

# STABILITY OF QUADRATIC MODULES

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ABSTRACT. A finitely generated quadratic module or preordering in the real polynomial ring is called *stable*, if it admits a certain degree bound on the sums of squares in the representation of polynomials. Stability, first defined explicitly in [PS], is a very useful property. It often implies that the quadratic module is closed; furthermore it helps settling the Moment Problem, solves the Membership Problem for quadratic modules and allows applications of methods from optimization to represent nonnegative polynomials.

We provide sufficient conditions for finitely generated quadratic modules in real polynomial rings of several variables to be stable. These conditions can be checked easily. For a certain class of semi-algebraic sets, we obtain that the nonexistence of bounded polynomials implies stability of every corresponding quadratic module. As stability often implies the non-solvability of the Moment Problem, this complements the result from [Sch3], which uses bounded polynomials to check the solvability of the Moment Problem by dimensional induction. We also use stability to generalize a result on the Invariant Moment Problem from [CKS].

## 1. INTRODUCTION

Preorderings and quadratic modules in the real polynomial ring are of great importance in real algebraic geometry. They correspond to semi-algebraic sets in a similar way as ideals correspond to algebraic sets. However, it is much more difficult to deal with preorderings and quadratic modules than with ideals in general. Nevertheless, substantial progress has been made in this field in the last fifteen years. The basic setup is the following. We take finitely many real polynomials  $f_1, \dots, f_s \in \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \dots, X_n]$  and consider the basic closed semi-algebraic set

$$\mathcal{S} = \mathcal{S}(f_1, \dots, f_s) := \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \dots, f_s(x) \geq 0\},$$

as well as the corresponding preordering

$$\text{PO}(f_1, \dots, f_s) := \left\{ \sum_{e \in \{0,1\}^s} \sigma_e f_1^{e_1} \cdots f_s^{e_s} \mid \sigma_e \in \mathbb{R}[\underline{X}]^2 \right\}$$

and the smaller quadratic module

$$\text{QM}(f_1, \dots, f_s) = \left\{ \sigma_0 + \sigma_1 f_1 + \cdots + \sigma_s f_s \mid \sigma_i \in \sum \mathbb{R}[\underline{X}]^2 \right\},$$

where  $\sum \mathbb{R}[\underline{X}]^2$  denotes the set of sums of squares of polynomials. Elements from the preordering are obviously nonnegative as polynomial functions on the semi-algebraic set. Now one can ask if the preordering or quadratic module contains

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*all* such nonnegative polynomials. Although this is not true in general, several Positivstellensätze give representations of nonnegative polynomials. For example, if the semi-algebraic set is compact, the preordering at least contains all strictly positive polynomials, by [Sch2]. For quadratic modules or noncompact sets, this result fails in general. See for example [M1, PD] for an extensive exposure of the field.

Another question concerns the Moment Problem. We say that the preordering/quadratic module  $M$  has the *Strong Moment Property*, if every linear functional on  $\mathbb{R}[X]$  which is nonnegative on  $M$  is integration with respect to a measure on the corresponding semi-algebraic set. The result from [Sch2] implies that every preordering describing a compact semi-algebraic set has the Strong Moment Property, and [Sch3] gives a criterion for the case of a noncompact set, see also [M1, N].

If one already knows that a polynomial belongs to the preordering or quadratic module, it is another problem how to find an explicit sums of squares representation. In general, the degree of the sums of squares used in the representation of some  $f$  can not be bounded by a function that only depends on the degree of  $f$ . For example, in the case of a compact set  $\mathcal{S}$ , one has to take into account the degree, the size of the coefficients and the minimum of  $f$  on  $\mathcal{S}$ , so be able to say something about the degree of the sums of squares (see [PD, Theorem 8.4.3] and [Sw1]). This is what makes it so difficult to find representations.

Now the notion of *stability* of a finitely generated preordering or quadratic module has first been introduced explicitly in [PS]. In the polynomial ring, stability means that every polynomial in the preordering has a representation, where the degree of the sums of squares can be bounded by a number depending only on the degree of the polynomial. The authors of [PS] give a strong geometric criterion for quadratic modules to be stable. Roughly speaking, if the set  $\mathcal{S}$  is big enough at infinity, then every corresponding finitely generated quadratic module is stable. The notion has also been dealt with in [P1, P2], where the geometric result from [PS] is applied and extended, for curves and surfaces mostly.

The importance of stability is evident from several results. First, as shown in [PS], stable quadratic modules are often closed (with respect to the finest locally convex topology). This was also shown in [KM], Theorem 3.5, in the case that  $\mathcal{S}$  contains a full dimensional cone, but without using the notion of stability explicitly. Similar arguments have been used in [PD], Proposition 6.4.5., and [Sch1], Section 11.6.

Second, stability often excludes the Strong Moment Property of quadratic modules. This useful fact was shown in [S1], generalizing an idea by Prestel and Berg. The result also shows that one can often not expect finitely generated quadratic modules to be stable.

A further reason making stable quadratic modules so interesting is that the degree bound condition allows the application of model theoretic methods. Indeed, the set of all polynomials of fixed degree which lie in the quadratic module can be defined by a first order logic formula then. This also solves the so called *Membership Problem* for stable quadratic modules. Whether the membership problem is solvable for arbitrary quadratic modules is still an open question. So far it is only known for finitely generated preorderings in the real polynomial ring of one variable, see [Au].

Also the question of finding an explicit representation of a polynomial in a stable quadratic module is easy to solve. Indeed, it can be translated into a semi-definite programming problem, which can be solved efficiently. See [L, Sw2, VB] for details.

Our contribution is the following. We define the notion of *stability with respect to a grading*, for quadratic modules (see Section 3). This notion of stability has a characterization which is of purely geometric nature (see Section 4). We then relate it to the notion of stability used in [PS, S1]. Indeed, this is the stability one is mostly interested in. Our results allow to obtain this stability in a lot of cases by checking some easy geometric or combinatorial properties (see Section 5 and the explicit examples in Section 6).

For a certain class of semi-algebraic sets we are able to prove that the absence of nontrivial bounded polynomials implies the stability of every corresponding finitely generated quadratic module (Theorem 5.4). Thus no such quadratic module can have the Strong Moment Property. This complements the result from [Sch3], that uses bounded polynomials to check the Strong Moment Property by dimensional induction.

Last, we use the notion of *strong stability* to improve upon a result from [CKS], while simplifying the proof. This is done in the last section.

