EXPOSED FACES OF SEMIDEFINITELY REPRESENTABLE SETS

TIM NETZER, DANIEL PLAUMANN, AND MARKUS SCHWEIGHOFER

ABSTRACT. A linear matrix inequality (LMI) is a condition stating that a symmetric matrix whose entries are affine linear combinations of variables is positive semidefinite. Motivated by the fact that diagonal LMIs define polyhedra, the solution set of an LMI is called a spectrahedron. Linear images of spectrahedra are called semidefinitely representable sets. Part of the interest in spectrahedra and semidefinitely representable sets arises from the fact that one can efficiently optimize linear functions on them by semidefinite programming, like one can do on polyhedra by linear programming.

It is known that every face of a spectrahedron is exposed. This is also true in the general context of rigidly convex sets. We study the same question for semidefinitely representable sets. Lasserre proposed a moment matrix method to construct semidefinite representations for certain sets. Our main result is that this method can only work if all faces of the considered set are exposed. This necessary condition complements sufficient conditions recently proved by Lasserre, Helton and Nie.

INTRODUCTION

A linear matrix polynomial is a symmetric matrix whose entries are real linear polynomials in n variables. Such a matrix can be evaluated in any point of $\mathbb{R}^n$, and the set of points where it is positive semidefinite is a closed convex subset of $\mathbb{R}^n$. If the matrix is diagonal, the resulting set is a polyhedron. Since sets defined by general linear matrix polynomials inherit certain properties from polyhedra, they are called *spectrahedra*. Sometimes also the term *LMI (representable) sets* has been used.

Spectrahedra have long been of interest in applications, see for example the book of Boyd, El Ghaoui, Feron, and Balakrishnan [5]. Most importantly, spectrahedra are the feasible sets of semidefinite programs, which have been much studied in recent years, as explained for example in Vandenberghe and Boyd [23]. Semidefinite programming is a generalization of linear programming for which there exist efficient algorithms.

Projections of spectrahedra will be called *semidefinitely representable sets*. They are still useful for optimization. Indeed, instead of optimizing a linear function on the projection, one can optimize the same function on the higher dimensional spectrahedron itself.

In recent years, the fundamental question to characterize spectrahedra and their projections geometrically has gained a lot of attention. Helton and Vinnikov have

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introduced the notion of rigid convexity, which is an obvious property of spectrahedra. They show that in dimension two this property characterizes spectrahedra, and conjecture that the same is true in arbitrary dimension [9]. As for semidefinitely representable sets, the only known property besides convexity is that they are semialgebraic, i.e. described by a boolean combination of polynomial inequalities. Indeed, Helton and Nie conjecture that every convex semialgebraic set is semidefinitely representable [7]. Lasserre proposed a construction to approximate convex semialgebraic sets by semidefinitely representable sets [12]. Under certain conditions this approximation is exact, i.e. the original set is semidefinitely representable itself. Helton and Nie have shown that these conditions are satisfied for a surprisingly large class of sets, see [9] Theorem 5.1. They also prove that Lasserre's method can be applied locally for compact sets. This allows them to show semidefinite representability for an even larger class of sets.

In this work, we investigate the facial geometry of spectrahedra, rigidly convex sets and semidefinitely representable sets. It is known that all faces of a spectrahedron are exposed. We review this fact in Section 2 and prove the same for rigidly convex sets, as a consequence of Renegar's result for hyperbolicity cones [20]. Our main result is Theorem 3.5 in Section 3. We prove that Lasserre's construction can only be exact if all faces of the considered convex set are exposed. This is a necessary condition which complements the sufficient conditions from the above mentioned literature. We use real algebra, basic model theory, and convex geometry in our proof.

1. Preliminaries

Let $\mathbb{R}[\underline{t}]$ denote the polynomial ring in n variables $\underline{t} = (t_1, \ldots, t_n)$ with coefficients in $\mathbb{R}$. A subset S of $\mathbb{R}^n$ is called *basic closed* if there exist polynomials $p_1, \ldots, p_m \in \mathbb{R}[\underline{t}]$ such that

$$S = \mathscr{S}(p_1, \ldots, p_m) = \{x \in \mathbb{R}^n \mid p_1(x) \geqslant 0, \ldots, p_m(x) \geqslant 0\}.$$

A *linear matrix polynomial* (of dimension k in the variables $\underline{t}$) is a linear polynomial whose coefficients are real symmetric $k \times k$-matrices, i.e. an expression $A(\underline{t}) = A_0 + t_1 A_1 + \cdots + t_n A_n$ with $A_0, \ldots, A_n \in \mathrm{Sym_k}(\mathbb{R})$. A subset S of $\mathbb{R}^n$ is called a *spectrahedron*, if it is defined by a linear matrix inequality, i.e. if there exists a linear matrix polynomial $A(\underline{t})$ such that

$$S = \mathscr{S}(A) = \{x \in \mathbb{R}^n \mid A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0\},$$

where $\succeq 0$ denotes positive semidefiniteness. It is obvious that spectrahedra are closed and convex. They are also basic closed: A real symmetric matrix is positive semidefinite if and only if the coefficients of its characteristic polynomial have alternating signs; write

$$\det(A(\underline{t}) - sI_k) = c_0(\underline{t}) + c_1(\underline{t})s + \cdots + c_{k-1}(\underline{t})s^{k-1} + (-1)^k s^k$$

with $p_i \in \mathbb{R}[\underline{t}]$, then

$$\mathscr{S}(A) = \mathscr{S}(c_0, -c_1, \ldots, (-1)^{k-1} c_{k-1}).$$

A further property of spectrahedra is their rigid convexity: A polynomial $p \in \mathbb{R}[\underline{t}]$ is called a *real zero polynomial with respect to $e \in \mathbb{R}^n$ (RZ_e-polynomial)* if $p(e) > 0$ and all zeros of the univariate polynomial $p(e+sv) \in \mathbb{R}[s]$ are real, for every $v \in \mathbb{R}^n \setminus \{0\}$. A set S is called *rigidly convex* if there exists $e \in S$ and an RZ_e-polynomial p such that S is the closure of the connected component of $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ containing e. Rigid convexity was introduced and studied by Helton and Vinnikov

