POSITIVE POLYNOMIALS AND PROJECTIONS OF SPECTRAHEDRA

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Abstract. This work is concerned with different aspects of spectrahedra and their projections, sets that are important in semidefinite optimization. We prove results on the limitations of so called Lasserre and theta body relaxation methods for semialgebraic sets and varieties. As a special case we obtain the main result of [19] on non-exposed faces. We also solve the open problems from that work. We further give a unified account of several results on convex hulls of curves and images of polynomial maps. We finally prove a Positivstellensatz for projections of spectrahedra, which exceeds the known results that only work for basic closed semialgebraic sets.

1. Introduction

Semidefinite programming has turned out to be a very important and valuable tool in polynomial optimization in recent times. It is concerned with finding optimal values of linear functions on certain convex sets. These sets, called *spectrahedra*, arise as linear sections of the cone of positive semidefinite matrices. Semidefinite programming generalizes linear programming. The importance of semidefinite programming comes from two facts. On one hand there exist efficient algorithms to solve semidefinite programming problems, see for example Ben-Tal and Nemirovski [1], Nesterov and Nemirovski [18], Nemirovski [17], Todd [30], Vandenberghe and Boyd [31] and Wolkowicz, Saigal and Vandenberghe [32]. On the other hand, a great amount of problems from various branches of mathematics can be approached using semidefinite programming. Examples come from combinatorial optimization, non-convex optimization and control theory; see for example Parrilo and Sturmfels [21], Gouveia, Parrilo and Thomas [4] and all of the above mentioned literature.

This brings up the theoretical question of how to characterize sets on which semidefinite programming can be performed, i.e. to characterize spectrahedra. Helton and Vinnikov [8] have done groundbreaking work towards this question. They show that spectrahedra are what they call *rigidly convex*, and this condition is sufficient in dimension two. This result also solves the Lax conjecture, as explained in Lewis, Parrilo and Ramana [15].

Observe that semidefinite programming can also be performed on projections of spectrahedra. One just has to optimize the objective function over a higher dimensional set. Up to now there are only two known necessary conditions for a set to be the projection of a spectrahedron: being convex and being semi-algebraic. Lasserre [14] has provided a method to prove for certain sets that they are the projection of a spectrahedron. Helton and Nie [6, 7] have applied the method to

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large classes of convex sets. They indeed conjecture that each convex semi-algebraic set is the projection of a spectrahedron.

This work is concerned with the question of how to write sets as projections of spectrahedra. Our contribution is the following. After introducing notation we review in Section 3 some of the methods to construct projections of spectrahedra. We give a unified account of some results spread across the literature on convex hulls of curves, and we prove a few simple facts for which we could not find a direct reference in the literature, for example, that the closure of the projection of a spectrahedron is again such a projection or that the closure of the convex hull of the image of a four-dimensional polyhedral cone by a quadratic map is the projection of a spectrahedron.

In Section 4 we analyze the Lasserre method, and the related theta body method from [4]. We present a general obstruction that explains all the common instances where we know these approaches to fail. As a special case we obtain a new proof for the main result from [19]. Moreover, this new proof relies on basic convex geometry and avoids the explicit use of model theory done in the original proof. We end that section by settling the open questions from that work by providing a series of examples.

Finally, in Section 5, we prove a Positivstellensatz for projections of spectrahedra. This is interesting in particular because such sets are usually not basic closed semialgebraic, so none from the large amount of known Positivstellensätze apply.

2. Notation

We will use the following notation. For $n \in \mathbb{N}$ let $\underline{X} = (X_1, \ldots, X_n)$ be an n-tuple of variables. Let $\mathbb{R}[\underline{X}]$ denote the real polynomial ring in these variables. By $\mathbb{R}[\underline{X}]_d$ we denote its finite dimensional subspace of polynomials of degree at most d. Let $\underline{p} = (p_1, \ldots, p_r)$ be an r-tuple of polynomials from $\mathbb{R}[\underline{X}]$. Then

$$\mathscr{S}(\underline{p}) := \{x \in \mathbb{R}^n \mid p_1(x) \geq 0, \ldots, p_r(x) \geq 0\} \subseteq \mathbb{R}^n$$

is the *basic closed semi-algebraic set* defined by $\underline{p}$. In the polynomial ring we have a corresponding *quadratic module*, defined as

$$\mathrm{QM}(\underline{p}) := \left\{\sigma_0 + \sigma_1 p_1 + \cdots + \sigma_r p_r \mid \sigma_i \in \sum \mathbb{R}[\underline{X}]^2\right\}.$$

Here we use the notation $\sum V^2$ for the set of all sums of squares of elements from a given subset V of a commutative ring R.

All elements from $\mathrm{QM}(\underline{p})$ are nonnegative as functions on $\mathscr{S}(\underline{p})$. There are also certain truncated parts of $\mathrm{QM}(\underline{p})$, defined as

$$\mathrm{QM}(\underline{p})_d := \left\{\sigma_0 + \sigma_1 p_1 + \cdots + \sigma_r p_r \mid \sigma_i \in \sum (\mathbb{R}[\underline{X}]_{d-h_i})^2\right\},$$

where $h_i = \lfloor \deg(p_i)/2 \rfloor$. $\mathrm{QM}(\underline{p})_d$ is contained in the finite dimensional space $\mathbb{R}[\underline{X}]_{2d}$, and in general, it will be strictly smaller than $\mathrm{QM}(\underline{p}) \cap \mathbb{R}[\underline{X}]_{2d}$. This follows for example from Theorem 5.4 in [26] (and is one of the reasons why Schmügdens famous theorem [28] works).

We denote by $\mathrm{M}_{k\times k}(V)$ the set of $k\times k$-matrices with entries from a given subset V of a commutative ring R. $\sum M_{k\times k}(V)^2$ is then the set of *sums of hermitian squares*, i.e. it contains the finite sums of elements of the form A^tA with $A \in \mathrm{M}_{k\times k}(V)$. We denote by $\mathrm{Sym}_k(V)$ the set of symmetric matrices from $\mathrm{M}_{k\times k}(V)$. The usual inner product $A \bullet B$ for $k \times k$-matrices $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$ is defined as

$$A \bullet B = \mathrm{Tr}(AB) = \sum_{i,j} a_{ij} b_{ij},$$

where Tr denotes the trace. For a matrix $A \in \mathrm{Sym}_k(\mathbb{R})$, $A \succeq 0$ means that A is positive semidefinite, i.e. $v^t A v \geq 0$ holds for every $v \in \mathbb{R}^k$. $A \succ 0$ means that A is positive definite, i.e. $v^t A v > 0$ holds for all $v \neq 0$.

A k-dimensional *linear matrix polynomial* is an affine linear polynomial

$$\mathscr{A}(\underline{X}) = A + X_1 B_1 + \cdots X_n B_n,$$

with $A, B_1, \ldots, B_n \in \mathrm{Sym}_k(\mathbb{R})$. It is called *strictly feasible* if there is a point $x \in \mathbb{R}^n$ with $\mathscr{A}(x) \succ 0$. The set

$$\mathscr{S}(\mathscr{A}) := \{x \in \mathbb{R}^n \mid \mathscr{A}(x) \succeq 0\}$$

is called a *spectrahedron.* It is a convex and basic closed semi-algebraic set, and a generalization of a polyhedron. This paper deals with projections of such spectrahedra, i.e. sets of the form

$$S = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m \ \mathscr{A}(x, y) \succeq 0\},$$

where $\mathscr{A}$ is a linear matrix polynomial in the variables $X_1, \ldots, X_n, Y_1, \ldots, Y_m$. So S is the image of the spectrahedron $\widetilde{S} \subseteq \mathbb{R}^{n+m}$ defined by $\mathscr{A}$, under the canonical projection $\mathbb{R}^{n+m} \to \mathbb{R}^n$. In some parts of the literature, projections of spectrahedra are defined as the image under any affine map of a spectrahedron. We should note that our seemingly more restrictive definition is actually equivalent as shown in the following elementary argument.