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## 2. NOTATIONS AND PRELIMINARIES

For this whole work, let  $A = \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \dots, X_n]$  be the real polynomial algebra in  $n$  variables. A subset  $M \subseteq A$  is called a *quadratic module*, if

$$1 \in M, \quad M + M \subseteq M \quad \text{and} \quad A^2 \cdot M \subseteq M$$

holds, where  $A^2$  denotes the set of squares in  $A$ . For elements  $f_1, \dots, f_s \in A$ ,  $\text{QM}(f_1, \dots, f_s)$ , defined in the introduction, is the smallest quadratic module containing  $f_1, \dots, f_s$ . It is called the quadratic module *generated by*  $f_1, \dots, f_s$ . We always assume generators of a quadratic module to be all non-zero. A quadratic module is called a *preordering*, if it is closed under multiplication, i.e. if  $M \cdot M \subseteq M$  holds. For  $f_1, \dots, f_s \in A$ ,  $\text{PO}(f_1, \dots, f_s)$ , again as in the introduction, is the smallest preordering containing  $f_1, \dots, f_s$ . It is called the preordering *generated by* these elements. For any quadratic module  $M$ ,  $\text{supp}(M) := M \cap -M$  is called the *support of*  $M$ . It is an ideal of  $A$ . For a quadratic module  $M$  in  $A$  write

$$\mathcal{S}(M) := \{x \in \mathbb{R}^n \mid f(x) \geq 0 \text{ for all } f \in M\}.$$

Of special interest is the case that  $M$  is finitely generated. If  $f_1, \dots, f_s$  are generators of  $M$ , then  $\mathcal{S}(M) = \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \dots, f_s(x) \geq 0\}$  is called *basic closed semi-algebraic*. We include the proof of the following proposition due to the lack of a good reference.

**Proposition 2.1.** *Let  $M$  be a finitely generated quadratic module in  $A$ . If  $\mathcal{S}(M)$  is Zariski-dense in  $\mathbb{R}^n$ , then  $\text{supp}(M) = \{0\}$ . If  $M$  is a finitely generated preordering, then  $\text{supp}(M) = \{0\}$  implies the Zariski-denseness of  $\mathcal{S}(M)$ .*

*Proof.* Take  $f \in \text{supp}(M)$ . Then  $f = 0$  on  $\mathcal{S}(M)$ , so  $f = 0$  by the Zariski-denseness.

Now suppose  $M$  is a finitely generated preordering with  $\text{supp}(M) = \{0\}$ . Suppose  $f = 0$  on  $\mathcal{S}(M)$  for some  $f \in A$ . By Theorem 4.2.11 from [PD] there are  $t_1, \tilde{t}_1, t_2, \tilde{t}_2 \in M$  and  $e_1, e_2 \in \mathbb{N}$  such that  $t_1 f = f^{2e_1} + t_2$  and  $\tilde{t}_1(-f) = f^{2e_2} + \tilde{t}_2$  holds. So  $t_1 \tilde{t}_1 f \in \text{supp}(M)$ , so  $f = 0$ . This shows the desired denseness.  $\square$

Following [S1], for any  $\mathbb{R}$ -subspace  $W$  of  $A$  we write  $\sum(W; f_1, \dots, f_s)$  for the set of all elements

$$\sigma_0 + \sigma_1 f_1 + \dots + \sigma_s f_s,$$

where  $\sigma_i \in \sum W^2$  for all  $i$ . Obviously each  $\sum(W; f_1, \dots, f_s)$  is contained in  $M = \text{QM}(f_1, \dots, f_s)$  and  $\sum(A; f_1, \dots, f_s) = M$ . If  $W$  is finite dimensional, then  $\sum(W; f_1, \dots, f_s)$  is contained in a finite dimensional subspace of  $A$ . The following definition is Definition 3.2 from [S1]:

**Definition 2.2.**  $M = \text{QM}(f_1, \dots, f_s)$  is called *stable*, if for every finite dimensional subspace  $U$  of  $A$  there is another finite dimensional subspace  $W$  of  $A$  such that

$$M \cap U \subseteq \sum(W; f_1, \dots, f_s)$$

holds.

The following two results show the importance of the notion:

**Theorem 2.3** (Powers, Scheiderer [PS]). *If  $M$  is stable and  $\mathcal{S}(M)$  is Zariski-dense in  $\mathbb{R}^n$ , then  $M$  is closed, i.e.  $M = M^{\vee\vee}$  holds, where  $M^{\vee\vee}$  denotes the double dual cone of  $M$ .*

**Theorem 2.4** (Scheiderer [S1]). *If  $M$  is stable and  $\mathcal{S}(M) \subseteq \mathbb{R}^n$  has dimension at least two, then  $M$  does not have the Strong Moment Property. In particular,  $M$  does not contain all polynomials that are nonnegative on  $\mathcal{S}(M)$ .*

For our approach towards stability, we need the notions of filtrations and gradings. So let  $(\Gamma, \leq)$  be an ordered Abelian group, i.e. an Abelian group  $\Gamma$  with a linear ordering, such that  $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$  holds for any  $\alpha, \beta, \gamma \in \Gamma$ .

**Definition 2.5.** A *filtration* of  $A$  is a family  $\{U_\gamma\}_{\gamma \in \Gamma}$  of linear  $\mathbb{R}$ -subspaces of  $A$ , such that for all  $\gamma, \gamma' \in \Gamma$

$$\begin{aligned} \gamma \leq \gamma' &\Rightarrow U_\gamma \subseteq U_{\gamma'}, \\ U_\gamma \cdot U_{\gamma'} &\subseteq U_{\gamma+\gamma'}, \\ \bigcup_{\gamma \in \Gamma} U_\gamma &= A \text{ and} \\ 1 &\in U_0 \end{aligned}$$

holds.

**Definition 2.6.** A *grading* of  $A$  is a decomposition of the  $\mathbb{R}$ -vector space  $A$  into a direct sum of linear subspaces:

$$A = \bigoplus_{\gamma \in \Gamma} A_\gamma,$$

such that  $A_\gamma \cdot A_{\gamma'} \subseteq A_{\gamma+\gamma'}$  holds for all  $\gamma, \gamma' \in \Gamma$ .

Any element  $0 \neq f \in A$  can then be written in a unique way as

$$f = f_{\gamma_1} + \dots + f_{\gamma_d}$$

for some  $d \in \mathbb{N}$  and  $0 \neq f_{\gamma_i} \in A_{\gamma_i}$ , where  $\gamma_1 < \gamma_2 < \dots < \gamma_d$ . Then  $\deg(f) := \gamma_d$  is called the *degree* of  $f$ , and  $f^{\max} := f_{\gamma_d}$  is called the *highest degree part* of  $f$ . Elements from  $A_\gamma$  are called *homogeneous of degree*  $\gamma$ . The degree of 0 is  $-\infty$ . One easily checks that  $1 \in A_0$ .

The following are some easy observations: If  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  is a grading, then

$$U_\tau := \bigoplus_{\gamma \leq \tau} A_\gamma$$

defines a filtration  $\{U_\tau\}_{\tau \in \Gamma}$  of  $A$ . If  $\nu: K \rightarrow \Gamma \cup \{\infty\}$  is a valuation of the quotient field  $K = \mathbb{R}(X_1, \dots, X_n)$  of  $A$  which is trivial on  $\mathbb{R}$ , then

$$U_\gamma := \{f \in A \mid \nu(f) \geq -\gamma\}$$

defines a filtration  $\{U_\gamma\}_{\gamma \in \Gamma}$  of  $A$ . If  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  is a grading, then

$$\nu\left(\frac{f}{g}\right) := \deg(g) - \deg(f)$$

defines a valuation on the quotient field  $K$ , trivial on  $\mathbb{R}$ . This valuation induces the same filtration on  $A$  as the grading. For any grading and all  $f, g \in A$  we have  $\deg(f \cdot g) = \deg(f) + \deg(g)$  and

$$\deg(f^2 + g^2) = \max\{\deg(f^2), \deg(g^2)\} = 2 \max\{\deg(f), \deg(g)\}.$$

### 3. DEFINITIONS OF STABILITY

**Definition 3.1.** Let  $\{U_\gamma\}_{\gamma \in \Gamma}$  be a filtration of  $A$  and  $f_1, \dots, f_s$  generators of the quadratic module  $M$ . We set  $f_0 = 1$ .

