SOS APPROXIMATIONS OF NONNEGATIVE POLYNOMIALS VIA SIMPLE HIGH DEGREE PERTURBATIONS

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ABSTRACT. We show that every real polynomial f nonnegative on $[-1, 1]^n$ can be approximated in the l_1-norm of coefficients, by a sequence of polynomials $\{f_{\varepsilon r}\}$ that are sums of squares. This complements the existence of s.o.s. approximations in the denseness result of Berg, Christensen and Ressel, as we provide a very simple and *explicit* approximation sequence.

Then we show that if the moment problem holds for a basic closed semi-algebraic set $K_S \subset \mathbb{R}^n$ with nonempty interior, then every polynomial nonnegative on K_S can be approximated in a similar fashion by elements from the corresponding preordering.

Finally, we show that the degree of the perturbation in the approximating sequence depends on ϵ as well as the degree and the size of coefficients of the nonnegative polynomial f, but *not* on the specific values of its coefficients.

1. Introduction

The question of representing or approximating real polynomials by sums of squares (s.o.s.) polynomials or elements from preorderings is a main topic in Real Algebra and Real Algebraic Geometry. Consider the following setup: Given a finite set S of real polynomials, consider the basic closed semi-algebraic set K_S and the preordering T_S defined by S. Then one wants to know more about the polynomials nonnegative on K_S, for example if they lie in T_S or can be approximated by elements from T_S in some sense.

There are classical results like the Positivstellensatz, which states that for each polynomial f strictly positive on K_S there exist elements t_1, t_2 from T_S such that $t_1 f = 1 + t_2$ holds (see e.g. [15]).

In the case that K_S is compact, Schmüdgen's famous theorem [18] says that every polynomial strictly positive on K_S is an element from T_S. So every nonnegative polynomial f is approximated by the polynomials $f + \varepsilon \quad (\varepsilon > 0)$, which all lie in T_S. Putinar [16] simplied this representation under an additional assumption by using quadratic modules instead of preorderings. Prestel and Jacobi [6] then developed a valuation theoretic method for testing if this additional assumption, namely that the quadratic module is archimedean, is fulfilled. For more information about this field see for example [15, 17].

But s.o.s. polynomials are also of primary importance for practical computation, especially in view of their numerous potential applications, notably in polynomial optimization; see e.g. [9, 14, 17, 19]. Indeed, in the computational complexity

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terminology, checking whether a given polynomial is nonnegative is a NP-hard problem, whereas checking whether it is s.o.s. reduces to solving a (convex) *semidefinite programming* (SDP) problem which (up to arbitrary precision) can be done in time polynomial in the input size of the problem; for more detail on semidefinite programming, the interested reader is refered to Vandenberghe and Boyd [20].

It has been known for some time that the cone of s.o.s. polynomials is *dense* (for the l_1-norm of coefficients) in the cone of polynomials nonnegative on the $\|\cdot\|_\infty$ unit ball $[-1,1]^n \subset \mathbb{R}^n$; see e.g. Berg, Christensen and Ressel [1] and Berg [2]. However, [1] is essentially an existence result.

On the other hand there is a negative result by Blekherman [3], which states that there are much more nonnegative polynomials than sums of squares.

Contribution. Our contribution is threefold:

(i) We first provide an explicit and very simple s.o.s. approximation of polynomials nonnegative on the $\|\cdot\|_\infty$ unit ball $[-1,1]^n$. Namely, let

$$\Theta_r := 1 + \sum_{j=1}^{n} X_j^{2r} \in \mathbb{R}[X_1,\ldots,X_n]. \tag{1}$$

Then, given $\varepsilon > 0$ and a polynomial $f \in \mathbb{R}[X_1,\ldots,X_n]$ nonnegative on $[-1,1]^n$, the polynomial $f_{\varepsilon r} := f + \varepsilon\Theta_r$ is s.o.s. provided r is large enough, say $r \geq r(f,\epsilon)$. Of course, $\|f_{\varepsilon r} - f\|_1 \to 0$ as $\varepsilon \to 0$. Although our result is not completely constructive (as $r(f,\epsilon)$ is not known), it complements the pure existence result [1].

If f is nonnegative on the ball $[-l,l]^n$ for some $l > 0$, then for every $\varepsilon > 0$, the polynomial $f + \varepsilon(1 + \sum_{j=1}^n (X_j/l)^{2r})$ is s.o.s. provided r is sufficiently large (just use $x \mapsto g(x) := f(lx) \geq 0$ on $[-1,1]^n$).

Note that the representation $f + \varepsilon\Theta_r = f_{\varepsilon r}$ for some s.o.s. polynomial $f_{\varepsilon r}$, is an obvious *certificate* of nonnegativity of f on $[-1,1]^n$. Indeed, for $x \in [-1,1]^n$, one has

$$f(x) + \varepsilon\Theta_r(x) = f_{\varepsilon r}(x) \geq 0,$$

provided r is big enough. As all Θ_r are bounded by $n+1$ on $[-1,1]^n$, letting $\varepsilon \downarrow 0$ yields $f(x) \geq 0$.

Our s.o.s. approximation result states that to approximate (uniformly on $[-1,1]^n$) a polynomial nonnegative on $[-1,1]^n$, it is enough to slightly perturb by a small $\varepsilon > 0$ its (maybe zero) coefficients of some even power of marginal monomials $\{X_i^{2r}\}$.

The method of the proof is quite different and much simpler than that of [11] for s.o.s. approximation of nonnegative polynomials; in particular, it does *not* use Nussbaum's deep result on moment sequences [13]. It also simplyfies the approximating sequence obtained in [12] in the spirit of [11].

In addition, if one fixes *à priori* the degree r of the perturbation Θ_r, we also characterize the minimum value ε_r^* of the parameter ε, to make $f + \varepsilon\Theta_r$ a s.o.s. It is given by

$$-\varepsilon_r^* := \min_L \left\{ L(f) \mid L : \mathcal{A}_{2r} \to \mathbb{R} \text{ linear},\ L(\Theta_r) \leq 1,\ L(h^2) \geq 0\ \forall h \in \mathcal{A}_r \right\},$$

where $\mathcal{A}_r$ is the finite dimensional vector space of polynomials of degree at most r.