[9]. Rigidly convex sets are convex (see Section 5.3 in [9]); they are also basic closed (see Remark 2.6 below). Furthermore, any spectrahedron with non-empty interior is rigidly convex. The principal reason is that if $A(\underline{t})$ is a linear matrix polynomial with $A_0 \succ 0$, then $p(\underline{t}) = \det(A(\underline{t}))$ is an RZ_0-polynomial defining $\mathscr{S}(A)$ (see [9], Thm. 2.2). A much harder question is whether every rigidly convex set is a spectrahedron. This has been shown for $n = 2$ and conjectured in general by Helton and Vinnikov in [9]. The question is closely related to the famous Lax-conjecture.

A subset S of $\mathbb{R}^n$ is called *semidefinitely representable* if it is the image of a spectrahedron S' in $\mathbb{R}^m$ under a linear map $\mathbb{R}^m \to \mathbb{R}^n$. A linear matrix representation of S' together with the linear map is called a *semidefinite representation* of S. In contrast to spectrahedra, no necessary conditions other than convexity are known for a semialgebraic set to be semidefinitely representable.

Various sufficient conditions have recently been given by Lasserre [12] as well as Helton and Nie [8], [7]. Moreover, it has been shown that various operations, like taking the interior or taking the convex hull of a finite union, preserve semidefinite representability, see [16] and [15].

2. Faces of spectrahedra and rigidly convex sets

In this section, we study the facial structure of spectrahedra and rigidly convex sets (see also [6] for a discussion of facial structures in a more abstract setting). We review the result of Ramana and Goldman that every spectrahedron has only exposed faces. We then discuss how the same result can be proven for rigidly convex sets, mostly by going back to Renegar's corresponding result for hyperbolicity cones.

Definitions 2.1. Let S be a closed convex subset of $\mathbb{R}^n$ with non-empty interior. A *supporting hyperplane* of S is an affine hyperplane H in $\mathbb{R}^n$ such that $S \cap H \neq \emptyset$ and $S \setminus H$ is connected (equivalently, the zero set of a linear polynomial $0 \neq \ell \in \mathbb{R}[\underline{t}]$ such that $\ell \geqslant 0$ on S and $\{\ell = 0\} \cap S \neq \emptyset$).

A *face* of S is a non-empty convex subset $F \subseteq S$ with the following property: For every $x, y \in S$, $\lambda \in (0, 1)$, if $\lambda x + (1 - \lambda)y \in F$, then $x, y \in F$.

A face F of S is called *exposed* if either $F = S$ or there exists a supporting hyperplane H of S such that $H \cap S = F$. The hyperplane H is said to *expose* F.

The *dimension* of a face F is the dimension of its affine hull.

Remarks 2.2.

(1) $H \cap S$ is an exposed face of S for any supporting hyperplane H of S.
(2) For every face $F \subsetneq S$ there exists a supporting hyperplane H of S such that $F \subseteq H$.
(3) Every face of S is closed (since S is closed).
(4) If F_1, F_2 are faces of S with $F_1 \subsetneq F_2$, then $\dim(F_1) < \dim(F_2)$.
(5) Let F be a face of S, and take x_0 in the relative interior of F. For any two points $x \neq y \in \mathbb{R}^n$, let $g(x, y)$ denote the line passing through x and y. Then F consists exactly of x_0 and those points $x \in S \setminus \{x_0\}$ such that x_0 lies in the relative interior of $g(x, x_0) \cap S$.

The following is a combination of Theorem 1 and Corollary 1 in [19] (see Corollary 1 in [6] for a more general statement).

Theorem 2.3 (Ramana and Goldman)**.** *Let $A(\underline{t})$ be a linear matrix polynomial of dimension k, $S = \mathscr{S}(A)$. For every linear subspace U of $\mathbb{R}^k$, the set*

$$F_U = \big\{x \in S \mid U \subseteq \ker\big(A(x)\big)\big\}$$

is a face of S or empty, and every face of S is of this form. Furthermore, every face of S is exposed.

A similar result can be proven for rigidly convex sets, by reducing to the results of Renegar on hyperbolicity cones that we now describe: A homogeneous polynomial P in $n+1$ variables is called *hyperbolic with respect to* $e \in \mathbb{R}^{n+1} \setminus \{0\}$ if $P(e) > 0$ and all zeros of the univariate polynomial $P(x - se) \in \mathbb{R}[s]$ are real, for every $x \in \mathbb{R}^{n+1}$. The *hyperbolicity cone of* P is the connected component of $\{P > 0\}$ containing e. It is a convex cone in $\mathbb{R}^{n+1}$. Its closure is called the *closed hyperbolicity cone of* P.

Theorem 2.4 (Renegar [20], Thm. 23). *The faces of a closed hyperbolicity cone are exposed.*

Corollary 2.5. *The faces of a rigidly convex set are exposed.*

Proof. It is well-known and easy to see that a polynomial $p \in \mathbb{R}[\underline{t}]$ is an RZ_e-polynomial if and only if the homogenisation $P(\underline{t}, u) = u^d p(\frac{t}{u})$ is hyperbolic with respect to $\widetilde{e} = (e, 1)$. Furthermore, the rigidly convex set $S \subseteq \mathbb{R}^n$ defined by p (i.e. the closure of the connected component of $\{p > 0\}$ containing e) is the intersection of C, the closed hyperbolicity cone of P in $\mathbb{R}^{n+1}$, with the hyperplane $H = \{u = 1\}$.