(1)  $f_1, \dots, f_s$  are called *stable generators of  $M$  with respect to the filtration*, if there is a monotonically increasing map  $\varrho: \Gamma \rightarrow \Gamma$ , such that

$$M \cap U_\gamma \subseteq \sum (U_{\varrho(\gamma)}; f_1, \dots, f_s)$$

holds for all  $\gamma \in \Gamma$ .

(2)  $f_1, \dots, f_s$  are called *strongly stable generators of  $M$  with respect to the filtration*, if there is a monotonically increasing map  $\varrho: \Gamma \rightarrow \Gamma$ , such that for all sums of squares  $\sigma_0, \dots, \sigma_s$ , where  $\sigma_i = g_{i,1}^2 + \dots + g_{i,k_i}^2$ , we have

$$\sum_{i=0}^s \sigma_i f_i \in U_\gamma \Rightarrow g_{i,j} \in U_{\varrho(\gamma)} \text{ for all } i, j.$$

Obviously, strongly stable generators of  $M$  are stable generators of  $M$ . The notion of strong stability has also been introduced in [P1], but under a different name. The following Lemma is essentially the same as [PS], Lemma 2.9.

**Lemma 3.2.** *If  $M$  has stable generators with respect to a given filtration, then any finitely many generators of  $M$  are stable generators with respect to that filtration.*

*Proof.* Suppose  $f_1, \dots, f_s$  are stable generators of  $M$  with stability map  $\varrho$  as in Definition 3.1(1). Let  $g_1, \dots, g_t$  be arbitrary generators of  $M$ . Then we find representations

$$f_i = \sum_{j=0}^t \sigma_j^{(i)} g_j,$$

where all  $\sigma_j^{(i)} \in \sum (U_\tau)^2$  for some big enough  $\tau \in \Gamma$ . Now take  $f \in M \cap U_\gamma$  for some  $\gamma$  and find a representation  $f = \sum_{i=0}^s \sigma_i f_i$  with  $\sigma_i \in \sum (U_{\varrho(\gamma)})^2$  for all  $i$ . Then

$$f = \sum_i \sigma_i f_i = \sum_i \sigma_i \sum_j \sigma_j^{(i)} g_j = \sum_j \left( \sum_i \sigma_i \sigma_j^{(i)} \right) g_j,$$

and all  $\sum_i \sigma_i \sigma_j^{(i)}$  are in  $\sum (U_{\varrho(\gamma)+\tau})^2$ . This shows that  $g_1, \dots, g_t$  are also stable generators of  $M$ , with stability map  $\gamma \mapsto \varrho(\gamma) + \tau$ .  $\square$

So it makes sense to talk about stability of a finitely generated quadratic module with respect to a filtration, without mentioning the generators. However, the stability map  $\varrho$  may depend on the generators in general.

Note that  $M$  is stable in the usual sense (defined in the previous section), if and only if it is stable with respect to a filtration consisting of finite dimensional subspaces  $U_\gamma$  of  $A$ .

Now suppose we are given a grading on  $A$ . We will talk about stable generators, strongly stable generators and stable quadratic modules *with respect to the grading*, and always mean these notions with respect to the induced filtration. However, things become easier to handle in this case.

**Lemma 3.3.** *Let  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a grading and let  $M$  be a finitely generated quadratic module in  $A$ . Then  $M$  has strongly stable generators with respect to the grading if and only if there is a monotonically increasing map  $\psi: \Gamma \rightarrow \Gamma$ , such that for all  $f, g \in M$*

$$\deg(f) \leq \psi(\deg(f + g))$$

*holds. In particular, if  $M$  has strongly stable generators, then any finitely many generators are strongly stable generators.*

*Proof.* Suppose  $f_1, \dots, f_s$  are strongly stable generators of  $M$  with stability map  $\varrho$ . Take  $f, g$  from  $M$  with representations  $f = \sum_i \sigma_i f_i, g = \sum_i \tau_i f_i$ . Then for all  $j$

$$\begin{aligned} \deg(\sigma_j f_j) &= \deg(\sigma_j) + \deg(f_j) \\ &\leq \deg(\sigma_j + \tau_j) + \deg(f_j) \\ &= \deg((\sigma_j + \tau_j) f_j) \\ &\leq \psi \left( \deg \left( \sum_i (\sigma_i + \tau_i) f_i \right) \right) \\ &= \psi(\deg(f + g)), \end{aligned}$$

where the last inequality is fulfilled with

$$\psi(\gamma) := 2\varrho(\gamma) + \max_i \deg(f_i),$$

by the strong stability of the  $f_i$ . So  $\deg(f) \leq \psi(\deg(f + g))$  holds. Note that  $\psi$  is monotonically increasing, as  $\varrho$  was.

So now suppose  $\deg(f), \deg(g) \leq \psi(\deg(f + g))$  for some suitable map  $\psi$  and all  $f, g \in M$ . Take any finitely many (non-zero) generators  $f_1, \dots, f_s$  and sums of squares  $\sigma_0, \dots, \sigma_s$ , where  $\sigma_j = p_{j,1}^2 + \dots + p_{j,k_j}^2$ . Set  $f_0 = 1$ . Then

$$\deg(\sigma_j f_j) \leq \psi \left( \deg \left( \sum_i \sigma_i f_i \right) \right)$$

for all  $j$ . Thus for all  $j, l$ ,

$$2 \deg(p_{j,l}) \leq \psi \left( \deg \left( \sum_i \sigma_i f_i \right) \right) - \min_i \deg(f_i).$$

So

$$\deg(p_{j,l}) \leq \max \left\{ 0, \psi \left( \deg \left( \sum_i \sigma_i f_i \right) \right) - \min_i \deg(f_i) \right\}$$

holds. Now  $\varrho(\tau) := \max \{0, \psi(\tau) - \min_i \deg(f_i)\}$  defines a monotonically increasing map, and whenever

$$f = \sum_i \sigma_i f_i \in \bigoplus_{\gamma \leq \tau} A_\gamma \text{ for some } \tau,$$

then  $\deg(p_{j,l}) \leq \varrho(\deg(f)) \leq \varrho(\tau)$ , which shows the strong stability of the  $f_1, \dots, f_s$ . We have used the fact that  $\varrho$  is monotonically increasing in the last inequality.

The proof shows that any finitely many generators are strongly stable generators in this case.  $\square$

So we can talk about strong stability of a finitely generated quadratic module with respect to a grading, without mentioning the generators. A very special case of strong stability is the following, which will have a nice characterization below.

**Definition 3.4.** Let  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a grading and let  $M \subseteq A$  be a finitely generated quadratic module.  $M$  is *totally stable with respect to the grading*, if

$$\deg(f) \leq \deg(f + g)$$

holds for all  $f, g \in M$ . The proof of Lemma 3.3 shows that this is equivalent to the fact that there are generators  $f_1, \dots, f_s$  of  $M$  such that

$$\deg(\sigma_j f_j) \leq \deg \left( \sum_i \sigma_i f_i \right)$$

holds for all  $\sigma_j \in \sum A^2$ . Any finite set of generators of  $M$  fulfills this condition then.