Observation. *If S has non-empty interior and is the image of a spectrahedron T under an affine map, then S is also a canonical projection of a spectrahedron T'. Furthermore, if T is a strictly feasible spectrahedron, T' can be chosen to be one too.*

Proof. Let $T = \{y \in \mathbb{R}^m \mid \mathscr{A}(y) := A + y_1 B_1 + ... + y_m B_m \succeq 0\}$ be a spectrahedron, $L : \mathbb{R}^m \to \mathbb{R}^n$ an affine map and $S = L(T)$. We can furthermore restrict L to be linear, since translations of projections of spectrahedra are still projections of spectrahedra, even in our restricted sense. We also know that L is onto, since S has nonempty interior. By reordering the variables y_i we can assume that $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}$, where L_1 is a $n \times n$ non-singular matrix. We can now consider the spectrahedron

$$T' = \{z \in \mathbb{R}^m \mid \mathscr{A}(L_1^{-1}(z_1, ..., z_n) - L_1^{-1} L_2(z_{n+1}, ..., z_m), z_{n+1}, ..., z_m) \succeq 0\},$$

which is easily seen to be strictly feasible if T was, and whose projection onto $z_1, ..., z_n$ equals S. $\square$

Note that the first part of the Lemma remains true even if S has empty interior.

For a convex set $S \subseteq \mathbb{R}^n$ let $\mathrm{Aff}(S)$ denote its affine hull, i.e. the smallest affine subspace of $\mathbb{R}^n$ containing S. A *face* of S is a nonempty convex subset $F \subseteq S$ which is extremal in the following sense: whenever $\lambda x + (1 - \lambda) y \in F$ for some $x, y \in S, \lambda \in (0, 1)$, then $x, y \in F$. For an affine linear polynomial $\ell \in \mathbb{R}[\underline{X}]_1$ that is nonnegative on S, the subset $\{x \in S \mid \ell(x) = 0\}$ is a face of S or empty. A face is called *exposed* if it is of such a form.

3. Some construction methods revisited

As indicated in the introduction, there is a large amount of work on the construction of spectrahedra that project to a given set. In this section we review some of them. We also provide proofs of some helpful facts that cannot be found in the existing literature.

3.1. **Polars and Closures.** We start by reviewing a result on polars by Nemirovski, and we deduce some helpful corollaries. We for example observe that the closure of the projection of a spectrahedron is again such a projection. The results on polars will also be very helpful in the subsequent section, when Lasserre relaxations are considered.

In [17], Section 4.1.1, Nemirovski proves the following result:

Proposition 3.1. *Let* $\mathscr{A}(\underline{X},\underline{Y}) = A + X_1B_1 + \cdots + X_nB_n + Y_1C_1 + \cdots + Y_mC_m$ *be a k-dimensional strictly feasible linear matrix polynomial. Let* $S := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m\ \mathscr{A}(x,y) \succeq 0\}$ *be the projection of the spectrahedron defined by* $\mathscr{A}$*, and let*

$$S^\circ := \{\ell \in \mathbb{R}[\underline{X}]_1 \mid \ell \geq 0 \text{ on } S\}$$

denote the convex cone of affine linear polynomials nonnegative on S*. Then*

$$\begin{aligned} S^\circ = \{l_0 + l_1X_1 + \cdots + l_nX_n \mid \exists U \in \mathrm{Sym}_k(\mathbb{R}) :\ & U \succeq 0,\ U \bullet A \leq l_0, \\ & U \bullet B_i = l_i \text{ for } i = 1, \ldots, n, \\ & U \bullet C_j = 0 \text{ for } j = 1, \ldots, m\}. \end{aligned}$$

In particular, S° *is again the projection of a spectrahedron.*

The result follows from the duality theory of conic programming, and is thus essentially a separation argument. The set S° is called the *polar* of S in Nemirovksi's work. A key observation is that, for our purposes, the imposition of strict feasibility is not necessary.

Proposition 3.2. *Let* $S \subseteq \mathbb{R}^n$ *be the projection of a spectrahedron. Then* $S^\circ = \{\ell \in \mathbb{R}[\underline{X}]_1 \mid \ell \geq 0 \text{ on } S\}$ *is again the projection of a spectrahedron.*

Proof. Let S be the projection of some spectrahedron T. We will first assume that S has nonempty interior in $\mathbb{R}^n$. By replacing the ambient space of T by the affine hull of T we can also assume that T has nonempty interior (but the projection might become an arbitrary linear map now). Note that for a spectrahedron, having nonempty interior is equivalent to being definable by a strictly feasible linear matrix polynomial, by Ramana and Goldman [24], Corollary 5. The observation from the last section now shows that we can apply Proposition 3.1, since S is the standard projection of a strictly feasible spectrahedron.

If S has empty interior, let V denote the affine hull of S. Then for $\ell \in \mathbb{R}[\underline{X}]_1$ the condition $\ell \geq 0$ on S is equivalent to $\ell|_V \geq 0$ on S. This shows that S° is the inverse image of a projection of a spectrahedron under a linear map, and so itself the projection of a spectrahedron. □

As an immediate corollary we get the following basic result, for which we could find no reference in the literature.

Corollary 3.3. *Let* $S \subseteq \mathbb{R}^n$ *be the projection of a spectrahedron. Then its closure* $\overline{S}$ *is again the projection of a spectrahedron.*

Proof. By Proposition 3.2, $(S^\circ)^\circ$ is the projection of a spectrahedron. But we have

$$\overline{S} = \{a \in \mathbb{R}^n \mid X_0 + a_1X_1 + \cdots + a_nX_n \in (S^\circ)^\circ\},$$

which proves the result. □

3.2. **Lasserre Relaxations.** In this subsection we review the method of Lasserre [14] to construct projections of spectrahedra, and use Proposition 3.2 to give an alternative explanation of the method.

We first observe that if $M \subseteq \mathbb{R}[\underline{X}]_1$ is the projection of a spectrahedron, then

$$\mathscr{L} := \{x \in \mathbb{R}^n \mid \ell(x) \geq 0 \text{ for all } \ell \in M\}$$

is also such a projection. This follows from Proposition 3.2, since $\mathscr{L}$ is M° intersected with a subspace. Now for a finite set of polynomials $p_1, \ldots, p_r \in \mathbb{R}[\underline{X}]$ let $S = \mathscr{S}(\underline{p})$ be the basic closed semi-algebraic set they define, $\mathrm{QM}(\underline{p})$ denote the corresponding quadratic module in $\mathbb{R}[\underline{X}]$ and $\mathrm{QM}(\underline{p})_d$ its truncated part, as defined in Section 2. It turns out that each $\mathrm{QM}(\underline{p})_d$ is the projection of a spectrahedron. One can for example use the following result, which is Theorem 1 from Ramana and Goldman [23]:

Theorem 3.4. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a quadratic polynomial map. Then the convex hull of the image $f(\mathbb{R}^n)$ is the projection of a spectrahedron.*

So note that each $\mathrm{QM}(\underline{p})_d$ is the convex hull of the image of a quadratic map. Indeed one just has to parametrize the coefficients occuring in the sums of squares used in the representations of its elements. Thus the sets $\mathrm{QM}(\underline{p})_d \cap \mathbb{R}[\underline{X}]_1$ are projections of spectrahedra and we finally obtain that each set

$$\mathscr{L}(\underline{p})_d := \{x \in \mathbb{R}^n \mid \ell(x) \geq 0 \text{ for all } \ell \in \mathrm{QM}(\underline{p})_d \cap \mathbb{R}[\underline{X}]_1\}$$

is the projection of a spectrahedron. The set $\mathscr{L}(\underline{p})_d$ is called a degree d Lasserre relaxation of S. Each $\mathscr{L}(\underline{p})_d$ is closed convex and contains S. The sequence of the $\mathscr{L}(\underline{p})_d$ is descending.

Note that our definition of a Lasserre relaxation differs slightly from the original one given in Lasserre [14]. There, the dual of $\mathrm{QM}(\underline{p})_d$ is projected to $\mathbb{R}^n$, whereas we intersect $\mathrm{QM}(\underline{p})_d$ with $\mathbb{R}[\underline{X}]_1$ and then consider the nonnegativity set in $\mathbb{R}^n$. However, the two definitions give rise to the same relaxations *up to closures*, at least if S has a nonempty interior. In fact, in this case the Lasserre relaxation as we define it is just the closure of the classical one. This can for example be checked with an argument as in the proof of Proposition 3.1 in Netzer, Plaumann and Schweighofer [19], using the closedness of $\mathrm{QM}(\underline{p})_d$.