(ii) We next obtain a similar approximation result for polynomials nonnegative on certain semi-algebraic sets. For a finite set $S \subset \mathbb{R}[X_1,\ldots,X_n]$ of polynomials, denote by K_S the associated basic closed semi-algebraic set in $\mathbb{R}^n$, and by T_S the preordering generated by S. Assume that K_S has nonempty interior and S has the so called *strong moment property*, that is, every linear form on $\mathbb{R}[X_1,\dots,X_n]$ which is nonnegative on T_S comes from a measure on K_S. Then every polynomial f nonnegative on K_S is approximated in the l_1-norm by the same sequence $\{f_{\varepsilon r}\}$, which now lies in T_S. In addition, if one uses the perturbation

$$\theta_r := \sum_{i=1}^{n}\sum_{k=0}^{r}\frac{X_i^{2k}}{k!} \quad \in \mathbb{R}[X_1,\dots,X_n], \tag{2}$$

instead of Θ_r as in (1), one obtains a certificate of nonnegativity on K_S. This is because when using θ_r, the fact that the (new) approximating sequence $\{f_{\varepsilon r}\}$ lies in T_S, also implies that f is nonnegative on K_S. Therefore, one may use this property to detect whether some given f is nonnegative on K_S.

(iii) Finally, we address the issue of identifying the factors that influence the degree r up to which one has to perturb f to obtain an s.o.s. We find that r depends only on ε, the dimension n, the degree and the size of the coefficients of f, but *not* on the explicit choice of f.

Link with related results. The s.o.s. approximation $f+\varepsilon\Theta_r$ in (1) resembles the one in (2) recently introduced by the first author in [11], for polynomials nonnegative on the *whole* $\mathbb{R}^n$; with θ_r instead of Θ_r, it is proven in [11] that given a globally nonnegative polynomial f and $\varepsilon > 0$, the polynomial $f+\varepsilon\theta_r$ is s.o.s. provided r is large enough (and we also have $\|f+\varepsilon\theta_r - f\|_1 \to 0$ as $\varepsilon \to 0$). Notice that this latter result is also a certificate of nonnegativity on $\mathbb{R}^n$ and is more than a denseness result for the l_1-norm. Indeed, it also shows that every nonnegative polynomial can be approximated by s.o.s. polynomials *uniformly on compact sets*, a nice additional property.

So a polynomial f nonnegative on $\mathbb{R}^n$ (hence also on $[-1,1]^n$) could be approximated either by $f_{\varepsilon r} = f+\varepsilon\Theta_r$ or by $f_{\varepsilon r} = f+\varepsilon\theta_r$ for sufficiently large $r\in\mathbb{N}$; in both cases $\|f-f_{\varepsilon r}\|_1 \to 0$ as $\varepsilon\to 0$. However, the former approximation is *not* a certificate of nonnegativity of f; in particular, it looses the nice property of uniform approximation on compact sets possessed by the latter.

In other words, the s.o.s. approximation $f+\varepsilon\Theta_r$ is indeed specific for polynomials nonnegative on $[-1,1]^n$. For polynomials nonnegative on $\mathbb{R}^n$, the s.o.s. approximation $f+\varepsilon\theta_r$ (although a little more complicated than $f+\varepsilon\Theta_r$) should be prefered.

Note that in [10] (and in a different way in [12]), approximation sequences for polynomials nonnegative on *algebraic* subsets of $\mathbb{R}^n$ are introduced. Such polynomials can be approximated by sums of squares plus elements from an ideal corresponding to the algebraic set.

For *semi-algebraic* subsets of $\mathbb{R}^n$, the question of approximating or representing nonnegative polynomials by elements from corresponding preorderings is of great interest in Real Algebra and Real Algebraic Geometry. The above mentioned Moment Problem for a finite set of polynomials $S \subset \mathbb{R}[X_1,\dots,X_n]$ is for example discussed in e.g. [7, 8], where the authors ask wether for each polynomial f nonnegative on the corresponding basic closed semi-algebraic set K_S, there exists some polynomial $q\in\mathbb{R}[X_1,\dots,X_n]$ such that for every $\varepsilon>0$, the polynomial $f+\varepsilon q$ lies in the preordering T_S generated by S. This is still an open problem. Our result is weaker, as the polynomial q ($=\Theta_r$ or θ_r) depends on ε via its degree r.

Finally, the degree bounds that we discuss here have been already investigated in [12] in a similar context, but for the approximations obtained in [11].

The paper is organized as follows. After introducing some notation and definitions in §2, our results are presented in §3.1 for s.o.s. approximations of polynomials nonnegative on $[-1,1]^n$, in §3.2 for related results on polynomials nonnegative on a basic closed semi-algebraic set $K_S \subset \mathbb{R}^n$, and in §3.3 for results on the degree bounds. For ease of exposition, some technical proofs have been postponed in an Appendix in §4.

2. Notations and definitions

Let $\mathbb{R}[X] := \mathbb{R}[X_1,\dots,X_n]$ denote the ring of real polynomials, $\mathcal{A}_r$ the finite dimensional subspace of polynomials of degree at most r and $s(r) = \binom{n+r}{n}$ its dimension. Let $\mathcal{A}_r^{\mathrm{sos}} \subset \mathcal{A}_r$ be the space of s.o.s. polynomials of degree as most r.

We always fix the canonical monomial basis for $\mathcal{A}_r$ and $\mathbb{R}[X_1,\dots,X_n]$, if we consider them as real vector spaces. For $\alpha \in \mathbb{N}^n$, we write X^α for $X_1^{\alpha_1}\cdots X_n^{\alpha_n}$, and $|\alpha|$ for $\sum_{i=1}^n \alpha_i$.

A linear form L on $\mathbb{R}[X]$ is said to have a *representing measure* μ if

$$L(f) = \int_{\mathbb{R}^n} f d\mu \qquad \forall f \in \mathbb{R}[X].$$

This is the same as saying that the sequence of values of L on the canonical monomial basis is the moment sequence of this measure μ.