Note that a quadratic module  $M$  in  $A$  which is totally stable with respect to a grading has trivial support. Indeed if  $f, -f \in M$ , then  $\deg(f) \leq \deg(f - f) = \deg(0) = -\infty$ , so  $f = 0$ .

If  $\nu: K \rightarrow \Gamma \cup \{\infty\}$  is the valuation corresponding to a given grading, then the notion of total stability is equivalent to saying that for any  $f, g \in M$ ,

$$\nu(f + g) = \min \{\nu(f), \nu(g)\}$$

holds. This is usually called *weak compatibility of  $\nu$  and  $M$* .

#### 4. CHARACTERIZATIONS OF STABILITY

Total stability with respect to a grading turns out to be well accessible. First, when checking total stability of a finitely generated quadratic module, one can apply an easy reduction result, to obtain possibly smaller quadratic modules. Therefore take generators  $f_1, \dots, f_s$  of  $M$ , define an equivalence relation on the generators by saying

$$f_i \equiv f_j :\Leftrightarrow \deg(f_i) \equiv \deg(f_j) \pmod{2\Gamma},$$

and group them into equivalence classes

$$\{f_{i1}, \dots, f_{is_i}\} \quad (i = 1, \dots, r).$$

Then total stability reduces to total stability of the quadratic modules generated by these equivalence classes:

**Proposition 4.1.** *M is totally stable with respect to the given grading if and only if all the quadratic modules*

$$M_i := \text{QM}(f_{i1}, \dots, f_{is_i})$$

*are totally stable.*

*Proof.* The "only if"-part is obvious. For the "if"-part take  $g, h \in M$  with representations  $g = \sigma_0 + \sigma_1 f_1 + \dots + \sigma_s f_s$  and  $h = \tau_0 + \tau_1 f_1 + \dots + \tau_s f_s$ . By grouping the terms with respect to the equivalence relation and using the total stability of the modules  $M_i$ , we get decompositions

$$g = g_1 + \dots + g_r, \quad h = h_1 + \dots + h_r$$

with  $g_i, h_i \in M_i$  and all the  $g_i$  (as well as the  $h_i$ ) have a different degree modulo  $2\Gamma$ . So if  $g$  and  $h$  have the same degree and  $\deg(g) = \deg(g_k), \deg(h) = \deg(h_l)$ , then  $k = l$  and the highest degree parts of  $g$  and  $h$  cannot cancel out, due to the total stability of  $M_k$ .  $\square$

Now total stability has the following easy characterization:

**Proposition 4.2.** *Let  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a grading and let  $M$  be a finitely generated quadratic module in  $A$ . Let  $f_1, \dots, f_s$  be generators of  $M$ . Then*

$$M \text{ is totally stable} \Leftrightarrow \text{supp}(\text{QM}(f_1^{\max}, \dots, f_s^{\max})) = \{0\}.$$

*Proof.* First suppose  $\text{supp}(\text{QM}(f_1^{\max}, \dots, f_s^{\max})) \neq \{0\}$ . So there are sums of squares  $\sigma_0, \dots, \sigma_s$ , not all zero, such that  $\sum_{i=0}^s \sigma_i f_i^{\max} = 0$ . Now

$$\begin{aligned} \deg\left(\sum_{i=0}^s \sigma_i f_i\right) &= \deg\left(\sum_{i=0}^s \sigma_i (f_i - f_i^{\max})\right) \\ &\leq \max_i \{\deg(\sigma_i (f_i - f_i^{\max}))\} \\ &< \max_i \{\deg(\sigma_i f_i)\}, \end{aligned}$$

so  $M$  is not totally stable. Conversely, for any sum of squares  $\sigma_j$ , the highest degree part of  $\sigma_j f_j$  lies in  $\text{QM}(f_1^{\max}, \dots, f_s^{\max})$ . So when adding elements of the form  $\sigma_i f_i$ , the highest degree parts cannot cancel out, if  $\text{supp}(\text{QM}(f_1^{\max}, \dots, f_s^{\max})) = \{0\}$ . So  $M$  is totally stable.  $\square$

The good thing about Proposition 4.2 is, that it allows to link total stability to a geometric condition, via Proposition 2.1:

**Theorem 4.3.** *Let  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a grading and  $M$  a finitely generated quadratic module in  $A$ . If for a set of generators  $f_1, \dots, f_s$  of  $M$ , the set*

$$\mathcal{S}(f_1^{\max}, \dots, f_s^{\max}) \subseteq \mathbb{R}^n$$

*is Zariski dense, then  $M$  is totally stable with respect to the grading. If  $M$  is closed under multiplication, then total stability implies the Zariski denseness for any finite set of generators of  $M$ .*

*Proof.* If  $\mathcal{S}(f_1^{\max}, \dots, f_s^{\max})$  is Zariski dense, then

$$\text{supp}(\text{QM}(f_1^{\max}, \dots, f_s^{\max})) = \{0\},$$

by Proposition 2.1. So Proposition 4.2 yields the total stability of  $M$ . If  $M$  is a preordering, generated by  $f_1, \dots, f_s$  as a quadratic module, and totally stable, then

$\text{QM}(f_1^{\max}, \dots, f_s^{\max})$  is also a preordering. So Propositions 4.2 and 2.1 imply the Zariski-denseness of  $\mathcal{S}(f_1^{\max}, \dots, f_s^{\max})$  in  $\mathbb{R}^n$ .  $\square$

Note that if  $M$  is a finitely generated quadratic module which is closed under multiplication, and  $f_1, \dots, f_t$  generate  $M$  as a *preordering*, then the products  $f_e := f_1^{e_1} \cdots f_t^{e_t}$  ( $e \in \{0, 1\}^t$ ) generate  $M$  as a quadratic module, and

$$\mathcal{S}(f_1^{\max}, \dots, f_t^{\max}) = \mathcal{S}(f_e^{\max} \mid e \in \{0, 1\}^t).$$

In the next section we will consider different kinds of gradings on  $A$ . The denseness condition from Theorem 4.3 will be translated into a geometric condition on the original set  $\mathcal{S}(M)$ .

Recall that we are mostly interested in stability of a finitely generated quadratic module in the sense of [PS] (see Definition 2.2), that is, stability with respect to a filtration of finite dimensional subspaces. Many of the later considered gradings do *not* induce such finite dimensional filtrations. Our goal is then to find stability with respect to enough different gradings, so that in the end the desired stability is still obtained. Therefore we consider the following setup: Let  $\Gamma, \Gamma_1, \dots, \Gamma_m$  be ordered Abelian groups and let

$$\{W_\gamma\}_{\gamma \in \Gamma}, \{U_\gamma^{(j)}\}_{\gamma \in \Gamma_j} \quad (j = 1, \dots, m)$$

be filtrations of  $A$ .