Let F_0 be a face of S. For any two points $x \neq y \in \mathbb{R}^{n+1}$, let $g(x, y)$ denote the line passing through x and y. Take x_0 in the relative interior of F_0, and let F be the set of all points $z \in C$ such that x_0 lies in the relative interior of $g(z, x_0) \cap C$. One checks that F is a face of C and that $F \cap H = F_0$ (see Remark 2.2 (5)). Since F is exposed by Thm. 2.4, so is F_0. □

The idea of the proof of Renegar's theorem is the following: Let P be a homogeneous polynomial in $n + 1$ variables $(\underline{t}, u)$ that is hyperbolic with respect to $e \in \mathbb{R}^{n+1} \setminus \{0\}$, and let C be the closed hyperbolicity cone of P. For every $k \geqslant 0$, put

$$P^{(k)}(\underline{t}, u) = \frac{d^k}{ds^k} P\big((\underline{t}, u) + se\big)\bigg|_{s=0}.$$

The polynomials $P^{(k)}$ are again hyperbolic with respect to e (by Rolle's theorem) and the corresponding closed hyperbolicity cones $C^{(k)}$ form an ascending chain $C = C^{(0)} \subseteq C^{(1)} \subseteq C^{(2)} \subseteq \cdots$. For $x \in C$, define $\operatorname{mult}(x)$ as the multiplicity of 0 as a zero of the univariate polynomial $P(x + se) \in \mathbb{R}[s]$. If $\operatorname{mult}(x) = m$, then x is a boundary point of $C^{(m-1)}$ and a regular point of $\{P^{(m-1)} = 0\}$, i.e. $(\nabla P^{(m-1)})(x) \neq 0$. Now if F is a face of C and x is in the relative interior of F, then the tangent space of $P^{(m-1)}$ in x exposes F as a face of $C^{(m-1)}$ and hence as a face of C.

This translates into the setting of rigid convexity as follows: Let $p \in \mathbb{R}[\underline{t}]$ be an RZ_0-polynomial of degree d, and let S be the corresponding rigidly convex set; write $p = \sum_{i=0}^d p_i$ with p_i homogeneous of degree i, and put $P(\underline{t}, u) = u^d p(\frac{t}{u}) = \sum_{i=0}^d p_{d-i}(\underline{t}) u^i$. Define $P^{(k)}$ for $k \geqslant 0$ as above and put $p^{(k)}(\underline{t}) = P^{(k)}(\underline{t}, 1)$, so that

$$p^{(k)}(\underline{t}) = \sum_{i=k}^{d} \frac{i!}{(i-k)!} p_{d-i}(\underline{t}).$$

The polynomials $p^{(k)}$ are again RZ_0-polynomials and the corresponding rigidly convex sets form an ascending chain $S = S^{(0)} \subseteq S^{(1)} \subseteq S^{(2)} \subseteq \cdots$. For any $x \in S$, we find that $\operatorname{mult}(x)$ is the multiplicity of 0 as a zero of the univariate polynomial $\sum_{i=0}^d p_{d-i}(x)(1 + s)^i \in \mathbb{R}[s]$. A simple computation shows that $\operatorname{mult}(x)$ is also the multiplicity of 1 as a zero of the univariate polynomial $p(sx) \in \mathbb{R}[s]$.

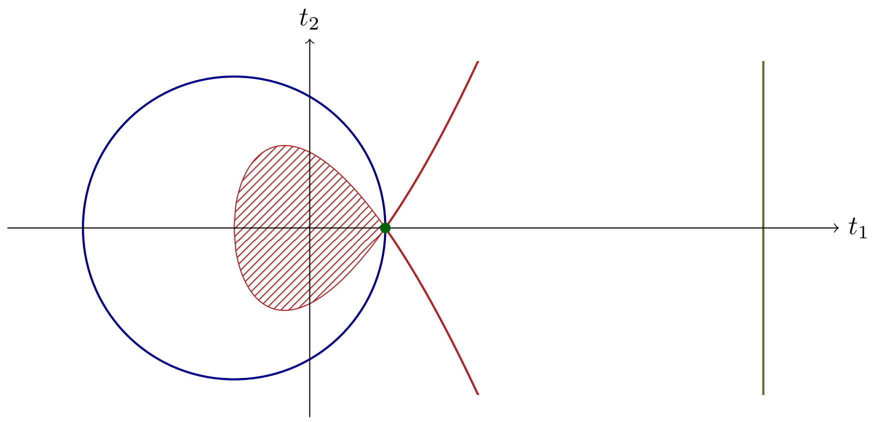
Now let F be a face of S, let x be a point in the relative interior of F, and put $m = \operatorname{mult}(x)$. Then x is a boundary point of $S^{(m-1)}$ and a regular point of $\{p^{(m-1)} = 0\}$. The tangent space $\{x + v \mid (\nabla p^{(m-1)}(x))^t v = 0\}$ exposes F as a face of S.

Remark 2.6. It follows from Renegar's construction that closed hyperbolicity cones and rigidly convex sets are basic closed semialgebraic sets. Namely, if C is the closed hyperbolicity cone of a hyperbolic polynomial P of degree d, then $C = \mathscr{S}(P, P^{(1)}, \ldots, P^{(d-1)})$; similarly, if S is a rigidly convex set corresponding to an RZ_0-polynomial p of degree d, then $S = \mathscr{S}(p, p^{(1)}, \ldots, p^{(d-1)})$.

Alternatively, one can use the fact that the closed hyperbolicity cone of P coincides with the set of all $x \in \mathbb{R}^{n+1}$ such that all zeros of $P(x - se) \in \mathbb{R}[s]$ are nonnegative. This translates to an alternating sign condition on the coefficients with respect to s, as explained in Section 1.

Example 2.7. Let $p = t_1^3 - t_1^2 - t_1 - t_2^2 + 1 \in \mathbb{R}[t_1, t_2]$. One checks that p is an irreducible RZ_0-polynomial. The corresponding rigidly convex set, i.e. the closure of the connected component of $\{p > 0\}$ containing 0, is the basic closed set $S = \mathscr{S}(p, 1 - t_1)$.