Of course not every linear form has a representing measure. However, there is a *sufficient* condition to ensure that it is indeed the case.

Definition 2.1. A function $\varphi\colon \mathbb{N}^n \to \mathbb{R}_+$ is called an *absolute value* if

(i) $\varphi(0) = 1$;
(ii) $\varphi(\alpha+\beta) \leq \varphi(\alpha)\varphi(\beta)$ for all $\alpha,\beta \in \mathbb{N}^n$.

The following result is stated in Berg et al. [1].

Theorem 2.2. *Let L be a linear form on $\mathbb{R}[X]$ such that $L(p^2) \geq 0$ for all $p \in \mathbb{R}[X]$. If there is an absolute value φ and a constant $C > 0$ such that $|L(X^\alpha)| \leq C\varphi(\alpha)$ for all $\alpha \in \mathbb{N}^n$, then L has exactly one representing measure μ on $\mathbb{R}^n$. The support of μ is contained in the set $\{x \in \mathbb{R}^n \mid |x^\alpha| \leq \varphi(\alpha)\ \forall \alpha \in \mathbb{N}^n\}$.*

For a finite set $S = \{g_1,\dots,g_s\}$ of polynomials, denote by K_S the basic closed semi-algebraic set $K_S := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0\ i = 1,\dots,s\}$, and by T_S the preordering generated by S, i.e the set of all finite sums of polynomials of the form

$$\sigma_e g_1^{e_1}\cdots g_s^{e_s},$$

where $e \in \{0,1\}^s$ and σ_e is s.o.s. Further, let T_r be the set of all finite sums of such elements $\sigma_e g_1^{e_1}\cdots g_s^{e_s}$ of degree at most r. Note that this is different from $T_S \cap \mathcal{A}_r$ in general, as cancellation of leading forms could result in a polynomial of degree at most r, without the single polynomials having this property.

For the degree bound issue addressed in §3.3, one needs some elementary notions from the theory of real closed fields and valuation theory. Given a real closed extension field R of $\mathbb{R}$, denote by $\mathcal{O}$ the convex hull of $\mathbb{Z}$ in R, i.e

$$\mathcal{O} = \{x \in R \mid \exists m \in \mathbb{N} :\ |x| \leq m\}.$$

$\mathcal{O}$ is a valuation ring of R with maximal ideal

$$\mathfrak{m} = \{x \in \mathcal{O} \mid \forall n \in \mathbb{N}\setminus\{0\} :\ |x| \leq \frac{1}{n}\}.$$

Let $\overline{R} := \mathcal{O}/\mathfrak{m}$ denote the residue field and $\sigma : \mathcal{O} \to \overline{R}$ the order preserving residue map. We have $\overline{R} = \mathbb{R}$ and σ is the identity on $\mathbb{R}$. In fact, for every $\beta \in \mathcal{O}$ there is exactly one $b \in \mathbb{R}$ such that $\beta \equiv b \mod \mathfrak{m}$.

3. Main results.

In this section we prove our main results, whereas for ease of exposition, some technical proofs are postponed in §4. We first consider polynomials nonnegative on $[-1,1]^n$.

3.1. **Nonnegativity on the $\|\cdot\|_\infty$ unit ball $[-1,1]^n$.** We begin with the following result of its own interest.

Theorem 3.1. *Let $f \in \mathbb{R}[X]$ be a polynomial of degree r_f, and let $\Theta_r \in \mathbb{R}[X]$ be as in (1). Let $r_f \leq 2r \in \mathbb{N}$ be fixed and consider the semidefinite program*

$$\min_L \left\{ L(f) \mid L : \mathcal{A}_{2r} \to \mathbb{R} \text{ linear, } L(\Theta_r) \leq 1,\ L(h^2) \geq 0\ \forall h \in \mathcal{A}_r \right\} =: \varepsilon_r^*. \tag{3}$$

Then

(i) *$\varepsilon_r^* \leq 0$ and (3) is solvable, i.e. $\varepsilon_r^* = L(f)$ for some feasible L.*
(ii) *The polynomial $f_{\varepsilon r} := f + \varepsilon\Theta_r$ is s.o.s. if and only if $\varepsilon \geq -\varepsilon_r^*$.*

(Note that the condition $L(h^2) \geq 0\ \forall h \in \mathcal{A}_r$ translates to the positive semidefiniteness of the matrix which represents the bilinear form $(p,q) \mapsto L(pq)$. Therefore (3) is an SDP.)

Proof. (i) The zero form is feasible for (3). So $\varepsilon_r^* \leq 0$. Furthermore, the set of feasible solutions is compact (if we consider each linear form on $\mathcal{A}_{2r}$ as the s(2r)-vector of its values on the monomial basis). Indeed, the constraint $L(\Theta_r) \leq 1$ implies that

$$L(1) \leq 1; \quad L(X_i^{2r}) \leq 1, \quad i = 1, \ldots, n.$$

As $L(p^2) \geq 0$ for all $p \in \mathcal{A}_r$, by Lemma 4.1 and Lemma 4.3 from the appendix, one has $|L(X^\alpha)| \leq 1$ for all $|\alpha| \leq 2r$. So the set of feasible solutions in $\mathbb{R}^{s(2r)}$ is bounded. As it is obviously closed as well, it is compact. Since the objective function is linear and therefore continuous, there always exists an optimal solution.

(ii) By definition, the minimum value ε_r' for which $f_{\varepsilon r}$ is s.o.s. is given by

$$\varepsilon_r' = \min_\varepsilon \left\{ \varepsilon \mid f + \varepsilon\Theta_r \in \mathcal{A}_{2r}^{\text{sos}} \right\}. \tag{4}$$

But (4) is an SDP whose dual reads

$$\max_L \left\{ L(f) \mid L : \mathcal{A}_{2r} \to \mathbb{R} \text{ linear, } -L(\Theta_r) \leq 1,\ L(h^2) \leq 0\ \forall h \in \mathcal{A}_r \right\}.$$

Equivalently, with the change of variable $L \to -L$,

$$-\min_L \left\{ L(f) \mid L : \mathcal{A}_{2r} \to \mathbb{R} \text{ linear, } L(\Theta_r) \leq 1,\ L(h^2) \geq 0\ \forall h \in \mathcal{A}_r \right\}. \tag{5}$$

One next proves that there is no *duality gap* between the respective primal and dual problems (4) and (5), that is, their respective optimal values are equal.

