with respect to $\mathfrak{m}$ is defined in [DuSc], IV.10. We have already seen that for a real valued step function s on S and $x \in S$ we have

$$\left(\int_S s d\mathfrak{m} \right)(x) = \int_S s d\mu_x.$$

So for any bounded measurable function h on S the same formula remains true. We will need it later in the proof of the main approximation theorem.

The semi-variation $||\mathfrak{m}||$ of $\mathfrak{m}$ is defined as

$$||\mathfrak{m}||(A) := \sup || \sum_{i=1}^{r} \alpha_i \mathfrak{m}(A_i) ||_\infty,$$

where the supremum ranges over all finite collections of scalars with $|\alpha_i| \leq 1$ and all partitions of A into a finite number of disjoint Borel sets A_i.

$$0 \leq ||\mathfrak{m}(A)||_\infty \leq ||\mathfrak{m}||(A) < \infty$$

holds for all Borel sets A. However, $||\mathfrak{m}||$ can not be expected to be a measure in general, i.e. it is usually not additive. But there always exists a finite positive measure λ on S such that $\lambda(A) \leq ||\mathfrak{m}||(A)$ and $\lambda(A) = 0 \Leftrightarrow ||\mathfrak{m}||(A) = 0$ for all A in $\mathcal{B}(S)$ ([DuSc], IV.10.5 Lemma 5). A Borel set A is called an $\mathfrak{m}$-*null set* if $||\mathfrak{m}||(A) = 0$ holds, or if A is a λ-null set, equivalently. The usual Theorem of Majorized Convergence is true for integration with respect to $\mathfrak{m}$ ([DuSc] IV 10.10 Theorem 10).

The following is the announced approximation result:

Theorem 7.1. *Let $S \subseteq \mathbb{R}^n$ be compact and Zariski dense. Let Φ be an S-nonnegativity preserver. Then Φ can be approximated by a sequence of operators from $\mathcal{W}$, with respect to the strong operator topology.*

Proof. First assume $A_1, \ldots, A_r$ are pairwise disjoint Borel sets with $A_1 \cup \cdots \cup A_r = S$, and let $D > 0$ be an upper bound for the diameter of all the A_i. Choose some $a_i \in A_i$ for all i.

Approximate the characteristic function $\mathbb{1}_{A_i}$ by a polynomial $f_i \geq 0$ on S, such that $\|\mathfrak{m}(A_i) - \int f_i d\mathfrak{m}\|_\infty \leq \frac{D}{r}$. This can be done by Urysohn's Lemma, the Stone-Weierstraß Theorem and the Theorem of Majorized Convergence for $\mathfrak{m}$. Then consider the operator

$$\Phi_{\underline{f},\underline{a}} \colon p \mapsto \sum_{i=1}^{r} p(a_i) \cdot \Phi(f_i),$$

which belongs to $\mathcal{W}$. For any polynomial p and any $x \in S$ we have

$$\begin{aligned}
|\Phi(p)(x) - \Phi_{\underline{f},\underline{a}}(p)(x)| &= \left| \int p d\mu_x - \sum_i p(a_i) \int f_i d\mu_x \right| \\
&\leq \left| \int p d\mu_x - \int \sum_i p(a_i) \mathbb{1}_{A_i} d\mu_x \right| \\
&\quad + \left| \int \sum_i p(a_i) \mathbb{1}_{A_i} d\mu_x - \int \sum_i p(a_i) f_i d\mu_x \right| \\
&\leq \mu_x(S) \cdot ||p - \sum_i p(a_i) \mathbb{1}_{A_i}||_\infty \\
&\quad + ||p||_\infty \cdot \sum_i \left| \int \mathbb{1}_{A_i} d\mu_x - \int f_i d\mu_x \right| \\
&\leq ||\Phi(1)||_\infty \cdot ||p - \sum_i p(a_i) \mathbb{1}_{A_i}||_\infty \\
&\quad + ||p||_\infty \cdot \sum_i \left| \mathfrak{m}(A_i)(x) - \left(\int f_i d\mathfrak{m} \right)(x) \right| \\
&\leq ||\Phi(1)||_\infty \cdot ||p - \sum_i p(a_i) \mathbb{1}_{A_i}||_\infty + D \cdot ||p||_\infty.
\end{aligned}$$