**Definition 4.4.** The filtration  $\{W_\gamma\}_{\gamma \in \Gamma}$  is *covered* by the filtrations

$$\{U_\gamma^{(j)}\}_{\gamma \in \Gamma_j} \quad (j = 1, \dots, m),$$

if there are monotonically increasing maps

$$\eta: \Gamma_1 \times \cdots \times \Gamma_m \rightarrow \Gamma, \quad \eta_j: \Gamma_j \rightarrow \Gamma \quad (j = 1, \dots, m),$$

such that for all  $\gamma \in \Gamma, \gamma_j \in \Gamma_j$  ( $j = 1, \dots, m$ ), the following holds:

$$W_\gamma \subseteq \bigcap_{j=1}^m U_{\eta_j(\gamma)}^{(j)} \quad \text{and}$$

$$\bigcap_{j=1}^m U_{\gamma_j}^{(j)} \subseteq W_{\eta(\gamma_1, \dots, \gamma_m)}.$$

For  $\eta$ , *monotonically increasing* refers to the *partial* ordering on the product group obtained by the componentwise orderings of the factors.

We will speak about covering of/by *gradings*, and mean the notion from Definition 4.4 applied to the induced filtrations. The next theorem makes clear why we are interested in coverings.

**Theorem 4.5.** *Suppose a quadratic module  $M$  in  $A$  has generators  $f_1, \dots, f_s$ , which are strongly stable generators with respect to all the filtrations*

$$\{U_\gamma^{(j)}\}_{\gamma \in \Gamma_j} \quad (j = 1, \dots, m).$$

*Then  $f_1, \dots, f_s$  are also strongly stable generators of  $M$  with respect to any filtration  $\{W_\gamma\}_{\gamma \in \Gamma}$  which is covered by these filtrations.*

*Proof.* For every  $j = 1, \dots, m$ , take a stability map  $\varrho_j$  for the generators with respect to the filtration  $\left\{U_\gamma^{(j)}\right\}_{\gamma \in \Gamma_j}$  (remember Definition 3.1(2)). As in Definition 4.4, the covering maps are denoted by  $\eta$  and  $\eta_j$ .

Take sums of squares  $\sigma_0, \dots, \sigma_s$ , where  $\sigma_i = g_{i,1}^2 + \dots + g_{i,k_i}^2$  and suppose  $\sum_{i=0}^s \sigma_i f_i \in W_\gamma$  for some  $\gamma \in \Gamma$ . Then  $\sum_{i=0}^s \sigma_i f_i \in U_{\eta_j(\gamma)}^{(j)}$  for all  $j$ . So by strong stability,

$$g_{i,l} \in U_{\varrho_j(\eta_j(\gamma))}^{(j)} \text{ for all } j, i, l.$$

But then

$$g_{i,l} \in W_{\eta(\varrho_1(\eta_1(\gamma)), \dots, \varrho_m(\eta_m(\gamma)))} \text{ for all } i, l,$$

which shows the strong stability with respect to  $\{W_\gamma\}_{\gamma \in \Gamma}$ .  $\square$

So we are taking the following approach towards stability in the sense of [PS]: First we use Theorem 4.3 for enough different gradings on  $A$ , to obtain conditions for total (and therefore strong) stability of a quadratic module with respect to each of the gradings. If the gradings are chosen in the right way, Theorem 4.5 yields total stability with respect to a filtration of finite dimensional subspaces, and therefore stability in the sense of [PS].

**Remark 4.6.** One checks that all the results hold in more general algebras than the polynomial algebra over  $\mathbb{R}$ . Indeed, for any real closed field  $R$  and any finitely generated  $R$ -algebra that is a real domain, the results remain valid. *Real* means, that a sum of squares  $a_1^2 + \dots + a_t^2$  in  $A$  can only be zero if all  $a_i$  are zero.  $A$  is called a *domain*, if it does not contain zero divisors. Note that we have used these two properties at several points in the previous proofs.

The notions of stable and strongly stable generators with respect to a filtration even make sense in arbitrary  $R$ -algebras. We come back to this in the last section of the paper, where we will generalize a result from [CKS].

## 5. EXAMPLES OF GRADINGS AND APPLICATIONS

As above, let  $A = \mathbb{R}[X] = \mathbb{R}[X_1, \dots, X_n]$  be the real polynomial algebra in  $n$  variables. For  $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{N}^n$  and  $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$  we write

$$\underline{X}^\delta := X_1^{\delta_1} \dots X_n^{\delta_n}$$

and

$$z \cdot \delta := z_1 \delta_1 + \dots + z_n \delta_n.$$

For  $d \in \mathbb{Z}$  define

$$A_d^{(z)} := \left\{ \sum_{\delta \in \mathbb{N}^n, z \cdot \delta = d} c_\delta \underline{X}^\delta \mid c_\delta \in \mathbb{R} \right\}.$$

Then

$$A = \bigoplus_{d \in \mathbb{Z}} A_d^{(z)}$$

is a grading indexed in the ordered group  $(\mathbb{Z}, \leq)$ , to which we will refer to as the  $z$ -grading. For example,  $z = (1, \dots, 1)$  gives rise to the usual degree-grading on  $A$ , whereas  $z = (1, 0, \dots, 0)$  defines the grading with respect to the usual degree in  $X_1$ . Note that the filtration induced by such a  $z$ -grading consists of finite dimensional linear subspaces of  $A$  if and only if all entries of  $z$  are positive.

We want to characterize the denseness condition from Theorem 4.3 for these  $z$ -gradings. For a compact set  $K$  in  $\mathbb{R}^n$  with nonempty interior, we define the *tentacle of  $K$  in direction of  $z$*  in the following way:

$$T_{K,z} := \{(\lambda^{z_1}x_1, \dots, \lambda^{z_n}x_n) \mid \lambda \geq 1, x = (x_1, \dots, x_n) \in K\}.$$

For  $z = (1, \dots, 1)$ , such a set is just a full dimensional cone in  $\mathbb{R}^n$ . For  $z = (1, 0, \dots, 0)$  it is a full dimensional cylinder going to infinity in the direction of  $x_1$ . For  $z = (1, -1) \in \mathbb{Z}^2$ , something like the set defined by  $xy \leq 2, xy \geq 1$  and  $x \geq 1$  would be such a set.

**Proposition 5.1.** *Let  $f_1, \dots, f_s$  be polynomials in the graded polynomial algebra  $A = \bigoplus_{d \in \mathbb{Z}} A_d^{(z)}$ , where  $z \in \mathbb{Z}^n$ . Then the set*

$$\mathcal{S}(f_1^{\max}, \dots, f_s^{\max}) \subseteq \mathbb{R}^n$$

*is Zariski-dense in  $\mathbb{R}^n$ , if and only if the set*

$$\mathcal{S}(f_1, \dots, f_s) \subseteq \mathbb{R}^n$$

*contains a tentacle  $T_{K,z}$  for some compact  $K \subseteq \mathbb{R}^n$  with nonempty interior.*

*Proof.* First suppose  $\mathcal{S}(f_1^{\max}, \dots, f_s^{\max})$  is Zariski-dense, which is equivalent to saying that there is a compact set  $K$  with nonempty interior, on which all  $f_i^{\max}$  are positive. Write each  $f_i$  as a sum of homogeneous elements (with respect to the  $z$ -grading), for example

$$f_1 = h_{d_1} + \dots + h_{d_t},$$

where  $d_1 < \dots < d_t$  and  $0 \neq h_{d_j} \in A_{d_j}^{(z)}$ . Then for  $x \in \mathbb{R}^n$  and  $\lambda > 0$

$$f_1(\lambda^{z_1}x_1, \dots, \lambda^{z_n}x_n) = \lambda^{d_1}h_{d_1}(x) + \dots + \lambda^{d_t}h_{d_t}(x).$$