Let μ be a measure on $\mathbb{R}^n$ with all moments up to order $2r$ finite and with a strictly positive density. One may scale μ to satisfy $\int_{\mathbb{R}^n} \Theta_r d\mu < 1$. Let L be integration with respect to μ. As μ has strictly positive density, we must have $L(p^2) > 0$ for all $p \in \mathcal{A}_r \setminus \{0\}$, and so L is a strictly feasible solution for the SDP in (5), that is, Slater's condition holds, which in turn implies that both SDP problems in (4) and (5) have the same optimal value $\varepsilon_r' = -\varepsilon_r^*$; see e.g. [20].

So the *only if* part in (ii) follows from the definition of ε_r'. Now let $\varepsilon \geq -\varepsilon_r^*$ and write

$$f + \varepsilon\Theta_r = f - \varepsilon_r^*\Theta_r + (\varepsilon + \varepsilon_r^*)\Theta_r,$$

and use that $f - \varepsilon_r^*\Theta_r$ as well as $(\varepsilon + \varepsilon_r^*)\Theta_r$ are s.o.s. to obtain the result. □

Observe that $\varepsilon_r^* = 0$ whenever f is a s.o.s., because then $L(f) \geq 0$ for every feasible L and the zero linear form is feasible. If f is not s.o.s. (so $\varepsilon_r^* < 0$), then the inequality constraint $L(\Theta_r) \leq 1$ in (3) can be replaced with the equality constraint $L(\Theta_r) = 1$, since by linearity, given a feasible solution L with $L(\Theta_r) < 1$ and with value $L(f) < 0$, one always obtains a better feasible solution $L' = \varrho L$ with $L'(\Theta_r) = 1$ (note that $L(\Theta_r) = 0$ implies $L = 0$).

Next, we obtain the following crucial result.

Theorem 3.2. *Let $f \in \mathbb{R}[X]$ be a polynomial of degree r_f, nonnegative on $[-1,1]^n$, and let $\Theta_r \in \mathbb{R}[X]$ be as in (1). Let ε_r^* be the optimal value of the semidefinite program defined in (3), for all $2r \geq r_f$. Then $\varepsilon_r^* \to 0$ as $r \to \infty$.*

Proof. From Theorem 3.1, $\varepsilon_r^* = L^{(r)}(f) \leq 0$ for some optimal solution $L^{(r)}$ of the semidefinite program (3), whenever $2r \geq r_f$. From the proof of Theorem 3.1, it follows that $|L^{(r)}(X^\alpha)| \leq 1$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2r$. Next, complete the vector of the values of $L^{(r)}$ on the monomial basis of $\mathcal{A}_r$ with zeros to make it an element in $\mathbb{R}^{\mathbb{N}^n}$, and in fact even an element of $[-1,1]^{\mathbb{N}^n}$. By Tychonoff's Theorem, we find a subsequence r_k such that the sequence $L^{(r_k)}$ converges to some $\mathbf{y}^* \in [-1,1]^{\mathbb{N}^n}$ in the product topology, and in particular pointwise convergence holds, i.e.

$$L^{(r_k)}(X^\alpha) \to \mathbf{y}_\alpha^* \qquad \forall \alpha \in \mathbb{N}^n. \tag{6}$$

Let L^* be the linear form on $\mathbb{R}[X]$ defined by $L^*(X^\alpha) := \mathbf{y}_\alpha^*$. From the pointwise convergence in (6) we obtain $L^*(p^2) \geq 0$ for all $p \in \mathbb{R}[X]$. This, together with $\mathbf{y}^* \in [-1,1]^{\mathbb{N}^n}$, implies that L^* has a representing measure μ^* with support contained in $[-1,1]^n$ (see Theorem 2.2). Now again from the pointwise convergence (6),

$$\varepsilon_{r_k}^* = L^{(r_k)}(f) \to L^*(f) = \int_{[-1,1]^n} f d\mu^* \geq 0,$$

where the inequality uses nonnegativity of f on $[-1,1]^n$. Since all $\varepsilon_r^* \leq 0$, we get $\varepsilon_{r_k}^* \to 0$. And as the converging subsequence r_k was arbitrary, this shows the desired result. □

Therefore, we finally obtain:

Corollary 3.3. *Let $f \in \mathbb{R}[X]$ be a polynomial nonnegative on $[-1,1]^n$ and let $\Theta_r \in \mathbb{R}[X]$ be as in (1). Let $\varepsilon > 0$ be fixed. Then there exists some $r(f,\varepsilon) \in \mathbb{N}$ such that for every $r \geq r(f,\varepsilon)$, the polynomial $f_{\varepsilon r} := f + \varepsilon\Theta_r$ is a s.o.s.*

Proof. From Theorem 3.2 we know that the sequence $\{\varepsilon_r^*\}$ with ε_r^* defined in (3) converges to 0 as $r \to \infty$. So there is an $r(f,\varepsilon)$ such that for all $r \geq r(f,\varepsilon)$ we have $\varepsilon_r^* \geq -\varepsilon$. By Theorem 3.1 the polynomial $f - \varepsilon_r^*\Theta_r$ is a s.o.s., and so

$$f + \varepsilon\Theta_r = f - \varepsilon_r^*\Theta_r + (\varepsilon + \varepsilon_r^*)\Theta_r$$

is a s.o.s. as well, since $(\varepsilon + \varepsilon_r^*)\Theta_r$ is also a s.o.s. ($\varepsilon_r^* \geq -\varepsilon$). □

Corollary 3.3 refines the *denseness* result of Berg [2], because it provides an *explicit* approximation sequence. In addition, this approximation sequence is extremely simple, as the perturbation polynomial Θ_r contains only the constant and the marginal monomials $X_i^{2r},\ i = 1, \ldots, n$. In addition, it provides a *certificate* of nonnegativity of f on $[-1, 1]^n$; indeed, if $x \in [-1, 1]^n$, then for every $r \geq r(f, \varepsilon)$ one has $f(x) + \varepsilon\Theta_r(x) \geq 0$. Letting $\varepsilon \to 0$ yields $f(x) \geq 0$.