By the mean value theorem applied to p we obtain

$$||p - \sum_i p(a_i) \mathbb{1}_{A_i}||_\infty \leq D \cdot ||J(p)||_\infty,$$

where $J(p) := 1 + \sum_{j=1}^n \left(\frac{\partial}{\partial X_j} p \right)^2$. So we have shown

$$||\Phi(p) - \Phi_{\underline{f},\underline{a}}(p)||_\infty \leq D \cdot (||\Phi(1)||_\infty \cdot ||J(p)||_\infty + ||p||_\infty)$$

for all $p \in \mathbb{R}[\underline{X}]$.

So if a sequence of partitionings of S is chosen such that the diameter bound D gets arbitrary small (which can obviously be done since S is compact), then the corresponding sequence of operators from $\mathcal{W}$ converges to Φ, pointwise on $\mathbb{R}[\underline{X}]$. □

8. Some open problems

We include a collection of open problems:

(1) For a given measure $\mathfrak{m}$ on $\mathcal{B}(S)$ with values in $\mathcal{F}(S)$ or $\mathbb{B}(S)$, find criteria for the operator defined by $\mathfrak{m}$ as in Theorem 5.2 and Theorem 6.3 to map polynomials to polynomials.

(2) Find S-nonnegativity preservers that are compact but do not have a finite dimensional range. For compact S this means to find an S-nonnegativity preserver Φ such that the family $(\Phi_A^*)_{A\in\mathcal{B}(S)}$ is equicontinuous, but not polynomial of bounded degree. Could it be true that compact implies finite dimensional range for nonnegativity preservers?

(3) The same question as in (2), but with weakly compact operators that are not compact. For compact S, find an nonnegativity preserver such that all Φ_A^* are continuous, but these functions do not form an equicontinuous family.

(4) Is every S-nonnegativity preserver with finite dimensional range of the form

$$\Phi\colon p\mapsto \sum_{i=1}^r f_i\cdot\int_S p d\nu_i$$

with nonnegative polynomials f_i and *nonnegative* Borel measures ν_i on S? In view of Haviland's Theorem, can the representation

$$\Phi=\Phi_{\underline{f},\underline{L}}$$

as given in the proof of Theorem 6.4 be chosen such that all $f_i\geq 0$ on S and all L_i map S-nonnegative polynomials to nonnegative reals? The following example might be interesting in this regard: Let λ denote the Lebesque measure. Consider the following operator on $\mathbb{R}[t]$:

$$\Phi\colon p\mapsto (t+2)\cdot\int_{-1}^1 pd\lambda\quad -t^2\cdot\int_0^1 pd\lambda.$$

Φ is $[-1,1]$-nonnegativity preserving, the polynomials $f_1=t+2$ and $f_2=t^2$ are nonnegative on $[-1,1]$, but the linear functional $L_2\colon p\mapsto -\int_0^1 pd\lambda$ is not integration with respect to a nonnegative measure on $[-1,1]$. However,

$$\Phi\colon p\mapsto (t+2)\cdot\int_{-1}^0 pd\lambda+(t+2-t^2)\cdot\int_0^1 pd\lambda$$

is a representation of Φ as desired.

(5) In case S is compact, which nonnegativity preservers can be approximated by elements from $\mathcal{W}$ with respect to the *operator norm* instead of the strong operator topology as in Theorem 7.1? As all elements from $\mathcal{W}$ are compact operators on $C(S)$, it is a well known fact that only compact operators can be approximated like that. Can every compact S-nonnegativity preserver be approximated?

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