The following Theorem is the key result on Lasserre relaxations. Part (i) is mainly Theorem 2 from Lasserre [14], and now clear from our above considerations. Part (ii) is proven as Proposition 3.1 (2) in Netzer, Plaumann and Schweighofer [19].

Theorem 3.5. *(i) If $\mathrm{QM}(\underline{p})_d$ contains all affine linear polynomials nonnegative on S, then $\mathscr{L}(\underline{p})_d = \overline{\mathrm{conv}(S)}$. In particular, $\overline{\mathrm{conv}(S)}$ is the projection of a spectrahedron then.*

(ii) If $\mathscr{L}_d(\underline{p}) = \overline{\mathrm{conv}(S)}$ and S has nonempty interior, then $\mathrm{QM}(\underline{p})_d$ contains all affine linear polynomials nonnegative on S.

Another possibility for obtaining semidefinite descriptions for convex sets is a different Lasserre-type relaxation hierarchy for convex hulls of algebraic sets, the *theta body* hierarchy introduced in Gouveia, Parrilo and Thomas [4]. Given an ideal

$I \subseteq \mathbb{R}[\underline{X}]$, we denote the set of all polynomials p such that $p - \sigma \in I$ for some sum of squares σ with $\deg(\sigma) \leq 2d$ by $\Sigma(d, I)$. Note that $\Sigma(d, I)$ intersected with any finite dimensional subspace of $\mathbb{R}[\underline{X}]$ is the projection of a spectrahedron. This follows since $I \cap W$ is a subspace in W, for each subspace W of $\mathbb{R}[\underline{X}]$.

Definition 3.6. Let $I \subseteq \mathbb{R}[\underline{X}]$ be an ideal. The d-th theta body of I, denoted by $\mathrm{TH}(I)_d$, is the intersection of all half-spaces $H_\ell := \{x \in \mathbb{R}^n \mid \ell(x) \geq 0\}$, where ℓ ranges over all linear polynomials in $\Sigma(d, I)$.

The theta body hierarchy for the ideal I approximates the convex hull of its real variety $\mathcal{V}_\mathbb{R}(I) = \{x \in \mathbb{R}^n \mid g(x) = 0 \text{ for all } g \in I\}$. An analogous result to Theorem 3.5 is true, with the condition of the ideal I being real radical replacing the condition of S having nonempty interior.

Theorem 3.7. *(i) If $\Sigma(d, I)$ contains all affine linear polynomials nonnegative on $\mathcal{V}_\mathbb{R}(I)$, then $\mathrm{TH}(I)_d = \overline{\mathrm{conv}(\mathcal{V}_\mathbb{R}(I))}$. In particular, $\overline{\mathrm{conv}(\mathcal{V}_\mathbb{R}(I))}$ is the projection of a spectrahedron then.*

(ii) If $\mathrm{TH}(I)_d = \overline{\mathrm{conv}(\mathcal{V}_\mathbb{R}(I))}$ and I is real radical, then $\Sigma(d, I)$ contains all affine linear polynomials nonnegative on $\mathcal{V}_\mathbb{R}(I)$.

Again, part (i) is immediate from the definition, while part (ii) is proven in Lemma 2.7 of [4]. In Section 4 we will study possible obstructions to these two methods. In particular we reprove the main result of Netzer, Plaumann and Schweighofer [19] and settle the open problems from that work.

3.3. **Images of Polynomial Maps.** In this subsection we want to give a unified account of several results on convex hulls of images under polynomial maps, including results by Lasserre, Parrilo, Ramana and Goldman, Henrion and Scheiderer. The results can all be deduced from the following principle:

Proposition 3.8. *Let $S \subseteq \mathbb{R}^n$ be a set and $V \subseteq \mathbb{R}[\underline{X}]$ a finite dimensional linear subspace containing* 1*. Assume the subset $P \subseteq V$ of all elements of V that are nonnegative on S is the projection of a spectrahedron. Then for any map $f = (f_1, \ldots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ with $f_i \in V$ for all i,*

$$\overline{\mathrm{conv}(f(S))} \subseteq \mathbb{R}^m$$

is the projection of a spectrahedron.

Proof. For any affine linear polynomial $\ell \in \mathbb{R}[Y_1, \ldots, Y_m]_1$ the polynomial $\ell(f_1, \ldots, f_m)$ belongs to V. Define $M := \{\ell \in \mathbb{R}[Y_1, \ldots, Y_m]_1 \mid \ell(f_1, \ldots, f_m) \in P\}$. One immediately checks that M is the projection of a spectrahedron (since P is) and contains only polynomials that are nonnegative on $f(S)$. Conversely, if ℓ is affine linear and nonnegative on $f(S)$, then $\ell(f_1, \ldots, f_m)$ is in P. Thus M is precisely the cone of affine linear polynomials nonnegative on $f(S)$, and by the arguments from the last section

$$\overline{\mathrm{conv}(f(S))} = \{x \in \mathbb{R}^m \mid \ell(x) \geq 0 \text{ for all } \ell \in M\}$$

is the projection of a spectrahedron. □

Example 3.9. Not very surprisingly, the Lasserre result can be recovered from Proposition 3.8. Indeed if there is some d such that $\mathrm{QM}(\underline{p})_d$ contains all affine linear polynomials that are nonnegative on S, then apply Proposition 3.8 with $V = \mathbb{R}[\underline{X}]_1$ and $f = \mathrm{id}$. $P = V \cap \mathrm{QM}(\underline{p})_d$ is the projection of a spectrahedron, as explained in the previous section.

Example 3.10. We also get that the closure of $\operatorname{conv}(f(\mathbb{R}^n))$ is the projection of a spectrahedron, for any quadratic map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ (which is of course also not a new result, in view of Theorem 3.4 and Corollary 3.3). Use the well-known fact that every globally nonnegative quadratic polynomial is a sum of squares of affine linear polynomials, and apply Proposition 3.8 with $S = \mathbb{R}^n$ and $V = \mathbb{R}[\underline{X}]_2$. Again recall that $P = \sum \mathbb{R}[\underline{X}]_1^2 \subseteq V$ is the projection of a spectrahedron.

In the following result, case (i) for a full rational curve is proven in Henrion [9], Theorem 1. In the version it is stated here it has also been the topic of a talk of Parrilo at a workshop in Banff in 2006, but there seems to be no suitable reference. Case (ii) relies on results of Scheiderer, as also explained in [25]. For an introduction to the algebraic geometry concepts present in this result [3] is a possible reference.

Corollary 3.11. *Let $S \subseteq \mathbb{R}^n$ be either*

(i) *a semi-algebraic subset of a rational curve, or*
(ii) *a smooth curve of genus* 1 *with at least one non-real point at infinity.*

Then for any rational map

$$f = \left(\frac{f_1}{g}, \dots, \frac{f_m}{g}\right) : \mathbb{R}^n \to \mathbb{R}^m$$

such that g does not vanish anywhere in S, we find that

$$\overline{\operatorname{conv}(f(S))}$$

is the projection of a spectrahedron.

Proof. First check that we can reduce to the case where the denominator is one i.e. where f is a polynomial map. Indeed for a general rational map f we can take without loss of generality a denominator g that is positive on S, and we can also prove the claim for the following map instead:

$$F\colon S \to \mathbb{R}^{m+1};\ x \mapsto \left(\frac{f_1(x)}{g(x)}, \dots, \frac{f_m(x)}{g(x)}, 1\right).$$

Then define

$$G\colon S \to \mathbb{R}^{m+1};\ x \mapsto g(x) \cdot F(x).$$

This map is polynomial and thus assume we already know that $\overline{\operatorname{conv}(G(S))}$ is the projection of a spectrahedron. By [20], Proposition 2.1, the conic hull of the projection of a spectrahedron is again such a projection. So together with Corollary 3.3 we get that $\overline{\operatorname{cone}(G(S))}$, the closure of the convex cone of $G(S)$, is the projection of a spectrahedron. But now one can check, by a simple argument of converging sequences, that

$$\overline{\operatorname{cone}(G(S))} \cap (\mathbb{R}^m \times \{1\}) = \overline{\operatorname{conv}(F(S))},$$

which finishes the reduction step. We now just have to show that for each case, we can for every d find a projected spectrahedron that contains all polynomials nonnegative on S of degree less or equal d so that we can apply Proposition 3.8.