It is straightforward to extend Corollary 3.3 to the case of a polynomial f nonnegative on the ball $[-l, l]^n \subset \mathbb{R}^n$ for some $l > 0$. Indeed, it suffices to apply Corollary 3.3 to the polynomial $x \mapsto g(x) := f(lx)$ which is nonnegative on $[-1, 1]^n$. In this case the polynomial $f + \varepsilon(1 + \sum_{j=1}^n (X_j/l)^{2r})$ provides an s.o.s. approximation.

In some specific examples, one may even obtain a more precise result with a slightly different perturbation. Namely, given r *fixed*, one may provide an explicit bound $\varepsilon_r > 0$, such that the perturbed polynomial $q_{\varepsilon r} := f + \varepsilon\Theta'_r$ is s.o.s. This is illustrated in the following nice two examples, kindly provided by Bruce Reznick.

Example 3.4. Consider the univariate polynomial $f = 1 - X^2$, obviously nonnegative on $[-1, 1]$. If $\varepsilon \geq \varepsilon_r^* := (r-1)^{r-1}/r^r$, the polynomial

$$q_{\varepsilon r} := 1 - X^2 + \varepsilon X^{2r}$$

is globally nonnegative and therefore a s.o.s. Indeed, its minimum occurs when $-2x + 2r\varepsilon x^{2r-1} = 0$, i.e. at $x_r := (1/r\varepsilon)^{1/(2r-2)}$. Hence, the value at x_r is

$$1 - x_r^2 + \varepsilon x_r^2 x_r^{2r-2} = 1 - x_r^2(r-1)/r,$$

which is nonnegative if and only if

$$x_r^2 \leq r/(r-1) \Leftrightarrow x_r^{2r-2} \leq (r/(r-1))^{r-1} \Leftrightarrow 1/(r\varepsilon) \leq (r/(r-1))^{r-1},$$

i.e. if and only if $\varepsilon \geq (r-1)^{r-1}/r^r = \varepsilon_r^*$.

Example 3.5. On the other hand, consider the Motzkin polynomial $f = 1 + X^2Y^2(X^2 + Y^2 - 3) \in \mathbb{R}[X, Y]$ which is nonnegative but *not* a s.o.s. Then, for all $r \geq 3$ and $\varepsilon := 2^{4-2r}$, the polynomial $q_{\varepsilon r} := f + \varepsilon X^{2r}$ is a s.o.s., and $\|f - q_{\varepsilon r}\|_1 \to 0$ as $r \to \infty$. To prove this, write

$$f = (XY^2 + X^3/2 - 3X/2)^2 + p,$$

where $p = 1 - (X^3/2 - 3X/2)^2 = (1 - X^2)^2(1 - X^2/4)$. Next, the univariate polynomial $q = p + 2^{4-2r}X^{2r}$ is nonnegative on $\mathbb{R}$, hence a sum of squares. Indeed, if $x^2 \leq 4$, then $p \geq 0$ and so $q \geq 0$. If $x^2 > 4$ then $|p(x)| \leq (x^2)^2x^2/4 = x^6/4$. From

$$q(x) \geq 2^{4-2r}x^{2r} - |p(x)| \geq \frac{x^6}{4}((x^2/4)^{r-3} - 1),$$

and the fact that $n \geq 3, x^2 > 4$, we deduce that $q(x) \geq 0$.

In Example 3.4, one approximates $1 - X^2$ (uniformly on $[-1, 1]$) by the s.o.s. $1 - X^2 + \varepsilon X^{2r}$. In Example 3.5, the Motzkin polynomial can also be approximated in the l_1-norm by $f + \varepsilon(X^{2r} + Y^{2r})$, but not uniformly on compact sets. For the latter property to hold, one needs the perturbation $f + \varepsilon \sum_{j=1}^n \sum_{k=0}^r X_i^{2k}/k!$ introduced in [11].

3.2. **Nonnegativity on basic closed semi-algebraic sets.** We next prove the second announced result, namely the approximation of polynomials nonnegative on basic closed semi-algebraic sets. Let $S \subset \mathbb{R}[X]$ be a finite set of polynomials and suppose that S has the strong moment property, which means that every linear form on $\mathbb{R}[X]$ which is nonnegative on the preordering T_S, is integration with respect to some measure on K_S. Further suppose K_S has nonempty interior, and let $f \in \mathbb{R}[X]$ be nonnegative on K_S.

With same notation as in §3.1, consider the semidefinite program

$$\varepsilon_r^* := \min_L \{L(f) \mid L\colon \mathcal{A}_{2r} \to \mathbb{R} \text{ linear},\ L(\Theta_r) \leq 1,\ L(t) \geq 0\ \forall t \in T_{2r}\}. \tag{7}$$

Its dual reads

$$\max_\varepsilon \{\varepsilon \mid f - \varepsilon\Theta_r \in T_{2r}\}. \tag{8}$$

Proceding exactly as in the proof of Theorem 3.1, one constructs a strictly feasible solution for (7) as integration with respect to some (suitably scaled) measure on a ball in K_S. Hence, with same arguments, the SDP (7) is also always solvable (note that $\mathcal{A}_{2r}^{\text{sos}} \subseteq T_{2r}$), and there is no duality gap between the SDPs (7) and (8), i.e., their optimal values are equal.