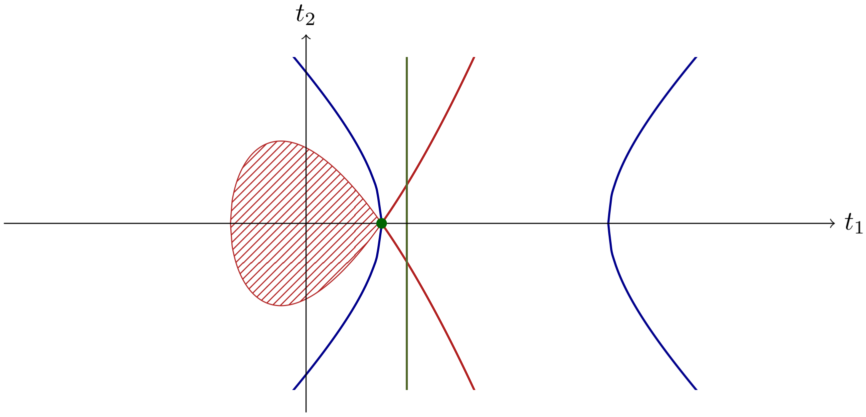
We have $\operatorname{mult}(x) = 1$ for every boundary point $x \in \partial S \setminus \{(1,0)\}$, and $\operatorname{mult}(1,0) = 2$. Furthermore, $p^{(1)} = -t_1^2 - t_2^2 - 2t_1 + 3$, $p^{(2)} = 6 - t_1$. Every $x \in \partial S \setminus \{(1,0)\}$ is a regular point of $\{p = 0\}$ and is exposed as a face of S by the tangent line to $\{p = 0\}$ in x. The point $(1,0)$ is a regular point of $\{p^{(1)} = 0\}$ and is exposed as a face of S by the tangent line to that curve in $(1,0)$, which is $t_1 = 1$. We also see that $S = \mathscr{S}(p, p^{(1)}, p^{(2)})$ (though $p^{(2)}$ is redundant):



By the theorem of Helton and Vinnikov, S is a spectrahedron. Explicitly, let $A(t_1, t_2) = A_0 + t_1 A_1 + t_2 A_2$ with

$$A_0 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -2 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the characteristic polynomial, one finds $\chi_A(s) = c_0 + c_1 s + c_2 s^2 - s^3$ with $c_0 = p$, $c_1 = -t_1^2 + 5t_1 + t_2^2 - 4$, $c_2 = 4 - 3t_1$. One checks that $S = \mathscr{S}(A) = \mathscr{S}(c_0, -c_1, c_2) = \mathscr{S}(c_0, -c_1)$. This gives an alternative description of S as a basic closed set.



3. Exposed faces and Lasserre relaxations

For a certain class of convex semialgebraic sets, Lasserre has given an explicit semidefinite representation [12] (see [10] for a less well known but related construction), as follows: Let $\underline{p} = (p_1, \ldots, p_m)$ be an m-tuple of real polynomials in n variables $\underline{t}$, and set $p_0 = 1$. Let $\mathrm{QM}(\underline{p})$ be the quadratic module generated by $\underline{p}$, i.e.

$$\mathrm{QM}(\underline{p}) = \left\{ \sum_{i=0}^{m} \sigma_i p_i \mid \sigma_i \in \sum \mathbb{R}[\underline{t}]^2 \right\}$$

where $\sum \mathbb{R}[\underline{t}]^2 = \{f_1^2 + \cdots + f_r^2 \,|r \geqslant 0, f_1, \ldots, f_r \in \mathbb{R}[\underline{t}]\}$. We denote by $\mathbb{R}[\underline{t}]_d$ the finite-dimensional vector space of polynomials of degree at most d, and write $\mathbb{R}[\underline{t}]_d^\vee$ for its (algebraic) dual. Define

$$\mathrm{QM}(\underline{p})_d = \left\{ \sum_{i=0}^{m} \sigma_i p_i \mid \sigma_i \in \sum \mathbb{R}[\underline{t}]^2;\ \sigma_i p_i \in \mathbb{R}[\underline{t}]_d \right\}.$$

Note that the inclusion $\mathrm{QM}(\underline{p})_d \subseteq \mathrm{QM}(\underline{p}) \cap \mathbb{R}[\underline{t}]_d$ is in general not an equality. Let

$$\mathscr{L}(\underline{p})_d = \left\{ L \in \mathbb{R}[\underline{t}]_d^\vee \mid L|_{\mathrm{QM}(\underline{p})_d} \geqslant 0, L(1) = 1 \right\}.$$

It is well-known that $\mathscr{L}(\underline{p})_d$ is a spectrahedron in $\mathbb{R}[\underline{t}]_d^\vee$ (see for example Marshall [14], 10.5.4). Now consider the projection $\pi \colon \mathbb{R}[\underline{t}]_d^\vee \to \mathbb{R}^n$, $L \mapsto (L(t_1), \ldots, L(t_n))$ and put

$$S(\underline{p})_d = \pi\big(\mathscr{L}(\underline{p})_d\big),$$

a semidefinitely representable subset of $\mathbb{R}^n$. The idea is to compare $S(\underline{p})_d$ with $S = \mathscr{S}(\underline{p})$, the basic closed set determined by $\underline{p}$. Note first that $S(\underline{p})_d$ contains S and therefore its convex hull: For if $x \in S$, let $L_x \in \mathbb{R}[\underline{t}]_d^\vee$ denote evaluation in x; then $L_x \in \mathscr{L}(\underline{p})_d$ and $\pi(L_x) = x$. Note also that the sets $S(\underline{p})_d$ form a decreasing sequence, i.e.

$$S(\underline{p})_{d+1} \subseteq S(\underline{p})_d$$

holds for all d.