As  $h_{d_t}(x) = f_1^{\max}(x) > 0$  if  $x$  is taken from  $K$ , the expression is positive for  $\lambda \geq N$  with  $N$  big enough. Thereby  $N$  can be chosen to depend only on the size of the coefficients  $h_{d_j}(x)$ . So  $N$  can be chosen big enough to make  $f_i(\lambda^{z_1}x_1, \dots, \lambda^{z_n}x_n)$  positive for all  $\lambda \geq N$ ,  $x \in K$  and all  $i = 1, \dots, s$ . Replacing  $K$  by

$$K' := \{(N^{z_1}x_1, \dots, N^{z_n}x_n) \mid x = (x_1, \dots, x_n) \in K\}$$

we find  $T_{K',z} \subseteq \mathcal{S}(f_1, \dots, f_s)$ .

Conversely, suppose  $\mathcal{S}(f_1, \dots, f_s)$  contains a tentacle  $T_{K,z}$ . Then all the highest degree parts of the  $f_i$  must be nonnegative on  $K$ , with the same argument as above. So  $\mathcal{S}(f_1^{\max}, \dots, f_s^{\max})$  contains  $K$  and is therefore Zariski-dense in  $\mathbb{R}^n$ .  $\square$

Combined with Theorem 4.3 we get:

**Theorem 5.2.** *Let  $f_1, \dots, f_s$  be polynomials in the graded polynomial algebra  $A = \bigoplus_{d \in \mathbb{Z}} A_d^{(z)}$ , where  $z \in \mathbb{Z}^n$ . If the set*

$$\mathcal{S}(f_1, \dots, f_s) \subseteq \mathbb{R}^n$$

*contains some tentacle  $T_{K,z}$  ( $K$  compact with nonempty interior), then the quadratic module  $M = \text{QM}(f_1, \dots, f_s)$  is totally stable. If  $M = \text{QM}(f_1, \dots, f_s)$  is closed under multiplication, then  $\mathcal{S}(f_1, \dots, f_s)$  must contain such a tentacle for  $M$  to be totally stable.*

For the  $z$ -gradings, we can also settle the questions of coverings:

**Proposition 5.3.** *Let  $z, z^{(1)}, \dots, z^{(m)} \in \mathbb{Z}^n$  and assume there exist numbers  $r_1, \dots, r_m, t_1, \dots, t_m \in \mathbb{N}$ , such that the following conditions hold (where  $v \succeq w$  means  $\geq$  in each component of the vectors  $v, w$  in  $\mathbb{Z}^n$ ):*

$$r_1 z^{(1)} + \dots + r_m z^{(m)} \succeq z \text{ and}$$

$$t_j z \succeq z^{(j)} \text{ for } j = 1, \dots, m.$$

*Then the  $z$ -grading on  $\mathbb{R}[\underline{X}]$  is covered by the  $z^{(j)}$ -gradings.*

*Proof.* We denote by  $\deg(f)$  and  $\deg^{(j)}(f)$  the degree of a polynomial  $f$  with respect to the  $z$ - and the  $z^{(j)}$ -grading, respectively. First take a polynomial  $f$  and suppose  $\deg(f) \leq d$  for  $d \in \mathbb{Z}$ . So for every monomial  $c\underline{X}^\delta$  occurring in  $f$  we have  $z \cdot \delta \leq d$ . Now for every  $j = 1, \dots, m$ ,

$$z^{(j)} \cdot \delta \leq t_j (z \cdot \delta) \leq t_j d,$$

so  $\deg^{(j)}(f) \leq t_j d$ . Thus  $\psi_j: \mathbb{Z} \rightarrow \mathbb{Z}; d \mapsto t_j d$  fulfills the condition from Definition 4.4.

Now suppose  $\deg^{(j)}(f) \leq d_j$  for  $d_j \in \mathbb{Z}$  and  $j = 1, \dots, m$ . So for every monomial  $c\underline{X}^\delta$  occurring in  $f$ ,

$$z \cdot \delta \leq r_1 (z^{(1)} \cdot \delta) + \dots + r_m (z^{(m)} \cdot \delta) \leq r_1 d_1 + \dots + r_m d_m$$

holds. So  $\psi: \mathbb{Z}^m \rightarrow \mathbb{Z}; (d_1, \dots, d_m) \mapsto r_1 d_1 + \dots + r_m d_m$  fulfills the other condition from Definition 4.4.  $\square$

For example, the usual grading ( $z = (1, \dots, 1)$ ) is covered by the gradings defined by

$$z^{(1)} = (1, 0, \dots, 0), z^{(2)} = (0, 1, 0, \dots, 0), \dots, z^{(n)} = (0, \dots, 0, 1).$$

For  $n = 2$ , the two gradings defined by

$$z^{(1)} = (0, 1), z^{(2)} = (1, -1)$$

also cover the usual grading.

The following Main Theorem merges the above explained results.

**Theorem 5.4.** *Let  $S \subseteq \mathbb{R}^n$  be a basic closed semi-algebraic set that contains tentacles  $T_{K_j, z^{(j)}}$ , where  $K_j$  is compact with nonempty interior and  $z^{(j)} \in \mathbb{Z}^n$  ( $j = 1, \dots, m$ ). If there exist  $r_1, \dots, r_m \in \mathbb{N}$  such that*

$$r_1 z^{(1)} + \dots + r_m z^{(m)} \succ 0,$$

*then any finitely generated quadratic module describing  $S$  is stable and closed. So if  $n \geq 2$ , such a quadratic module does never have the Strong Moment Property.*

*Such natural numbers  $r_i$  exist, if and only if the only polynomial functions bounded on*

$$\bigcup_{j=1}^m T_{K_j, z^{(j)}}$$

*are the reals.*

*Proof.* The first part of the theorem is clear from the above results. We only have to prove the part concerning the bounded polynomial functions. Note that a polynomial  $f$  is bounded on a tentacle  $T_{K, z}$  if and only if it has degree less or equal to 0 with respect to the  $z$ -grading. This follows easily, using the ideas from

the proof of Proposition 5.1, and the fact that  $K$  is compact and has nonempty interior. So in case there are natural numbers  $r_1, \dots, r_m \in \mathbb{N}$  with

$$r_1 z^{(1)} + \dots + r_m z^{(m)} \succ 0,$$

there is no nontrivial monomial  $\underline{X}^\delta$  that has degree less or equal to 0 with respect to all the  $z^{(j)}$ -gradings. As all the monomials are homogeneous elements, there can be no nontrivial polynomial bounded on  $\bigcup_{j=1}^m T_{K_j, z^{(j)}}$ .