For (i) it is clearly enough to consider the case of S being a semialgebraic subset of a straight line, which is covered by Kuhlmann, Marshall and Schwartz [13] Theorem 4.1 (see also the paper by Scheiderer [22]). They prove that for any such set S and degree d there is a d' such that the truncated quadratic module $\operatorname{QM}(\underline{p})_{d'}$ contains all polynomials of degree at most d that are nonnegative on S.

For (ii), the results on the existence of sums of squares representations in Scheiderer [27], Theorem 4.10 (a), plus the degree bounds explained in Scheiderer [25] imply the intended result, by showing that if I is the vanishing ideal of such a curve S, for any degree bound d we can find a d' such that $\Sigma(d', I)$ contains all polynomials of degree at most d that are nonnegative on S. □

These results give us a tool for quickly justifying for some classes of sets that they are projections of spectrahedra. Note that they can be applied even in some cases where the curvature results from Helton and Nie [6] [7] and Lasserre's direct approach from [14] do not apply, and a more *ad hoc* method would have had to be used.

Example 3.12. The basic closed semi-algebraic set $S = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, -1 \leq x, y^2 - x^3 \geq 0\}$ is bounded by segments of rational curves. The convex hull of each such segment is the projection of a spectrahedron, by Corollary 3.11. The set S is the convex hull of all of these convex hulls combined and thus also the projection of a spectrahedron. The results from Helton and Nie do not apply since $Y^2 - X^3$ is neither strictly quasi-concave on S, nor sos-concave. Also it is singular at the origin. The standard Lasserre method does not apply since S has a nonexposed face, see for example Theorem 4.2 below. One could also replace the part of the set on the left hand side of the y-axis by a half disk. The resulting set is not even basic closed, and still the result applies.

Example 3.13. Let $S \subseteq \mathbb{R}^2$ be defined by the inequality $y^2 \leq 1 - x^4$. The boundary is a smooth genus one curve with a non-real point at infinity. Thus S is the projection of a spectrahedron. Applying the polynomial map $(x, y) \mapsto (x^2, xy^2)$ sends this curve to the boundary of the convex set $y^2 \leq x - 2x^3 + x^5$ which has a singularity at the point $(1, 0)$ as seen in Figure 1. Still Corollary 3.11 guarantees that this set is the projection of a spectrahedron.

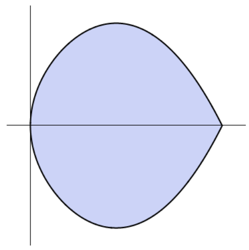

FIGURE 1. Convex Hull of the image of the curve $1 - x^4 - y^2 = 0$ by the map $(x, y) \mapsto (x^2, xy^2)$.

We state some more corollaries of Proposition 3.8. The following result is Henrion [9], Theorem 1:

Corollary 3.14. *Let either $f : \mathbb{R}^3 \to \mathbb{R}^m$ be homogeneous of degree 4 or $f : \mathbb{R}^2 \to \mathbb{R}^m$ componentwise of degree 4 (but not necessarily homogeneous). Then the closure of the convex hull of the image of f is the projection of a spectrahedron .*

Proof. We can apply Proposition 3.8, using Hilbert's result that every globally nonnegative homogeneous degree 4 polynomial in three variables and every globally nonnegative degree 4 polynomial in two variables is a sum of squares. □

We get another result that has to our knowledge not been observed before:

Corollary 3.15. *Let $f : \mathbb{R}^4 \to \mathbb{R}^m$ be homogeneous quadratic. Let $C \subseteq \mathbb{R}^4$ be any polyhedral cone. Then $\overline{\operatorname{conv}(f(C))}$ is the projection of a spectrahedron.*

Proof. Every polyhedral cone in $\mathbb{R}^4$ is a finite union of cones that can be transformed by a linear automorphism to the first orthant in some $\mathbb{R}^k$ with $k \le 4$. This follows from Caratheodory's Theorem for cones. If $C = C_1 \cup \dots \cup C_m$ then

$$\overline{\mathrm{conv}(f(C))} = \overline{\mathrm{conv}(\overline{f(C_1)} \cup \dots \cup \overline{f(C_m)})}.$$

So by the convex hull result from Helton and Nie [7] (see also [20]) and Corollary 3.3 it is enough to prove the Corollary for the first orthant C in $\mathbb{R}^4$.

Every quadratic form in 4 variables that is nonnegative on the first orthant belongs to the quadratic module generated by the pairwise products of the variables X_iX_j. This is just a slight reformulation of the main result from Diananda [2]. But then a degree bound condition on the sums of squares is fulfilled for any such representation, since no degree cancellation can occur when adding polynomials that are nonnegative on the first orthant. So in fact each such nonnegative quadratic form is a positive combination of the X_iX_j plus a sums of squares of linear forms. Now apply Proposition 3.8 with $\underline{p} = \{X_iX_j \mid 1 \le i, j \le n\}$ and V the space spanned by the quadratic forms and 1. □

4. Obstructions to the relaxation methods

In this section we examine the assumption from Lasserre's Theorem, as stated in Theorem 3.5 above. That means, we want to know whether there exists some d such that the truncated quadratic module $\mathrm{QM}(\underline{p})_d$ contains all nonnegative linear polynomials. Note here that this condition is absolutely not necessary for $\overline{\mathrm{conv}(S)}$ to be the projection of a spectrahedron. This is for example shown by Example 3.7 in [19] (that we will discuss in more detail below). But in view of Theorem 3.5, the condition is necessary and sufficient for the Lasserre approach to work. This brings up the question when this so called *bounded degree representation property* for affine polynomials is fulfilled.

A necessary condition is given by the following result:

Proposition 4.1. *Let $p_1, ..., p_r \in \mathbb{R}[\underline{X}]$, $S = \mathscr{S}(\underline{p})$ and L be a line in $\mathbb{R}^n$ such that $S \cap L$ has non-empty interior relative to L. Let $a \in S$ be a point that belongs to the relative boundary of $\overline{\mathrm{conv}(S)} \cap L$ in L. Assume that for all p_i with $p_i(a) = 0$ the vector $\nabla p_i(a)$ is orthogonal to L. Then, for all d, the Lasserre relaxation $\mathscr{L}(\underline{p})_d$ strictly contains $\overline{\mathrm{conv}(S)}$.*

Proof. By applying a linear transformation we may assume L to be the X_1-axis, a to be the origin and $\overline{\mathrm{conv}(S)} \cap L$ to be on the positive half axis. Let $p_1', \dots, p_r' \in \mathbb{R}[X_1]$ be the polynomials obtained from $p_1, \dots, p_r$ by setting the last $n-1$ variables to zero. We have $\mathscr{L}(\underline{p}')_d \subseteq \mathscr{L}(\underline{p})_d \cap \mathbb{R}$ for any d. This inclusion follows from the fact that each polynomial $f \in \mathrm{QM}(\underline{p})_d$ ends up in $\mathrm{QM}(\underline{p}')_d$ when setting the last $n-1$ variables to zero.