We call the set $S(\underline{p})_d$ the d-th *Lasserre relaxation* of $\mathrm{conv}(S)$ with respect to $\underline{p}$. If there exists $d \geqslant 0$ such that $S(\underline{p})_d = \mathrm{conv}(S)$, we say that $\mathrm{conv}(S)$ possesses an *exact Lasserre relaxation* with respect to $\underline{p}$. The existence of an exact Lasserre relaxation is a sufficient condition for the semidefinite representability of $\mathrm{conv}(S)$.

A characterization for exactness of Lasserre relaxations is the following proposition. The implication (2)⇒(1) is [12], Thm. 2.

Proposition 3.1. *Assume that $S = \mathscr{S}(\underline{p})$ has non-empty interior. For $d \in \mathbb{N}$, the following are equivalent:*

(1) $\operatorname{conv}(S) \subseteq S(\underline{p})_d \subseteq \overline{\operatorname{conv}(S)}$;
(2) *Every $\ell \in \mathbb{R}[\underline{t}]_1$ with $\ell|_S \geqslant 0$ is contained in $\operatorname{QM}(\underline{p})_d$.*

Proof. We include the proof of (2)⇒(1) for the sake of completeness. So assume that (2) holds and suppose that there exists $x \in S(\underline{p})_d \setminus \overline{\operatorname{conv}(S)}$. Thus there is $\ell \in \mathbb{R}[\underline{t}]_1$ with $\ell|_S \geqslant 0$ and $\ell(x) < 0$. Furthermore, there exists a linear functional $L\colon \mathbb{R}[\underline{t}]_d \to \mathbb{R}$ such that $L|_{\operatorname{QM}(\underline{p})_d} \geqslant 0$, $L(1) = 1$, and $x = \big(L(t_1), \ldots, L(t_n)\big)$. By assumption, ℓ belongs to $\operatorname{QM}(\underline{p})_d$, so $0 \leqslant L(\ell) = \ell(L(t_1), \ldots, L(t_n)) = \ell(x) < 0$, a contradiction.

For the converse, assume that (1) holds, and suppose that there exists $\ell \in \mathbb{R}[\underline{t}]_1$ with $\ell|_S \geqslant 0$ but $\ell \notin \operatorname{QM}(\underline{p})_d$. Since S has non-empty interior, $\operatorname{QM}(\underline{p})_d$ is a closed convex cone in $\mathbb{R}[\underline{t}]_d$ (see for example Marshall [14], Lemma 4.1.4, or Powers and Scheiderer [17], Proposition 2.6). Thus there exists a linear functional $L\colon \mathbb{R}[\underline{t}]_d \to \mathbb{R}$ such that $L|_{\operatorname{QM}(\underline{p})_d} \geqslant 0$, $L(1) = 1$, and $L(\ell) < 0$ (note that $L(1) = 1$ is non-restrictive; see the little trick in Marshall [13], proof of Theorem 3.1). Since $x = (L(t_1), \ldots, L(t_n)) \in S(\underline{p})_d \subseteq \overline{\operatorname{conv}(S)}$, we have $0 \leqslant \ell(x) = L(\ell) < 0$, a contradiction. □

An immediate consequence is that if $\operatorname{conv}(S)$ is closed (for example if S is compact or convex), then (2) implies that $\operatorname{conv}(S)$ is semidefinitely representable. Lasserre shows that (2) is satisfied for certain classes of sets, for example if all p_i are linear or concave and quadratic. These results have been extended substantially by Helton and Nie [8, 7].

In the following, we will give a necessary condition for (2) in the case that S is convex. Namely, all faces of S must be exposed. The following lemma and its proof are a special case of Prop. II.5.16 in Alfsen [1].

Lemma 3.2. *Let S be a closed convex subset of $\mathbb{R}^n$. A face F of S is exposed if and only if for every $x \in S \setminus F$ there exists a supporting hyperplane H of S with $F \subseteq H$ and $x \notin H$.*

Proof. Necessity is obvious. To prove sufficiency, write $F = \bigcap_{k \geqslant 1} U_k$ with U_k open subsets of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus U_k$ is compact for every $k \geqslant 1$ (note that F is closed by Remark 2.2 (3)). Fix $k \geqslant 1$. For each $x \in S \setminus U_k$, we can choose by hypothesis a linear polynomial $\ell_x \in \mathbb{R}[\underline{t}]$ such that $\{\ell_x = 0\}$ is a supporting hyperplane of S with $\ell_x|_F = 0$ und $\ell_x(x) > 0$. Since $S \setminus U_k$ is compact, we may choose $x_1, \ldots, x_m \in S \setminus U_k$ such that $\ell_k := \sum_{i=1}^m \ell_{x_i}$ is strictly positive on $S \setminus U_k$. Clearly, $\ell_k|_F = 0$. Put

$$\ell := \sum_{k=1}^{\infty} \frac{\ell_k}{2^k \cdot ||\ell_k||},$$

where $||\cdot||$ is a norm on the space of linear polynomials. Then $\{\ell = 0\}$ is a supporting hyperplane of S that exposes F. □

Lemma 3.3. *Let S be a closed convex subset of $\mathbb{R}^n$ with non-empty interior. A face F of S is exposed if and only if $F \cap U$ is an exposed face of $S \cap U$ for every affine-linear subspace U of $\mathbb{R}^n$ containing F with* $\dim(U) = \dim(F) + 2$ *and* $U \cap \operatorname{int}(S) \neq \emptyset$.

Proof. Note first that the condition is empty if F is of dimension $\geqslant n-1$. Indeed, F is always exposed in that case by Remark 2.2 (2),(4). Thus we may assume that $n \geqslant 2$ and $\dim(F) \leqslant n-2$.