Again, every sequence of optimal solutions for (7) (with r growing) has a subsequence that converges pointwise to some $\mathbf{y}^* \in [-1,1]^{\mathbb{N}^n}$ wich is the moment sequence of some measure on K_S, this time using the fact that S has the strong moment property. So, as in the proof of Theorem 3.2, the sequence $\{\varepsilon_r^*\}$ converges to 0, since f is nonnegative on K_S. Hence, as in Corollary 3.3, we get the following result:

Corollary 3.6. *Let $S \subset \mathbb{R}[X]$ be a finite set of polynomials and suppose that S has the strong moment property. Further, suppose that K_S has a nonempty interior. Let $f \in \mathbb{R}[X]$ be nonnegative on K_S and let $\Theta_r \in \mathbb{R}[X]$ be as in* (1)*. Let $\varepsilon > 0$ be fixed. Then there is some $r(f,\varepsilon,S)$ such that for every $r \geq r(f,\varepsilon,S)$, the polynomial $f_{\varepsilon r} := f + \varepsilon\Theta_r$ lies in T_S.*

Note that the pointwise limit $\mathbf{y}^*$ from above is the moment sequence of a measure on K_S as S has the strong moment property, but on the other hand it is also the moment sequence of a measure on $[-1,1]^n$, as $\mathbf{y}^* \in [-1,1]^{\mathbb{N}^n}$ (Theorem 2.2). But by Theorem 2.2, $\mathbf{y}^*$ is the moment sequence of exactly one measure. So the measure must be supported by $K_S \cap [-1,1]^n$. This leads to the fact that in Corollary 3.6, the polynomial f must only be nonnegative on $K_S \cap [-1,1]^n$ for the statement to hold. So for example if $[-1,1]^n \cap K_S = \emptyset$, it holds for every polynomial f.

However, notice that "$f + \epsilon\Theta_r$ *lies in* T_S" provides a certificate of nonnegativity of f on $K_S \cap [-1,1]^n$ only, and *not* on K_S. So Corollary 3.6 is useful when one already knows that f is nonnegative on K_S and one wishes to obtain an l_1-norm approximation in T_S. If one wishes to test whether f is indeed nonnegative on K_S, then the following result provides a certificate of nonnegativity on K_S.

Corollary 3.7. *Let $S \subset \mathbb{R}[X]$ be a finite set of polynomials and suppose that S has the strong moment property. Further, suppose that K_S has a nonempty interior. Let $f \in \mathbb{R}[X]$ be nonnegative on K_S and let $\theta_r \in \mathbb{R}[X]$ be as in* (2)*. Let $\varepsilon > 0$ be fixed. Then there is some $r(f,\varepsilon,S)$ such that for every $r \geq r(f,\varepsilon,S)$, the polynomial $f_{\varepsilon r} := f + \varepsilon\theta_r$ lies in T_S.*

The proof is similar to that of Corollary 3.6, except that in the semidefinite program (7) we now have the constraint $L(\theta_r) \le 1$ (instead of $L(\Theta_r) \le 1$). In this case, every sequence of optimal solutions for (7) (with r growing) has a subsequence that converges pointwise to some $\mathbf{y}^* \in \mathbb{R}^{\mathbb{N}^n}$ (rather than $\mathbf{y}^* \in [-1,1]^{\mathbb{N}^n}$). To prove this result, and as one cannot use Theorem 2.2 any more, one now invokes Nussbaum's result [13] on moment sequences, which, in the present context, states that if

$$\sum_{i=1}^{n}\sum_{k=1}^{\infty} L(X_i^{2k})^{-1/2k} \;=\; +\infty, \qquad i = 1,\dots,n,$$

then L is integration with respect to some measure on $\mathbb{R}^n$; see also Berg [2, Theorem 8]. The rest of the proof is identical.

That Corollary 3.7 provides a certificate of nonnegativity of f on K_S follows from the fact that $\theta_r(x)$ is bounded by $\sum_{i=1}^n \exp(x_i^2)$, for all $x \in \mathbb{R}^n$. Therefore, fix $x \in K_S$; as $f + \epsilon\theta_r$ lies in T_S, one has $f(x) + \epsilon\theta_r(x) \ge 0$. Letting $\epsilon \to 0$ yields $f(x) \ge 0$, the desired result.

The result in Corollary 3.6 (resp. in Corollary 3.7) is weaker than the condition $f + \varepsilon q \in T_S$ for some fixed q and all $\varepsilon > 0$, as our Θ_r (resp. θ_r) depends on ε (via r). Whether the moment property in general implies even this stronger version is an open problem, see for example [7, 8]. If the basic closed semi-algebraic set K_S is compact, then by Schmüdgens Theorem, combined with Havilands result [4, 5], S has the strong moment property, but even more, every polynomial strictly positive on K_S belongs to T_S. So in this case $q = 1$ can be chosen in the approximating sequence $f + \varepsilon q$.

3.3. **The degree of the perturbation.** We are now concerned with the last announced result. We prove that the degree $r(f, \varepsilon)$ in Corollary 3.3 does *not* depend on the *explicit choice* of the polynomial f but only on

- ε and the dimension n,
- the degree and the size of the coefficients of f.

Therefore, if we fix these four parameters, we find an r such that the statement of Corollary 3.3 holds for any f nonnegative on $[-1,1]^n$, whose degree and size of the coefficients do not exceed the fixed parameters.

We first generalize Corollary 3.3 to real closed extension fields of $\mathbb{R}$ and then use the result in an ultrapower of $\mathbb{R}$. This approach towards degree bounds is similar to the one in [15].

Let Θ_r be as in (1). We first write the fact that there is no duality gap between the SDP problems (4) and (5) as a first order logic formula in the language of ordered rings with coefficients from $\mathbb{R}$. We just say that for every polynomial f of some fixed maximum degree $2r$, there is a linear form L on $\mathcal{A}_{2r}$ (indeed a $s(2r)$-tuple of values) which is nonnegative on $\mathcal{A}_{2r}^{\text{sos}}$ and which is less than or equal to 1 on Θ_r. We also demand that all the values of L on the monomial basis are bounded by 1 (as we have seen, this follows from the other conditions anyway). Further, we say that there exists some ε such that $f + \varepsilon\Theta_r$ is a s.o.s and $\varepsilon = -L(f)$ with L from above. All this can be done, using the known fact that every polynomial in $\mathcal{A}_{2r}^{\text{sos}}$ is already a sum of $s(2r)$ squares of polynomials from $\mathcal{A}_r$.