Let $S' = \mathscr{S}(\underline{p}')$, so $S' = S \cap \mathbb{R}$. If $\mathrm{conv}(S')$ is some closed interval $[0, c]$ then let $p_{r+1}' = c - X_1$, otherwise (i.e. if $\mathrm{conv}(S') = [0, \infty)$) let $p_{r+1}' = 1$ (just to keep the notation uniform). Then $\mathscr{S}(\underline{p}') = \mathscr{S}(\underline{p}', p_{r+1}')$. Note that $\mathscr{L}(\underline{p}', p_{r+1}')_d \subseteq \mathscr{L}(\underline{p}')_d$, and since S' has an interior point $\mathscr{L}(\underline{p}', p_{r+1}')_d = \mathrm{conv}(S')$ holds if and only if every on S' nonnegative affine linear polynomial from $\mathbb{R}[X_1]$ belongs to $\mathrm{QM}(\underline{p}', p_{r+1}')_d$, by Theorem 3.5. Consider the polynomial X_1, that is nonnegative on S', and suppose there exists a representation

$$X_1 = \sigma + \sum_{i \in I} \sigma_i p_i' + \sum_{j \in J} \sigma_j p_j',$$

where $i \in I$ if $p_i(0) > 0$ and $i \in J$ otherwise, and the $\sigma, \sigma_i, \sigma_j$ are sums of squares. For $i \in I$, p_i' has a positive constant term, so σ_i cannot have a constant term, and its homogeneous part of minimal degree must be at least quadratic. The same is true for σ. So none of the elements σ and $\sigma_i p_i'$ where $i \in I$ contains the monomial X_1. But by hypothesis, $\triangledown p_j(0)$ is orthogonal to the x_1-axis for $j \in J$, which implies that the terms of p_j' have all degree at least 2. This is a contradiction. So $\mathscr{L}(\underline{p}', p_{r+1}')_d$ is not $\operatorname{conv}(S')$. Since $p_{r+1}' \in \operatorname{QM}(\underline{p}', p_{r+1}')_d$ this implies the existence of some negative b with $b \in \mathscr{L}(\underline{p}', p_{r+1}')_d \subseteq \mathscr{L}(\underline{p}')_d \subseteq L \cap \mathscr{L}(\underline{p})_d$. But since $b \notin \overline{\operatorname{conv}(S)}$ by hypothesis, this implies $\mathscr{L}(\underline{p})_d \neq \overline{\operatorname{conv}(S)}$. □

This gives an alternative and more elementary proof to Theorem 3.5 in [19]:

Theorem 4.2. *Let $p_1, ..., p_r \in \mathbb{R}[\underline{X}]$ be such that $S = \mathscr{S}(\underline{p})$ is convex and has non-empty interior. If S has a non-exposed face, then for all d, the Lasserre relaxation $\mathscr{L}(\underline{p})_d$ strictly contains S.*

Proof. Let $F \subseteq S$ be a non-exposed face. Then there exists some face F_1 of S, such that $F \subsetneq F_1$ and for all supporting hyperplanes H containing F, $F_1 \subseteq H$. Let a be a point in the relative interior of F and L a line passing through a and some point in the relative interior of F_1. By convexity and closedness of S we have that a belongs to the relative boundary of $\overline{\operatorname{conv}(S)} \cap L$, and we just have to verify the gradient condition at a.

Suppose $p_j(a) = 0$, and consider $v := \triangledown p_j(a)$. For any $x \in \mathbb{R}^n$ the product $v \cdot (x - a)$ equals the derivative of p_j at a in direction of $(x - a)$, so by convexity of S we get $v \cdot (x - a) \geq 0$ whenever $x \in S$. Hence the linear polynomial $\ell := v_1(X_1 - a_1) + \cdots + v_n(X_n - a_n)$ is nonnegative on S. Since ℓ vanishes at a, which lies in the relative interior of F, it vanishes on the whole of F and thus also on F_1. Then, since it vanishes in two points of L, it must vanish on the entire line, which implies that v is orthogonal to L, and Proposition 4.1 gives us the result. □

The lemma also shows us the following result:

Theorem 4.3. *Let $p_1, \ldots, p_r \in \mathbb{R}[\underline{X}]$ and let $S := \mathscr{S}(\underline{p}) \subseteq \mathbb{R}^n$ have non-empty interior. Let a be point in the boundary of $\overline{\operatorname{conv}(S)}$ that is also in S, and suppose that all active constrains at a are singular. Then for all d, the Lasserre relaxation $\mathscr{L}(\underline{p})_d$ strictly contains $\overline{\operatorname{conv}(S)}$.*

Note that the active constraints are those p_i with $p_i(a) = 0$, and singular means that the gradient of p_i vanishes at a.

Proof. Just consider a line L passing through a and through the interior of S and apply Proposition 4.1. □

Example 4.4. Consider the semi-algebraic set $S = \{(x, y) \in \mathbb{R}^2 : p(x, y) := -x^4 + x^3 - y^2 \geq 0\}$. The convex hull of S is intersected by the x-axis in the segment $[0, 1]$, which has non-empty interior. Furthermore p has a singularity at the origin, hence we are in the conditions of Proposition 4.1 and the Lasserre hierarchy does not converge in finitely many steps, although it does approximate the set $\operatorname{conv}(S)$ as shown in Figure 2.

The same general idea we used for the Lasserre relaxations can also be applied to the theta body construction. To do that, however, we need some auxiliary definitions. Let I be any ideal, and p a point in $\mathcal{V}_{\mathbb{R}}(I)$. The **tangent space** $T_p(\mathcal{V}_{\mathbb{R}}(I))$ to $\mathcal{V}_{\mathbb{R}}(I)$ at p is the affine space through p that is orthogonal to the space

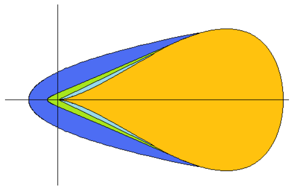

FIGURE 2. From the smallest to the largest: the sets S, $\mathrm{conv}(S)$, $\mathscr{L}(\underline{p})_2$ and $\mathscr{L}(\underline{p})_1$.

spanned by the gradients at p of all polynomials in $I(\mathcal{V}_{\mathbb{R}}(I))$. Here, $I(\mathcal{V}_{\mathbb{R}}(I)$ denotes the vanishing ideal of $\mathcal{V}_{\mathbb{R}}(I)$; it is precisely the real radical of I. We say that a point $p \in \mathcal{V}_{\mathbb{R}}(I)$ on the boundary of $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$ is **convex-non-singular** if $T_p(\mathcal{V}_{\mathbb{R}}(I))$ is tangent to $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$, i.e. if it does not intersect its relative interior. Otherwise, i.e. if a vector from $T_p(\mathcal{V}_{\mathbb{R}}(I))$ points into the relative interior, we say that p is **convex-singular**.

Theorem 4.5. *Let I be any ideal such that $\mathcal{V}_{\mathbb{R}}(I)$ has a convex-singular point, then for all d, $\mathrm{TH}_d(I)$ strictly contains $\overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$.*

Proof. Let J be the vanishing ideal of $\mathcal{V}_{\mathbb{R}}(I)$. Since I is contained in J, $\mathrm{TH}_d(J) \subseteq \mathrm{TH}(I)_d$, so it is enough to show that $\mathrm{TH}_d(J) \neq \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$. Suppose we have equality. Since J is real radical, Theorem 3.7 tell us that any linear polynomial that is nonnegative in $\mathcal{V}_{\mathbb{R}}(I)$ must be in $\Sigma(d, J)$. Let p be the convex-singular point of $\mathcal{V}_{\mathbb{R}}(I)$. Since p is in the boundary of $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$ there exists a linear polynomial ℓ that is zero in p and positive on the relative interior of $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$. Therefore $\ell = \sigma + g$ where σ is a sum of squares and $g \in J$. Let q be a point in $T_p(\mathcal{V}_{\mathbb{R}}(I))$ that is in the relative interior of $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$. We have

$$(q-p)\cdot\nabla\ell(p) = (q-p)\cdot\nabla\sigma(p) + (q-p)\cdot\nabla g(p).$$

But since σ is a sum of squares vanishing at p, it must have a double zero there so its gradient also vanishes there, and since q belongs to $T_p(\mathcal{V}_{\mathbb{R}}(I))$ then for all $g \in J$, $(q-p)$ is orthogonal to their gradients at p, so we have that the derivative of ℓ in the direction of $(q-p)$ is zero. Since ℓ is linear, this implies that it vanishes at q, which is a contradiction. □

Remark 4.6. Note that if $J = I(\mathcal{V}_{\mathbb{R}}(I))$ is generated by a single polynomial (so $\mathcal{V}_{\mathbb{R}}(I)$ is a hypersurface), then any singular point p from $\mathcal{V}_{\mathbb{R}}(I)$ that belongs to the boundary of $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$ is convex-singular. This is clear since the tangent space at p is the whole of $\mathbb{R}^n$ in that case.