If H exposes F and $U \cap \operatorname{int}(S)$ is non-empty, then $H \cap U$ exposes F in $S \cap U$. Conversely, assume that $F \cap U$ is an exposed face of $S \cap U$ for every U satisfying the hypotheses. We want to apply the preceding lemma. Let $x \in S \setminus F$, then we must produce a supporting hyperplane H of S containing F with $x \notin H$. Choose U to be an affine-linear subspace of $\mathbb{R}^n$ of dimension $\dim(F) + 2$ containing F such that $x \in U$ and $U \cap \operatorname{int}(S) \neq \emptyset$. By hypothesis, there exists a supporting hyperplane G of $S \cap U$ in U that exposes F as a face of $S \cap U$. In particular, $x \notin G$. Since $G \cap S = F$, it follows that $G \cap \operatorname{int}(S) = \emptyset$, hence by separation of disjoint convex sets (see e.g. Barvinok [3], Thm. III.1.2), there exists a hyperplane H that satisfies $G \subseteq H$ and $H \cap \operatorname{int}(S) = \emptyset$. Since $U \cap \operatorname{int}(S) \neq \emptyset$, it follows that $G \subseteq H \cap U \subsetneq U$, hence $G = H \cap U$. Thus H is a supporting hyperplane of S containing F with $x \notin H$. □

We need the following technical lemma.

Lemma 3.4. *Let S be a convex subset and U be an affine-linear subspace of $\mathbb{R}^n$ intersecting the interior of S. Suppose that $\ell\colon \mathbb{R}^n \to \mathbb{R}$ is an affine linear function such that $\ell \geqslant 0$ on $S \cap U$. Then there exists an affine linear function $\ell'\colon \mathbb{R}^n \to \mathbb{R}$ such that $\ell' \geqslant 0$ on S and $\ell'|_U = \ell|_U$.*

Proof. Let $N := \{x \in U \mid \ell(x) < 0\}$ and S' be the convex hull of $\{x \in U \mid \ell(x) \geqslant 0\} \cup S$. Then N and S' are convex sets that we now prove to be disjoint.

Assume for a contradiction that there are $\lambda \in [0, 1]$, $x \in U$ and $y \in S$ such that $\ell(x) \geqslant 0$ and $\lambda x + (1 - \lambda)y \in N$. Since neither x nor y lies in N, we have $\lambda \notin \{0, 1\}$. Since U is an affine linear subspace, $\lambda x + (1 - \lambda)y \in U$ now implies $y \in U$ and therefore $\ell(y) \geqslant 0$, leading to the contradiction $0 > \ell(\lambda x + (1 - \lambda)y) = \lambda\ell(x) + (1 - \lambda)\ell(y) \geqslant 0$.

Without loss of generality $N \neq \emptyset$ (otherwise $\ell|_U = 0$ and we can take $\ell' = 0$). Then by separation of non-empty disjoint convex sets (e.g., Thm. III.1.2 in Barvinok [3]), we get an affine linear $\ell'\colon \mathbb{R}^n \to \mathbb{R}$, not identically zero, such that $\ell' \geqslant 0$ on S' and $\ell' \leqslant 0$ on N. In particular, $\ell' \geqslant 0$ on S and ℓ' cannot vanish at an interior point of S. Since U intersects by hypothesis the interior of S, it is not possible that ℓ' vanishes identically on U. Moreover, all $x \in U$ with $\ell(x) = 0$ lie at the same time in S' and in the closure of N, implying that $\ell'(x) = 0$. This shows that the restrictions of ℓ and ℓ' on U are the same up to a positive factor which we may assume to be 1 after rescaling. □

We are now ready for the main result:

Theorem 3.5. *Let $S = \mathscr{S}(\underline{p})$ be a basic closed convex subset of $\mathbb{R}^n$ with non-empty interior. Suppose that there exists $d \geqslant 1$ such that the d-th Lasserre relaxation of S with respect to $\underline{p}$ is exact, i.e.*

$$S(\underline{p})_d = S$$

holds. Then all faces of S are exposed.

In view of Proposition 3.1, we have the following equivalent formulation of the same theorem:

Theorem (Alternative formulation)**.** *Let $S = \mathscr{S}(\underline{p})$ be a basic closed convex subset of $\mathbb{R}^n$ with non-empty interior. Suppose that there exists $d \geqslant 1$ such that every linear polynomial ℓ with $\ell \geqslant 0$ on S is contained in $\mathrm{QM}(\underline{p})_d$. Then all faces of S are exposed.*

Proof. We begin by showing that it is sufficient to prove that all faces of dimension $n-2$ are exposed. Let F be a face of S of dimension e. For $e \geqslant n-1$ there is nothing to show, so assume $e \leqslant n-2$. If F is not exposed, then by Lemma 3.3 there exists an affine-linear subspace U of $\mathbb{R}^n$ containing F with $\dim(U) = e+2$ and $U \cap \operatorname{int}(S) \neq \emptyset$ and such that F is a non-exposed face of $S \cap U$. Furthermore, by Lemma 3.4, for every linear polynomial ℓ that is psd on $S \cap U$ there exists a linear polynomial ℓ' that is psd on S and agrees with ℓ on U. Upon replacing $\mathbb{R}^n$ by U and S by $S \cap U$, we reduce to the case $e = n-2$.

Now assume for contradiction that $d \geqslant 1$ as in the statement exists and that F is a face of dimension $n-2$ that is not exposed.

Step 1. There is exactly one supporting hyperplane H of S that contains F. For if ℓ_1, ℓ_2 are non-zero linear polynomials with $\ell_i|_F = 0$ and $\ell_i|_S \geqslant 0$, put $W := \{\ell_1 = 0\} \cap \{\ell_2 = 0\}$. Then $\ell := \ell_1 + \ell_2$ defines a supporting hyperplane $\{\ell = 0\}$ of S with $\{\ell = 0\} \cap S = W \cap S$. If ℓ_1, ℓ_2 are linearly independent, then $\dim(W) = n-2 = \dim(F)$, hence $F = \{\ell = 0\} \cap S$, contradicting the fact that F is not exposed.