So, by Tarski's Transfer Principle, for every $r \in \mathbb{N}$, this formula holds in every real closed extension field of $\mathbb{R}$. We use this in the following theorem:

Theorem 3.8. *Let R be a real closed extension field of $\mathbb{R}$, and denote by $\mathcal{O}$ the convex hull of $\mathbb{Z}$ with respect to the unique ordering in R. Let $\mathfrak{m}$ denote the unique maximal ideal in the valuation ring $\mathcal{O}$, and fix some $\varepsilon \in R, \varepsilon > 0$ and $\varepsilon \notin \mathfrak{m}$. Suppose $f \in \mathcal{O}[X]$ is nonnegative on $[-1,1]^n \subset R^n$. Then there exists $r \in \mathbb{N}$ such that the polynomial $f_{\varepsilon r} = f + \varepsilon\Theta_r$ is a s.o.s. in $R[X]$.*

Proof. Let $\overline{f}$ be the real polynomial obtained from f by applying the residue map $\sigma\colon \mathcal{O} \to \mathcal{O}/\mathfrak{m} = \mathbb{R}$ to the coefficients of f. As $f \geq 0$ on $[-1,1]^n \subset R^n$, we have $\overline{f} \geq 0$ on $[-1,1]^n \subset \mathbb{R}^n$.

Next, consider the SDP problems from (3) associated with $\overline{f}$. From Theorem 3.2, there exists some r such that $\varepsilon_r^* > -\sigma(\varepsilon)$ ($\varepsilon > 0, \varepsilon \notin \mathfrak{m}$ implies $\sigma(\varepsilon) > 0$). With that r fixed, we now use that the formula described above holds in R. That is, we first get a linear form L on the subspace of polynomials of $R[X]$ with degree at most $2r$, whose values on the monomial basis are bounded by 1 (and therefore, are in $\mathcal{O}$), which is nonnegative on the s.o.s. polynomials. Further, we also have $L(\Theta_r) \leq 1$. In addition, we get an ε' such that $f + \varepsilon'\Theta_r$ is a s.o.s. in $R[X]$ and $\varepsilon' = -L(f)$.

But now, we can apply the residue map σ to the values of L on the monomial basis and get a linear form $\overline{L}$ wich is feasible for the optimization problem from (3) associated with $\overline{f}$ and r. So

$$-\sigma(\varepsilon) < \varepsilon_r^* \leq \overline{L}(\overline{f}) = \sigma(L(f)) = -\sigma(\varepsilon').$$

This shows $\varepsilon' < \varepsilon$, and as $f + \varepsilon'\Theta_r$ is a s.o.s. in $R[X]$, so is $f + \varepsilon\Theta_r$. □

Once we have this result, the rest follows from a standard ultrapower argument. We use the result in

$$\mathbb{R}^* = (\prod_{\mathbb{N}} \mathbb{R})/\mathcal{U},$$

where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$.

Fix some $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, and define by a first order logic formula Φ in the language of ordered rings, the set of all polynomials f of degree at most d, with coefficients bounded by some $N \in \mathbb{N}$, and which are nonnegative on $[-1,1]^n$.

Next, for every $r \in \mathbb{N}$, define by a formula φ_r, the set of all polynomials f of degree at most d, such that $f + \varepsilon\Theta_r$ is a s.o.s.

Notice that boundedness of the coefficients of a polynomial f by some $N \in \mathbb{N}$, implies $f \in \mathcal{O}[X]$, and so, by Theorem 3.8, one has

$$\Phi \to \bigvee_{r \in \mathbb{N}} \varphi_r.$$

Now the $\aleph_1$-saturation of $\mathbb{R}^*$ yields

$$\Phi \to \varphi_{r'}$$

for some r' depending on the formulas used, i.e. on d, N, n, ε. Therefore, in $\mathbb{R}^*$ one may choose the degree r in Theorem 3.8 to depend only on d, N, n, ε. As this can be again formulated as a first order logic formula, it holds in $\mathbb{R}$ as well:

Theorem 3.9. *Let $n, N, d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}_{>0}$ be given. Then there exists $r = r(n, N, d, \varepsilon) \in \mathbb{N}$ such that for every $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most d, with* *coefficients bounded by N, and nonnegative on $[-1,1]^n$, the polynomial $f+\varepsilon\Theta_r$ is a s.o.s. (and so are $f+\epsilon\Theta_{r'}$ for all $r'\geq r$).*

4. APPENDIX

In this section we derive auxiliary results that are helpful in the proofs of the main section.

Lemma 4.1. *Let $n=1$ and let $L\colon \mathcal{A}_{2r}\to\mathbb{R}$ be a linear form such that $L(p^2)\geq 0$ for all $p\in\mathcal{A}_r$. Then $L(X^{2k})\leq\max[L(1),L(X^{2r})]$ for all $k=0,\dots,r$.*

Proof. The proof is by induction on r. Indeed for $r=0$ and $r=1$ the statement is trivial. So we assume the statement of Lemma 4.1 is true for some r and we prove it for $r+1$.

Let L be a linear form on $\mathcal{A}_{2r+2}$ as stipulated. From $L(p^2)\geq 0$ for all $p\in\mathcal{A}_{r+1}$ we have

$$L(X^{2r})^2\leq L(X^{2r+2})L(X^{2r-2}). \tag{9}$$

By the induction hypothesis, we have

$$L(X^{2k})\leq\max[L(1),L(X^{2r})],\qquad k=0,\dots,r.$$

Suppose first that $L(1)=\max[L(1),L(X^{2r})]$. Then obviously $L(X^{2k})\leq\max[L(1),L(X^{2r+2})]$ for all $k\leq r+1$ and we are done. Next, suppose $L(X^{2r})=\max[L(1),L(X^{2r})]$. Then from (9) we obtain

$$L(X^{2r})^2\leq L(X^{2r+2})L(X^{2r-2})\leq L(X^{2r+2})L(X^{2r}),$$

so that $L(X^{2r})\leq L(X^{2r+2})$. Therefore again $L(X^{2k})\leq\max[L(1),L(X^{2r+2})]$ for all $k=0,\dots,r+1$, the desired result. □

Lemma 4.2. *Let $n=2$ and $L:\mathcal{A}_{2r}\to\mathbb{R}$ be a linear form and suppose $L(p^2)\geq 0$ for all $p\in\mathcal{A}_r$. Then all values $L(X^{2\alpha})$ where $0\leq|\alpha|\leq r$ are bounded by $\max_{k=0,\dots,r}\max\{L(X_1^{2k}),L(X_2^{2k})\}$.*

Proof. What we will actually show is that all $L(X^{2\alpha})$, where $|\alpha|=k$, are bounded by $\max\{L(X_1^{2k}),L(X_2^{2k})\}$.