Example 4.7. (i) An example illustrating the above remark is the (compact) Zitrus surface defined by $x^2+z^2+(y^2-1)^3 = 0$ in $\mathbb{R}^3$. It has a singularity at $(0,1,0)$, which belongs to the boundary of the convex hull, and thus each theta body relaxation strictly contains the convex hull of the surface. The boundary equations for the convex hull of this surface have been examined in detail by Sturmfels and Ranestad in [29], Section 4.2.

(ii) Consider the variety $V_{\mathbb{R}}(I)$ in $\mathbb{R}^3$ defined by the ideal

$$I = \langle x^2+y^2+z^2-4, (x-1)^2+y^2-1\rangle.$$

It has a singularity at the point $p = (2,0,0)$, which belongs to the boundary of the convex hull of $\mathcal{V}_{\mathbb{R}}(I)$. This singularity is however not convex-singular, as one easily checks. And indeed already the first theta body relaxation equals $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$.

To see this first note that I can also be defined by $p_1 = (x-1)^2 + y^2 - 1$ and $p_2 = 2x + z^2 - 4$. Write $I_1 = \langle p_1 \rangle$ and $I_2 = \langle p_2 \rangle$. Then we claim that

$$\operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I)) = \operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I_1)) \cap \operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I_2)).$$

To see this note that the variety $V_{\mathbb{R}}(I)$ can be written as

$$\{(x, \pm\sqrt{1-(x-1)^2}, \pm\sqrt{4-2x}) \mid 0 \le x \le 2\}.$$

In particular for each fixed x we get four points, hence the rectangle they form must be contained in the convex hull. This means

$$\{(x,y,z) \in \mathbb{R}^3 \mid |y| \le \sqrt{1-(x-1)^2}, |z| \le \sqrt{4-2x}, 0 \le x \le 2\} \subseteq \operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I)),$$

but it is clear that this set can be rewritten as

$$\{(x,y,z) \in \mathbb{R}^3 \mid y^2 \le 1-(x-1)^2, z^2 \le 4-2x\} = \operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I_1)) \cap \operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I_2)),$$

which contains $\operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I))$, so we get the intended equality.

Since $\mathrm{TH}_d(I) \subseteq \mathrm{TH}_d(I_1) \cap \mathrm{TH}_d(I_2)$ holds obviously, it is enough to show that the theta body relaxations for I_1 and I_2 are exact in the first step. But this follows for example from Lemma 5.5. in [4], since p_1 and p_2 are convex quadrics. The example shows that the notion of a convex-singular point is crucial in Theorem 4.5.

(iii) A third example is the bean curve, defined by the polynomial $p = x(x^2 + y^2) - x^4 - x^2y^2 - y^4$. This curve was for example examined in Henrion [10], where it was already proven that none of the relaxations is exact. We can deduce this fact also from the above results, since the origin is a convex-singular point. It is in particular interesting that the origin is not a singular point of the boundary of $\operatorname{conv}(\mathcal{V}_{\mathbb{R}}(p))$.

We go back to Theorem 4.2. It says that a convex basic closed set S can only equal some relaxation $\mathscr{L}(\underline{p})_d$ if all of its faces are exposed. In [19] the question is raised whether this can be generalized:

Question 4.8. [19, Remark 3.8]

(i) Is Theorem 4.2 still true with S replaced by $\overline{\operatorname{conv}(S)}$, if S is non-convex?
(ii) More generally, are all faces of $\mathscr{L}(\underline{p})_d$ exposed for all d and $\underline{p}$?

One can also ask if Theorem 4.2 can be generalized to the theta body relaxations:

Question 4.9. Let $I \subseteq \mathbb{R}[\underline{X}]$ be an ideal such that $\mathrm{TH}(I)_d = \overline{\operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I))}$. Are all faces of $\mathrm{TH}(I)_d$ exposed faces in this case?

The answer to all these questions is negative, as we will show.

Proposition 4.10. *Let $p_1 = Y, p_2 = 1-Y, p_3 = Y - X^3, p_4 = 1 + X$ define the set $S = \mathscr{S}(\underline{p}) \subseteq \mathbb{R}^2$. Then $\mathscr{L}(\underline{p})_2$ is the convex hull of $S \cup \{(1/3, 0)\}$.*

Proof. Let $C = \operatorname{conv}(S \cup \{(1/3,0)\})$. Then C is cut out by the infinitely many affine linear inequalities

$$\{Y \ge 0, 1-Y \ge 0, 1+X \ge 0, Y - 3a^2X + 2a^3 \ge 0 \mid a \in [1/2, 1]\},$$

since the polynomial $\ell_a := Y - 3a^2X + 2a^3$ defines the half-plane containing S and tangent to the curve $Y = X^3$ at the point (a, a^3). To prove $\mathscr{L}(\underline{p})_2 \subseteq C$ it is thus enough to show that the polynomials ℓ_a belong to $\mathrm{QM}(\underline{p})_1$ for all $a \ge 1/2$. To see this, note that

$$\ell_a = (\sqrt{2a-1}(X-a))^2 + (Y - X^3) + (X-a)^2(X+1).$$

To prove the inclusion $C \subseteq \mathscr{L}(\underline{p})_2$, using the fact that $\mathscr{L}(\underline{p})_2$ is convex and contains S, it is enough to show that $(1/3, 0) \in \mathscr{L}(\underline{p})_2$. Since translations commute with taking Lasserre relaxations, we will instead consider the set of polynomials $p'_1 = Y, p'_2 = 1 - Y, p'_3 = X + 4/3, p'_4 = Y - X^3 - X^2 - X/3 - 1/27$ obtained from the p_i by replacing X by $X + 1/3$, and prove that $(0, 0) \in \mathscr{L}(\underline{p}')_2$. Suppose that is not the case. Then there must exist $\varepsilon, \mu > 0$ such that $\ell = Y - \mu X - \varepsilon$ belongs to $\mathrm{QM}(\underline{p}')_2$. This means

$$\ell = \sigma_0 + \sigma_1 Y + \sigma_2(1 - Y) + \sigma_3(X + 4/3) + c(Y - X^3 - X^2 - X/3 - 1/27),$$

where c is simply a nonnegative constant, since $\deg(p'_4) = 3$. Note that σ_0 has at most degree 2, as do σ_1, σ_2 and σ_3.

Let $\sigma_3 = a_1X^2 + a_2X + a_3 + a_4Y^2 + a_5XY + a_6Y$. In order to cancel the X^3 term of the entire expression, we must have $a_1 = c$. The coefficient for X^2 will then be $a - c + 4/3c + a_2$, where a is a nonnegative number which is the sum of the coefficients of X^2 in σ_0 and σ_2. This implies $a_2 \leq -c/3$, which by using the fact that σ_3 is a sum of squares, implies $a_3 \geq c/36$ (just consider a Hankel matrix for this sum of squares and analyze the submatrix indexed by 1 and x).

Now checking the constant coefficient, we will have it to be $b - c/27 + 4a_3/3$, where b is the nonnegative constant term of $\sigma_0 + \sigma_2$. Since this must be $-\varepsilon$, we have $-c/27 + 4a_3/3 < 0$ which since $a_3 \geq c/36$ is impossible. Hence $\ell \notin \mathrm{QM}(\underline{p}')_2$, and $(0, 0)$ is in $\mathscr{L}(\underline{p})_2$ as intended. □

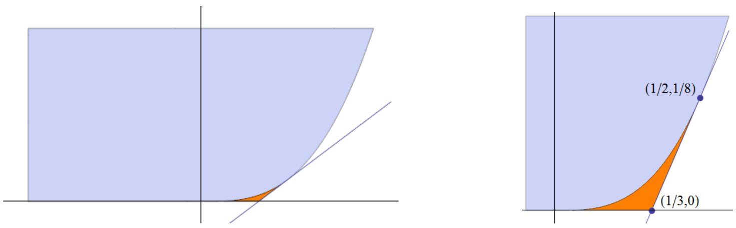

FIGURE 3. Comparison between S and $\mathscr{L}(\underline{p})_2$, where $p_1 = Y, p_2 = 1 - Y, p_3 = 1 + X, p_4 = Y - X^3$. Full region on the left, close up on the right.