We may assume after an affine change of coordinates that $H = \{t_1 = 0\}$, $t_1 \geqslant 0$ on S, and that 0 lies in the relative interior of F. Note that any supporting hyperplane of S containing 0 must contain F and therefore coincide with H.

Since F is not exposed, $F_0 = H \cap S$ is a face of dimension $n-1$ with F contained in its relative boundary. In particular, it follows that F is also contained in the closure of $\partial S \setminus H$.

Step 2. By the curve selection lemma (see e.g. Thm. 2.5.5. in Bochnak, Coste, and Roy [4]), we may choose a continuous semialgebraic path $\gamma\colon [0,1] \to \partial S$ such that $\gamma(0) = 0 \in F$, $\gamma\big((0,1]\big) \cap H = \emptyset$. We relabel $p_0, \ldots, p_m$ into two groups $f_1, \ldots, f_r$, $g_1, \ldots, g_s$ as follows:

$$\begin{aligned} f_i|_{\gamma([0,1])} &= 0 \quad (i = 1, \ldots, r) \\ g_j|_{\gamma((0,1])} &> 0 \quad (j = 1, \ldots, s) \end{aligned}$$

(Indeed, after restricting γ to $[0,\alpha]$ for suitable $\alpha \in (0,1]$ and reparametrizing, we can assume that each p_i falls into one of the above categories.)

We claim that there exists an expression

$$(*) \qquad t_1 = \sum_{i=1}^{r} \rho_i f_i + \sum_{j=1}^{s} \sigma_j g_j$$

with $\rho_i, \sigma_j \in \sum \mathbb{R}[\underline{t}]^2$ and such that $\sigma_j(0) = 0$ for all $j = 1, \ldots, s$.

To prove the existence of the expression $(*)$, consider the following statement:

(†) For each $\lambda \in (0,1]$ there exists a linear polynomial $\ell_\lambda \in \mathbb{R}[\underline{t}]_1$ such that $\ell_\lambda(\gamma(\lambda)) = 0$, $\ell_\lambda \geqslant 0$ on S, and $||\ell_\lambda|| = 1$. For this ℓ_λ, there exist $\rho_i^{(\lambda)}, \sigma_j^{(\lambda)} \in \sum \mathbb{R}[\underline{t}]_d^2$ such that

$$\ell_\lambda = \sum_{i=1}^{r} \rho_i^{(\lambda)} f_i + \sum_{j=1}^{s} \sigma_j^{(\lambda)} g_j$$

and such that

$$\sigma_j^{(\lambda)}(\gamma(\lambda)) = 0$$

for all $j = 1, \ldots, s$.

The statement (†) is true, with $d \geqslant 1$ not depending on λ: For $\lambda \in (0,1]$, let $\ell_\lambda \in \mathbb{R}[\underline{t}]_1$ be such that $\{\ell_\lambda = 0\}$ is a supporting hyperplane of S passing through $\gamma(\lambda)$, and such that $||\ell_\lambda|| = 1$ and $\ell_\lambda|_S \geqslant 0$. By hypothesis, $\ell_\lambda \in \operatorname{QM}(\{f_i\}, \{g_j\})_d$

with d not depending on λ, which yields the desired representation. Note that $\sigma_j^{(\lambda)}(\gamma(\lambda)) = 0$ is automatic, since $g_j(\gamma(\lambda)) \neq 0$, but $\ell_\lambda(\gamma(\lambda)) = 0$.

Furthermore, because the degree-bound d is fixed, (†) can be expressed as a first-order formula in the language of ordered rings. Thus (†) holds over any real closed extension field R of $\mathbb{R}$, by the model-completeness of the theory of real closed fields. Let R be any proper (hence non-archimedean) extension field and let $\varepsilon \in R$, $\varepsilon > 0$, be an infinitesimal element with respect to $\mathbb{R}$. We apply (†) with $\lambda = \varepsilon$ and get

$$(\ddagger) \qquad \ell_\varepsilon = \sum_{i=1}^{r} \rho_i^{(\varepsilon)} f_i + \sum_{j=1}^{s} \sigma_j^{(\varepsilon)} g_j$$

with

$$\sigma_j^{(\varepsilon)}(\gamma(\varepsilon)) = 0$$

for all $j = 1, \dots, s$. Let $\mathcal{O}$ be the convex hull of $\mathbb{R}$ in R, a valuation ring with maximal ideal $\mathfrak{m}$. Since $\mathrm{int}(S) \neq \emptyset$, the quadratic module $\mathrm{QM}(\{f_i\}, \{g_j\})$ has trivial support. As $||\ell_\varepsilon|| = 1$, it follows that all coefficients of the polynomials in (‡) must lie in $\mathcal{O}$ (see e.g. the proof of Lemma 8.2.3 in Prestel and Delzell [18]). We can therefore apply the residue map $\mathcal{O} \to \mathcal{O}/\mathfrak{m} \cong \mathbb{R}$, $a \mapsto \overline{a}$ to the coefficients of (‡). From the uniqueness of the supporting hyperplane $H = \{t_1 = 0\}$ in 0 (Step 1), it follows that $\overline{\ell_\varepsilon} = c \cdot t_1$ for some $c \in \mathbb{R}_{>0}$. This yields the desired expression $(*)$.