Conversely, assume there do *not* exist suitable numbers  $r_i$ . Then, by a Theorem of the Alternative (see for example [Ad], Lemma 1.2), there must be  $\delta \in \mathbb{N}^n \setminus \{0\}$ , such that

$$\delta \cdot z^{(j)} \leq 0$$

for all  $j$ . But this means that the (nontrivial) monomial  $\underline{X}^\delta$  is bounded on the set  $\bigcup_{j=1}^m T_{K_j, z^{(j)}}$ .  $\square$

Another class of gradings on the polynomial algebra  $A$  is given by term-orders. A term order is a linear ordering  $\leq$  on  $\mathbb{N}^n$  which fulfills

$$\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$$

for all  $\alpha, \beta, \gamma \in \mathbb{N}^n$ . Such a term order extends in a canonical way to an ordering of the Abelian group  $\mathbb{Z}^n$ . Indeed write  $\gamma \in \mathbb{Z}^n$  as a difference  $\alpha - \beta$  of elements from  $\mathbb{N}^n$ ; then define  $\gamma \geq 0$  if and only if  $\alpha \geq \beta$ .

We have a grading

$$A = \bigoplus_{\gamma \in \mathbb{Z}^n} A_\gamma^{(\leq)},$$

where  $A_\gamma^{(\leq)} := \mathbb{R} \cdot \underline{X}^\gamma$  if  $\gamma \in \mathbb{N}^n$  and  $A_\gamma^{(\leq)} := \{0\}$  otherwise. We refer to this grading as the  $\leq$ -grading. The decomposition of a polynomial  $f \in \mathbb{R}[\underline{X}]$  is

$$f = c_{\gamma_1} \underline{X}^{\gamma_1} + \dots + c_{\gamma_t} \underline{X}^{\gamma_t},$$

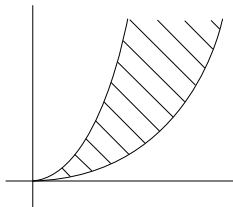
where  $c_{\gamma_i} \neq 0$  are the coefficients of  $f$  and  $\gamma_1 < \dots < \gamma_t$  with respect to the term order. The degree of  $f$  is  $\gamma_t$  then, and the highest degree part is the monomial  $c_{\gamma_t} \underline{X}^{\gamma_t}$ . Now for these term order gradings, the question of total stability is easy to solve. First we apply the reduction result from Proposition 4.1 to the generators of the quadratic module. So we can assume that all the generators have the same degree mod  $2\mathbb{Z}^n$ . The highest degree parts of the generators are then monomials  $c_\gamma \underline{X}^\gamma$ , where all the  $\gamma$  are congruent modulo  $2\mathbb{Z}^n$ . So obviously the quadratic module is totally stable if and only if all the occurring coefficients  $c_\gamma$  have the same sign, and are positive in case the  $\gamma$  are congruent 0 modulo  $2\mathbb{Z}^n$ . This gives an easy to apply method to decide total stability of a quadratic module with respect to a term order grading.

Note that not all of these  $\leq$ -gradings induce filtrations with finite dimensional linear subspaces. For example, a lexicographical ordering on  $\mathbb{N}^n$  does not. However, if we first sort by the usual total degree and then lexicographically, the subspaces are finite dimensional.

These term order gradings can show stability of quadratic modules, where the purely geometric conditions derived above and in [PS] do not apply. So they allow to take into account the difference between quadratic modules and preorderings.

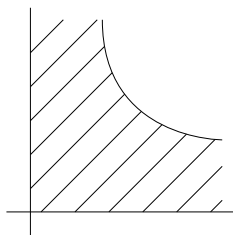
## 6. EXAMPLES

We start with some examples for the geometric stability result of Theorem 5.4. The first set we look at is defined by the inequalities  $0 \leq x, x^2 \leq y, y \leq 2x^2$  in  $\mathbb{R}^2$ .



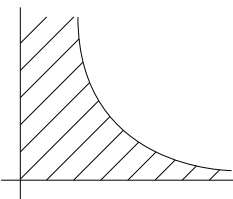
It contains a tentacle  $T_{K,(1,2)}$ . Therefore every finitely generated quadratic module describing this set is stable, thus also closed and does not have the Strong Moment Property.

The second set is described by  $0 \leq x, 0 \leq y, (x-1)(y-1) \leq 1$ .



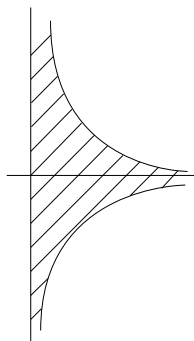
It contains a full dimensional cylinder in each direction of coordinates (that is, sets  $T_{K_1,(1,0)}$  and  $T_{K_2,(0,1)}$ ), and so every finitely generated quadratic module describing it is stable, closed and can not have the Strong Moment Property. This is one way to answer Open Question 4 from [KMS]. Another way to solve this open question is due to Claus Scheiderer (unpublished). One applies Theorem 3.10 from [PS].

We can weaken the geometric situation and still obtain stability. Look at the inequalities  $0 \leq x, 0 \leq y, (x-1)y \leq 1$ .



This set contains a full dimensional cylinder in direction of  $y$  (a set  $T_{K_1,(0,1)}$ ) and a set  $T_{K_2,(1,-1)}$ . The  $(0,1)$ - and the  $(1,-1)$ -gradings cover the usual grading, by Proposition 5.3 (or the fact that there are no nontrivial bounded polynomials; see Theorem 5.4). So every finitely generated quadratic module describing this set is stable, therefore also closed and can not have the Strong Moment Property.

We can still go one step further in narrowing the tentacles going to infinity. Look at the semi-algebraic set defined by  $0 \leq x, x^2 y \leq 1, -1 \leq xy$ .

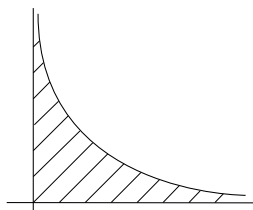


It contains a set  $T_{K_1,(-1,2)}$  (corresponding to the tentacle going to infinity in positive direction of  $y$ ), and a set  $T_{K_2,(1,-1)}$  (corresponding to the part of the tentacle going to infinity in direction of  $x$  that lies below the  $x$ -axis). As

$$2 \cdot (-1, 2) + 3 \cdot (1, -1) = (1, 1)$$

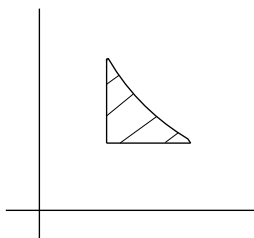
is positive in each coordinate, every finitely generated quadratic module describing this set is stable, and therefore also closed and does not have the Strong Moment Property. The considerations also show that there are no nontrivial bounded polynomials on this set, which is not completely obvious in this case.

We conclude the section with two non-geometric stability results. First, look at the semi-algebraic set defined by  $0 \leq x, 0 \leq xy \leq 1$ .



The geometric tentacle result does not apply to this set, and for example the pre-ordering generated by  $X, Y, 1 - XY$  indeed has the Strong Moment Property (see [KMS], Example 8.4). So it can not be stable. However, to the quadratic module  $M_1 = \text{QM}(X, Y, 1 - XY)$  we can apply the above explained results. Take the monomial ordering that first sorts by the usual total degree and then lexicographically with  $X > Y$ . No two of the generators of  $M_1$  have the same degree modulo  $2 \cdot (\mathbb{Z} \oplus \mathbb{Z})$ . So  $M_1$  is stable, closed and does not have the Strong Moment Property.