Corollary 4.11. *For $p_1 = Y, p_2 = 1 - Y, p_3 = 1 + X, p_4 = Y - X^3$, $\mathscr{L}(\underline{p})_2$ has a non-exposed face at* $(1/2, 1/8)$.

Proof. Immediate, from Figure 3. □

This shows that general Lasserre relaxations might have non-exposed faces, giving a negative answer to Question 4.8 (ii). In fact, this can happen even for very "well-behaved" semialgebraic sets. If in Proposition 4.10 we change the defining polynomials $\underline{p}$ to $\underline{p}'$ by replacing Y with $Y - 1/10$, we get a semialgebraic set that has only exposed faces (it can even be shown that $\mathscr{S}(\underline{p}') = \mathscr{L}(\underline{p}')_3$). However, our proof still works in this case, showing that $\mathscr{L}(\underline{p}')_2 = \mathscr{L}(\underline{p})_2 \cap \{(x, y) \mid y > 1/10\}$ has a non-exposed face.

In the next proposition we show that when $\mathscr{S}(\underline{p})$ is not convex, even if one of its Lasserre relaxations is tight (meaning $\mathscr{L}(\underline{p})_d = \overline{\mathrm{conv}(S)}$ for some d), $\mathscr{L}(\underline{p})_d$ might still have non-exposed faces.

Proposition 4.12. *For* $p := -X^4 - Y^4 - 2X^2Y^2 + 4X^2 \in \mathbb{R}[X,Y]$ *we find*

$$\mathscr{L}(p)_2 = \operatorname{conv}(\mathscr{S}(p)).$$

Proof. The set $S = \mathscr{S}(p)$ is the union of two disks of radius 1 with centers $(-1,0)$ and $(1,0)$. By symmetry, it is enough to show that any linear polynomial tangent to the left circle and non-negative on both disks belongs to $\mathrm{QM}(p)_2$. The points on the left circle that are on the boundary of $\operatorname{conv}(S)$ are of the form $z_\vartheta := (\cos(\vartheta) - 1, \sin(\vartheta))$, for some $\vartheta \in [\pi/2, 3\pi/2]$, and an affine linear polynomial ℓ_ϑ defining the tangent to z_ϑ such that $\ell_\vartheta \geq 0$ on S is given by $\ell_\vartheta = 1 - \cos(\vartheta) - \cos(\vartheta)X - \sin(\vartheta)Y$. Since $\cos(\vartheta) \leq 0$ it is enough to check that the equality

$$\begin{aligned}(8 - 8\cos(\vartheta))\ell_\vartheta &= p + (X^2 + Y^2 - 2 + 2\cos(\vartheta))^2 + \\ &+ \left(2\sqrt{1-\cos(\vartheta)}(Y - \sin(\vartheta))\right)^2 + \\ &+ \left(2\sqrt{-\cos(\vartheta)}(X - \cos(\vartheta) + 1)\right)^2\end{aligned} \tag{4.13}$$

holds, thus proving the result. □

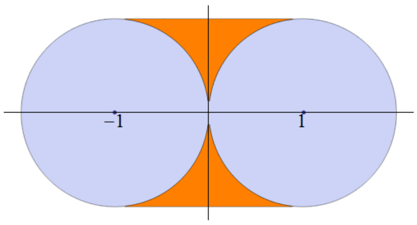

FIGURE 4. Comparison between S and $\mathscr{L}(p)_2 = \operatorname{conv}(S)$, where $p = -X^4 - Y^4 - 2X^2Y^2 + 4X^2$.

Corollary 4.14. *For* $p = -X^4 - Y^4 - 2X^2Y^2 + 4X^2$, $\mathscr{L}(p)_2 = \operatorname{conv}(\mathscr{S}(p))$ *has a non-exposed face.*

Proof. Just note that the four points $(\pm 1, \pm 1)$ are all non-exposed faces of $\mathscr{L}(p)_2 = \operatorname{conv}(\mathscr{S}(p))$ as it can be seen in Figure 4. □

Note that the proof of Proposition 4.12 not only completes the answer to Question 4.8 (i), but also answers Question 4.9. Our representation (4.13) shows that if we consider the ideal $I = \langle p \rangle$, then $\mathrm{TH}(I)_2 = \operatorname{conv}(V_{\mathbb{R}}(I))$ has non-exposed faces.

5. A Positivstellensatz for projections of spectrahedra

In this section we describe a quadratic module that is assigned to the projection of a spectrahedron. This quadratic module will in general not be finitely generated, but still its elements can be described almost constructively. The module will turn out to be archimedean whenever the set is bounded and has nonempty interior, and it will thus provide us with a Positivstellensatz for projections of spectrahedra. This is in particular interesting since such projections are usually *not* basic closed semialgebraic. So none from the large amount of present Positivstellensätze applies to this setup.

So for this whole section let $\mathscr{A}(\underline{X}, \underline{Y}) = A + X_1B_1 + \cdots + X_nB_n + Y_1C_1 + \cdots + Y_mC_m$ be a strictly feasible k-dimensional linear matrix polynomial. Let $\widetilde{S} \subseteq \mathbb{R}^{n+m}$ be the spectrahedron defined by $\mathscr{A}$, and $S = \operatorname{pr}(\widetilde{S}) \subseteq \mathbb{R}^n$ its projection. We will write $\mathscr{A}'(\underline{X})$ for $\mathscr{A}(\underline{X}, 0)$.

Recall that any linear polynomial $\ell \in \mathbb{R}[\underline{X}]_1$ that is nonnegative on S is of the form

$$\ell = U \bullet \mathscr{A}'(\underline{X}) + r,$$

with some $r \geq 0$ and a positive semidefinite $k \times k$-matrix U that fulfills $U \bullet C_i = 0$ for all $i = 1, \ldots, m$. This is precisely the statement of Proposition 3.1. By Cholesky decomposition of U this is the same as saying

$$\ell = \sum_j v_j^t \mathscr{A}'(\underline{X}) v_j + r$$

for finitely many vectors $v_j \in \mathbb{R}^k$ fulfilling $\sum_j v_j^t C_i v_j = 0$ for all $i = 1, \ldots, m$.

If we now want to construct a quadratic module containing all the nonnegative linear polynomials on S, we can use polynomial vectors q_j instead of real vectors v_j only. We allow for sums of arbitrary length. Formally, define

$$\mathrm{QM}(\mathscr{A}) := \left\{ \sum_j q_j^t \mathscr{A}'(\underline{X}) q_j + \sigma \mid q_j \in \mathbb{R}[\underline{X}]^k, \sum_j q_j^t C_i q_j = 0 \text{ for } i = 1, \ldots m, \right. \\ \left. \sigma \in \sum \mathbb{R}[\underline{X}]^2 \right\}.$$

It is easy to see that $\mathrm{QM}(\mathscr{A})$ is a quadratic module. The following main result now follows easily. In the case of a bounded set S it provides the announced Positivstellensatz.

Theorem 5.1. $\mathrm{QM}(\mathscr{A})$ *contains only polynomials that are nonnegative on* S*, and the set of points in* $\mathbb{R}^n$ *where all elements from* $\mathrm{QM}(\mathscr{A})$ *are nonnegative is precisely* $\overline{S}$*. If* S *is bounded then* $\mathrm{QM}(\mathscr{A})$ *is archimedean, and thus contains all polynomials* p *with* $p > 0$ *on* $\overline{S}$*.*

Proof. The first statements follows immediately from the fact that each element from $\mathrm{QM}(\mathscr{A})$ is in particular of the form

$$\sum_j q_j^t \mathscr{A}(\underline{X}, \underline{Y}) q_j + \sigma,$$

and from the definition of S. The second statement is then clear from the fact that all nonnegative linear polynomials are contained in $\mathrm{QM}(\mathscr{A})$, by Proposition 3.1. In the case of a bounded set S we have $N \pm X_i \in \mathrm{QM}(\mathscr{A})$ for all i and some sufficiently large number N. As for example explained in Marshall [16], Corollary 5.2.4, $\mathrm{QM}(\mathscr{A})$ is archimedean. Then Jacobi's Representation Theorem [11, Theorem 4] implies the statement about strictly positive polynomials. □

Note again that we assume $\mathscr{A}$ to be strictly feasible. This implies that the set S has nonempty interior. Note also that in the case of a bounded nonempty spectrahedron, Helton, Klep, and McCullough [5] have already proven $\mathrm{QM}(\mathscr{A})$ to be archimedean, using results about completely positive maps. They use this to obtain a Positivstellensatz for matrix polynomials, see their Theorem 1.3. The case of an empty spectrahedron is more complicated. The situation is examined in [12].