Step 3. The existence of $(*)$ leads to a contradiction: Substituting $t_1 = 0$ in $(*)$ gives

$$0 = \sum_{i=1}^{r} \rho_i(0, \underline{t}') f_i(0, \underline{t}') + \sum_{j=1}^{s} \sigma_j(0, \underline{t}') g_j(0, \underline{t}')$$

in $\mathbb{R}[\underline{t}']$, with $\underline{t}' = (t_2, \dots, t_n)$. Since all $f_i(0, \underline{t}'), g_j(0, \underline{t}')$ are non-negative on F_0, which has non-empty interior in H, it follows that $\rho_i(0, \underline{t}') = 0$ whenever $f_i(0, \underline{t}') \neq 0$. In other words, if t_1 does not divide f_i, then t_1^2 divides ρ_i in $\mathbb{R}[\underline{t}]$.

Going back to $(*)$ and substituting $t_2 = \cdots = t_n = 0$ now gives

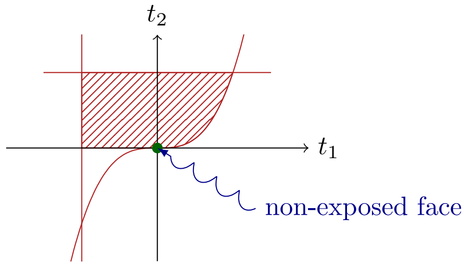
$$t_1 = \sum_{i=1}^{r} \rho_i(t_1, 0) f_i(t_1, 0) + \sum_{j=1}^{s} \sigma_j(t_1, 0) g_j(t_1, 0)$$

Since $\sigma_j(0) = 0$ for all $j = 1, \dots, s$, we now know that t_1^2 divides all terms on the right-hand side, except possibly $\rho_i(t_1, 0) f_i(t_1, 0)$ for such i where $t_1 | f_i$. In the latter case, write $f_i = t_1 \widetilde{f_i}$ and note that $\widetilde{f_i}$ vanishes on $\gamma((0,1])$ since f_i does and t_1 does not. Thus $\widetilde{f_i}(0) = 0$ by continuity which implies $t_1 | \widetilde{f_i}(t_1, 0)$, so $t_1^2 | f_i(x_1, 0)$ after all. It follows that t_1^2 divides t_1, a contradiction. □

Remarks 3.6. (1) Note that whether the faces of S are exposed is a purely geometric condition, independent of the choice of the polynomials $\underline{p}$. Thus if S has a non-exposed face, there do not exist polynomials $\underline{p}$ defining S that yield an exact Lasserre relaxation for S.

(2) The theorem does *not* imply that a basic closed convex set with a non-exposed face cannot be semidefinitly representable, as we will see in the example below. We have only shown that Lasserre's explicit approach does not work in that case.

Example 3.7. Consider the basic closed semialgebraic set S defined by $p_1 = t_2 - t_1^3$, $p_2 = t_1 + 1$, $p_3 = t_2$, $p_4 = 1 - t_2$.



The point $(0, 0)$ is a non-exposed face of S since the only supporting hyperplane of S passing through $(0, 0)$ is the vertical line $\{t_2 = 0\}$, whose intersection with S is strictly bigger than $\{(0, 0)\}$. Therefore, there do not exist polynomials $\underline{p}$ with $S = \mathscr{S}(\underline{p})$ such that all linear polynomials that are non-negative on S belong to $\mathrm{QM}(\underline{p})_d$ for some *fixed* value of d. On the other hand, the preordering generated by p_1, p_2, p_3, p_4 as above (i.e. the quadratic module generated by all products of the p_i) contains all polynomials that are non-negative on S. This follows from results of Scheiderer. Indeed, by the local-global principle [22, Corollary 2.10] it suffices to show that the preordering generated by the p_i is locally saturated. At the origin this follows from the results in [21] (in particular, Theorem 6.3 and Corollary 6.7). At all other points it follows already from [22], Lemma 3.1.

However, from the result of Helton and Nie, we can deduce that S is in fact semidefinitely representable: For S is the (convex hull of) the union of the sets $S_1 = [-1, 0] \times [0, 1]$ and $S_2 = \mathscr{S}(t_2 - t_1^3, t_1, 1 - t_2)$. The set S_1 is obviously semidefinitely representable (even a spectrahedron), while S_2 possesses an exact Lasserre-relaxation: More precisely, we claim that $\mathrm{QM}(t_2 - t_1^3, t_1, 1 - t_2)_3$ contains all linear polynomials $\ell \in \mathbb{R}[t_1, t_2]$ such that $\ell|_{S_2} \geqslant 0$. It suffices to show this for the tangents $\ell_a = t_2 - 3a^2 t_1 + 2a^3$ to S_2 passing through the points (a, a^3), $a \in [0, 1]$ (The claim then follows from Farkas's lemma). Write $\ell_a = t_1^3 - 3a^2 t_1 + 2a^3 + (t_2 - t_1^3)$. The polynomial $t_1^3 - 3a^2 t_1 + 2a^3 \in \mathbb{R}[t_1]$ is non-negative on $[0, \infty)$ and is therefore contained in $\mathrm{QM}(t_1)_3 \subseteq \mathbb{R}[t_1]$ (see Kuhlmann, Marshall, and Schwartz [11], Thm. 4.1), which implies the claim.

Remark 3.8. We do not know if the conclusion of Theorem 3.5 remains true for $\mathrm{conv}(S)$ in place of S, if S is not assumed to be convex. It seems unlikely that our proof can be extended to that case. More generally, is every face of any Lasserre relaxation exposed?

Note added in proof: João Gouveia [2] showed that our Theorem 3.5 is optimal in the sense that the questions in Remark 3.8 have negative answers. He also gave an alternative proof of our main theorem which is yet unpublished.

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Fachbereich Mathematik, Universität Konstanz, 78457 Konstanz, Germany
E-mail address: `daniel.plaumann@uni-konstanz.de`

Fachbereich Mathematik, Universität Konstanz, 78457 Konstanz, Germany
E-mail address: `tim.netzer@uni-konstanz.de`

Université de Rennes 1, Laboratoire de Mathématiques, Campus de Beaulieu, 35042 Rennes cedex, France
E-mail address: `markus.schweighofer@univ-rennes1.fr`