Let $p\in\mathbb{N}$ be such that either $k=2p$ (if k is even) or $k=2p+1$ (if k is odd) and define $\Gamma:=\{\ (2a,2b)\mid a+b=k;\ a,b\neq 0\}$. One has $\Gamma=\Gamma_1\cup\Gamma_2$ where

$$\begin{aligned}\Gamma_1 &:= \{\ (k,0)+(k-2i,2i)\mid i=1,...,p\ \}\\ \Gamma_2 &:= \{\ (0,k)+(2j,k-2j)\mid j=1,...,p\}.\end{aligned}$$

If k is odd, then this union is disjoint, else $\Gamma_1\cap\Gamma_2=\{(2p,2p)\}$. For $s:=\max\{L(X^\gamma)\mid\gamma\in\Gamma\ \}$, we get $s=L(X^{\gamma^*})$ for some $\gamma^*\in\Gamma_1$ or $\gamma^*\in\Gamma_2$.

From $L(p^2)\geq 0$ for all $p\in\mathcal{A}_r$ we have

$$L(X^{\alpha+\beta})^2\leq L(X^{2\alpha})L(X^{2\beta}).$$

So in our case, we obtain

$$L\left(X_1^{2k}\right)\cdot L\left(X_1^{2k-4i}X_2^{4i}\right)\ \geq\ L\left(X_1^{2k-2i}X_2^{2i}\right)^2,\ i=1,...,p \tag{10}$$

$$L\left(X_2^{2k}\right)\cdot L\left(X_1^{4j}X_2^{2k-4j}\right)\ \geq\ L\left(X_1^{2j}X_2^{2k-2j}\right)^2,\ j=1,...,p. \tag{11}$$

With $s_k:=\max\{L\left(X_1^{2k}\right),L\left(X_2^{2k}\right)\}$, by (10) and (11), one gets either

$$s_k\cdot s\geq L\left(X_1^{2k}\right)\cdot L\left(X^{\gamma^*}\right)\geq L\left(X^{\gamma^*}\right)^2=s^2$$

or

$$s_k \cdot s \geq L\left(X_2^{2k}\right) \cdot L\left(X^{\gamma^*}\right) \geq L\left(X^{\gamma^*}\right)^2 = s^2.$$

In any case $s_k \geq s$. □

Lemma 4.3. *Let n be arbitrary and $L : \mathcal{A}_{2r} \to \mathbb{R}$ be a linear form and suppose $L(p^2) \geq 0$ for all $p \in \mathcal{A}_r$. Assume that for all i=1,...,n and k=0,...,r, the values $L(X_i^{2k})$ are bounded by some τ. Then all values $L(X^\alpha)$, where $|\alpha| \leq 2r$, satisfy $|L(X^\alpha)| \leq \tau$.*

Proof. We only need to show that all values $L\left(X^{2\alpha}\right)$, where $|\alpha| \leq r$, are bounded by τ. Indeed, from $L(p^2) \geq 0$ for all $p \in \mathcal{A}_r$ we have $L(X^{\alpha+\beta})^2 \leq L(X^{2\alpha})L(X^{2\beta})$, and therefore, if all the values $L\left(X^{2\gamma}\right)$ are bounded by τ, one gets $|L\left(X^\alpha\right)| \leq \tau$ for all $0 \leq |\alpha| \leq 2r$.

The proof is by induction on the number n of variables.

$n = 1$: Nothing is to be shown in this case, as all the values $L\left(X^{2\alpha}\right)$ are bounded by τ by the assumption.

$n = 2$: This is an immediate result of Lemma 4.2.

$n-1 \rightsquigarrow n, n > 2$: By the induction hypothesis, the claim is true for all $L\left(X^{2\alpha}\right)$, where $|\alpha| \leq r$ and some $\alpha_i = 0$. Indeed, L restricts to a linear form on the ring of polynomials with $n-1$ indeterminates and satisfies all the assumptions needed. So the induction hypothesis gives the boundedness of all those values $L\left(X^{2\alpha}\right)$.

Now take $L\left(X^{2\alpha}\right)$, where $|\alpha| \leq r$ and all $\alpha_i \geq 1$. With no loss of generality, assume $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$. Consider the two elements

$$\gamma := (2\alpha_1, 0, \alpha_3 + \alpha_2 - \alpha_1, \alpha_4, ..., \alpha_n) \in \mathbb{N}^n \text{ and}$$

$$\gamma' := (0, 2\alpha_2, \alpha_3 + \alpha_1 - \alpha_2, \alpha_4, ..., \alpha_n) \in \mathbb{N}^n.$$

We have $|\gamma|, |\gamma'| \leq r$ and $\gamma_2 = \gamma'_1 = 0$. Therefore, by the above result, we get

$$L\left(X^{2\gamma}\right) \leq \tau \text{ and } L\left(X^{2\gamma'}\right) \leq \tau.$$

As $L(p^2) \geq 0$ for all $p \in \mathcal{A}_r$ one has

$$L\left(X^{2\alpha}\right)^2 = L\left(X^{\gamma+\gamma'}\right)^2 \leq L\left(X^{2\gamma}\right) \cdot L\left(X^{2\gamma'}\right) \leq \tau^2,$$

which yields

$$|L\left(X^{2\alpha}\right)| \leq \tau.$$

□

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