Exactly the same argument shows that the quadratic module  $M_2 = \text{QM}(X - \frac{1}{2}, Y - \frac{1}{2}, 1 - XY)$  is stable. In contrast to  $M_1$ , it describes a compact set:



This quadratic module is Example 6.3.1 from [PD], for a non-archimedean quadratic module describing a compact set. We can see here that  $M_2$  is not only non-archimedean, but indeed does not have the Strong Moment Property, which is stronger.

## 7. STRONG STABILITY AND THE INVARIANT MOMENT PROBLEM

We conclude this section with a generalization of Theorem 6.23 from [CKS]. First note that the definition of filtrations and strongly stable generators of a quadratic module with respect to a filtration make sense in arbitrary  $\mathbb{R}$ -algebras. Of course, if the algebra is not reduced or real, strong stability will only occur in degenerate situations.

If  $\iota: B \rightarrow A$  is a homomorphism of  $\mathbb{R}$ -algebras, then a filtration on  $A$  induces a canonical one on  $B$ . If for some  $b_1, \dots, b_s \in B$  the elements  $\iota(b_1), \dots, \iota(b_s) \in A$  are strongly stable generators with respect to a given filtration on  $A$ , then obviously  $b_1, \dots, b_s$  are strongly stable generators with respect to that induced filtration.

We now briefly recall the setup of [CKS] and refer the reader to it for more detailed information. Consider a finitely generated and reduced  $\mathbb{R}$ -algebra  $A$  with affine  $\mathbb{R}$ -variety  $\mathcal{V}_A$ . Denote the set of real points by  $\mathcal{V}_A(\mathbb{R})$ . Then  $A$  equals  $\mathbb{R}[\mathcal{V}_A]$ , the algebra of real regular functions on  $\mathcal{V}_A$ . Let  $\mathcal{G}$  be a linear algebraic group defined over  $\mathbb{R}$ , acting on  $\mathcal{V}_A$  by means of  $\mathbb{R}$ -morphisms. Then  $\mathcal{G}(\mathbb{R})$  acts canonically on  $A = \mathbb{R}[\mathcal{V}_A]$ , and if  $\mathcal{G}(\mathbb{R})$  is compact, the set of invariant regular functions, denoted by  $B = \mathbb{R}[\mathcal{V}_A]^{\mathcal{G}}$ , is a finitely generated  $\mathbb{R}$ -algebra. So it corresponds to an affine  $\mathbb{R}$ -variety  $\mathcal{V}_B$  and the inclusion  $\iota: B = \mathbb{R}[\mathcal{V}_A]^{\mathcal{G}} \hookrightarrow \mathbb{R}[\mathcal{V}_A] = A$  corresponds to a morphism  $\mathcal{V}_A \rightarrow \mathcal{V}_B$ . The restricted morphism  $\iota^*: \mathcal{V}_A(\mathbb{R}) \rightarrow \mathcal{V}_B(\mathbb{R})$  can be seen as the orbit map of the group action, by a Theorem by Procesi, Schwarz and Bröcker. Indeed, the nonempty fibers are precisely the  $\mathcal{G}(\mathbb{R})$ -orbits. Furthermore, for any basic closed semi-algebraic set  $\mathcal{S}$  in  $\mathcal{V}_A(\mathbb{R})$ , the set  $\iota^*(\mathcal{S})$  is basic closed semi-algebraic in  $\mathcal{V}_B(\mathbb{R})$ . The affine variety  $\mathcal{V}_B$  is denoted by  $\mathcal{V}_A//\mathcal{G}$ .

Now suppose  $\mathcal{S} \subseteq \mathcal{V}_A(\mathbb{R})$  is  $\mathcal{G}$ -invariant. Then one can look at the *Invariant Moment Problem* for  $\mathcal{S}$ . That is, one wants to find a finitely generated quadratic module  $M \subseteq \mathbb{R}[\mathcal{V}_A]^{\mathcal{G}}$ , such that every linear functional  $L$  on  $A$  that is invariant under the action of  $\mathcal{G}(\mathbb{R})$  and nonnegative on  $M$  is integration with respect to a measure on  $\mathcal{S}$ . One of the main results from [CKS] concerning the Invariant Moment Problem is, that this is possible if and only if  $M$  defines  $\iota^*(\mathcal{S})$  in  $\mathcal{V}_B(\mathbb{R})$  and has the Strong Moment Property in  $B$  (Lemma 6.9 in [CKS]). The situation in  $\mathcal{V}_B(\mathbb{R})$  is often simpler than the one in  $\mathcal{V}_A(\mathbb{R})$ , and so the Invariant Moment Problem can be solved in cases where the Strong Moment Problem can not.

However, Theorem 6.23 in [CKS] yields a negative result about the Invariant Moment Problem. Roughly spoken, it says that if the Moment Problem for  $\mathcal{S}$  is not solvable due to some geometric conditions on  $\mathcal{S}$ , then the Invariant Moment Problem is not solvable either. The result is proven for finite groups  $\mathcal{G}$  and irreducible varieties only. The following result holds for arbitrary compact groups.

**Theorem 7.1.** *Let the compact group  $\mathcal{G}$  act on the affine variety  $\mathcal{V}_A$  and let  $\mathcal{S}$  be a  $\mathcal{G}(\mathbb{R})$ -invariant basic closed semi-algebraic set in  $\mathcal{V}_A(\mathbb{R})$ . Fix a filtration of finite dimensional subspaces of  $A$ , and assume that every finitely generated quadratic module in  $A$  describing  $\mathcal{S}$  has only strongly stable generators with respect to that filtration. Then every finitely generated quadratic module in  $B = \mathbb{R}[\mathcal{V}_A]^{\mathcal{G}}$  describing*

$\iota^*(\mathcal{S})$  has only strongly stable generators with respect to the induced filtration on  $B$  (which consists of finite dimensional subspaces as well).

In particular, if  $\dim(\iota^*(\mathcal{S})) \geq 2$ , then no finitely generated quadratic module in  $B$  describing  $\iota^*(\mathcal{S})$  can have the Strong Moment Problem. So the Invariant Moment Problem is not finitely solvable for  $\mathcal{S}$ .

*Proof.* If  $\text{QM}(f_1, \dots, f_s) \subseteq B$  describes  $\iota^*(\mathcal{S})$ , then

$$\text{QM}(\iota(f_1), \dots, \iota(f_s)) \subseteq A$$

describes  $\mathcal{S} = (\iota^*)^{-1}(\iota^*(\mathcal{S}))$ . This uses that the fibres of  $\iota^*$  are precisely the  $\mathcal{G}(\mathbb{R})$ -orbits and that  $\mathcal{S}$  is  $\mathcal{G}(\mathbb{R})$ -invariant. So the assumption implies that  $\iota(f_1), \dots, \iota(f_s)$  are strongly stable generators in  $A$ , and so are  $f_1, \dots, f_s$  in  $B$ . The result concerning the Moment Problem follows from [S1] now.  $\square$

One checks that the geometric conditions from Theorem 6.24 in [CKS] imply, that the conditions from our Theorem 7.1 are fulfilled. Note also that the geometric conditions obtained in Theorem 5.4 above always imply the strong stability of any finite set of generators for  $\mathcal{S}$ . So Theorem 7.1 yields a negative result concerning the Invariant Moment Problem in all of these cases.

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