Note also that in our result we cannot expect $\mathrm{QM}(\mathscr{A})$ to be a finitely generated quadratic module in general. This would imply that $\overline{S}$ is basic closed semi-algebraic, i.e. defined by finitely many simultaneous polynomial inequalities. This is clearly not true for all projections of spectrahedra.

Example 5.2. Consider the example from Proposition 4.12, the convex hull of two disks in the plane. In contrast to the above example, we denote by S the full convex hull. Note that S is an example of a closed semi-algebraic set that is not basic closed. Since S is the union of disks shifted along the x-axis, one immediately checks that it has the following representation:

$$S = \{(x, y) \in \mathbb{R}^2 \mid \exists z \in [-1, 1] \ (x - z)^2 + y^2 \leq 1\}.$$

The defining condition of S can now be stated as positive semidefiniteness of the following linear matrix polynomial:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + X \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + Y \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + Z \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

So by Theorem 5.1, every polynomial that is strictly positive on S is a sum of squares plus a polynomial of the following form:

$$\sum_j \left(q_{1j}^2 + q_{2j}^2 + q_{3j}^2 + q_{4j}^2 + X(q_{3j}^2 - q_{4j}^2) + Y(q_{1j}^2 - q_{2j}^2) \right),$$

where $q_{ij} \in \mathbb{R}[X, Y]$ with $\sum_j \left(2q_{1j}q_{2j} - q_{3j}^2 + q_{4j}^2\right) = 0$.

Example 5.3. This last example shows that we cannot weaken the assumption $p > 0$ on $\overline{S}$ to $p > 0$ on S. Consider

$$S = (0, 1] \times (0, 1] \subseteq \mathbb{R}^2.$$

It can be realized as the projection of a spectrahedron, using for example

$$\mathscr{A}(X_1, X_2, Y) = \begin{pmatrix} X_1 & & & & & & & \\ & 1 - X_1 & & & & & & \\ & & X_2 & & & & & \\ & & & 1 - X_2 & & & & \\ & & & & X_1 & 1 & & \\ & & & & 1 & Y & & \\ & & & & & & X_2 & 1 \\ & & & & & & 1 & Y \end{pmatrix}.$$

It is easily checked that $\mathrm{QM}(\mathscr{A})$ is the standard quadratic module generated by $X_1, 1 - X_1, X_2, 1 - X_2$. An elementary computation then shows that X_1X_2 does not belong to $\mathrm{QM}(\mathscr{A})$, although it is strictly positive on S.

References

[1] A. Ben-Tal and A. Nemirovski. *Lectures on modern convex optimization.* MPS/SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Analysis, algorithms, and engineering applications.

[2] P. H. Diananda. On non-negative forms in real variables some or all of which are non-negative. *Proc. Cambridge Philos. Soc.*, **58**, 17–25, 1962.

[3] W. Fulton. *Algebraic curves. An introduction to algebraic geometry.* W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[4] J. Gouveia, P. A. Parrilo, and R. R. Thomas. Theta bodies for polynomial ideals. *SIAM J. Optim.*, **20 (4)**, 2097–2118, 2010.

[5] J. W. Helton, I. Klep, and S. McCullough. The matricial relaxation of a linear matrix inequality. Preprint.

[6] J. W. Helton and J. Nie. Semidefinite representation of convex sets. *Math. Program.*, **122 (1)**, 21–64, 2010.

[7] ———. Sufficient and necessary conditions for semidefinite representability of convex hulls and sets. *SIAM J. Optim.*, **20**, 759–791, 2009

[8] J. W. Helton and V. Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, **60 (5)**, 654–674, 2007.

[9] D. Henrion. Semidefinite representation of convex hulls of rational varieties. *LAAS-CNRS Research Report No. 09001*, January 2009.
[10] D. Henrion. On semidefinite representations of plane quartics. *LAAS-CNRS Research Report No. 08444*, September 2008.
[11] T. Jacobi. A representation theorem for certain partially ordered commutative rings. *Math. Z.*, **237 (2)**, 259–273, 2001.
[12] I. Klep and M. Schweighofer. Infeasibility certificates for linear matrix inequalities. work in progress.
[13] S. Kuhlmann, M. Marshall, and N. Schwartz. Positivity, sums of squares and the multi-dimensional moment problem. II. *Adv. Geom.*, **5 (4)**, 583–606, 2005.
[14] J. B. Lasserre. Convex sets with semidefinite representation. *Math. Program.*, **120 (2)**, 457 -477, 2009.
[15] A. S. Lewis, P. A. Parrilo, and M. V. Ramana. The Lax conjecture is true. *Proc. Amer. Math. Soc.*, **133 (9)**, 2495–2499 (electronic), 2005.
[16] M. Marshall. *Positive polynomials and sums of squares*, vol. 146 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.
[17] A. Nemirovski. Advances in convex optimization: conic programming. In *International Congress of Mathematicians. Vol. I*, pp. 413–444. Eur. Math. Soc., Zürich, 2007.
[18] Y. Nesterov and A. Nemirovski. *Interior-point polynomial algorithms in convex programming*, vol. 13 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.
[19] T. Netzer, D. Plaumann, and M. Schweighofer. Exposed faces of semidefinitely representable sets. *Siam Journal on Optimization*, **20 (4)**, 1944–1955, (2010).
[20] T. Netzer and R. Sinn. A note on the convex hull of finitely many projections of spectrahedra. ArXiv:0908.3386v1 [math.OC].
[21] P. A. Parrilo and B. Sturmfels. Minimizing polynomial functions. In *Algorithmic and quantitative real algebraic geometry (Piscataway, NJ, 2001)*, vol. 60 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pp. 83–99. Amer. Math. Soc., Providence, RI, 2003.
[22] A. Prestel and C. N. Delzell. *Positive polynomials*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001.
[23] M. Ramana and A. Goldman. Quadratic maps with convex images. Tech. rep., Rutgers Center for Operations Research, 1995.
[24] M. Ramana and A. J. Goldman. Some geometric results in semidefinite programming. *J. Global Optim.*, **7 (1)**, 33–50, 1995.
[25] C. Scheiderer. Convex hulls of curves of genus one. *Preprint*.
[26] ———. Non-existence of degree bounds for weighted sums of squares representations. *Journal of Complexity*, **21 (6)**, 823–844, 2005.
[27] ———. Sums of squares of regular functions on real algebraic varieties. *Trans. Amer. Math. Soc.*, **352 (3)**, 1039–1069, 2000.
[28] K. Schmügden. The K-moment problem for compact semialgebraic sets. *Math. Ann.*, **289 (1)**, 203-206, 1991.
[29] B. Sturmfels and K. Ranestad. The convex hull of a variety. To appear in the Julius Borcea Memorial Volume, (eds. Petter Branden, Mikael Passare and Mihai Putinar), Trends in Mathematics, Birkhauser Verlag.
[30] M. Todd. Semidefinite Optimization *Acta Numerica*, **10**, 515–560, 2001.
[31] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Rev.*, **38 (1)**, 49–95, 1996.
[32] H. Wolkowicz, R. Saigal, and L. Vandenberghe (editors). *Handbook of semidefinite programming*. International Series in Operations Research & Management Science, 27. Kluwer Academic Publishers, Boston, MA, 2000. Theory, algorithms, and applications.

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