A NOTE ON THE CONVEX HULL OF FINITELY MANY PROJECTIONS OF SPECTRAHEDRA

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Abstract. A spectrahedron is a set defined by a linear matrix inequality. A projection of a spectrahedron is often called a *semidefinitely representable set.* We show that the convex hull of a finite union of such projections is again a projection of a spectrahedron. This improves upon the result of Helton and Nie [3], who prove the same result in the case of bounded sets.

1. Introduction

Let $A_0, A_1, \dots, A_n$ be real symmetric $k \times k$ matrices. The set

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid A_0 + x_1 A_1 + \dots + x_n A_n \succeq 0\},$$

where $\succeq 0$ means positive semidefiniteness, is called a *spectrahedron.* Spectrahedra are generalizations of polyhedra and occur as feasible sets for semidefinite optimization.

A projection of a spectrahedron to a subspace of $\mathbb{R}^n$ is often called a *semidefinitely representable set.* Helton and Nie [3] conjecture that every convex semialgebraic set is such a projection. See for example [1, 2, 3, 4, 5, 6, 7] for more detailed information on spectrahedra and their projections.

We prove that the convex hull of finitely many projections of spectrahedra is again a projection of a spectrahedron. This generalizes Theorem 2.2 from Helton and Nie [3], which is the same result in the case that all sets are bounded or that the convex hull is closed.

2. Result

Proposition 2.1. *If $S \subseteq \mathbb{R}^n$ is a projection of a spectrahedron, then so is* $\mathrm{cc}(S)$*, the conic hull of* S*.*

Proof. Since S is a projection of a spectrahedron we can write

$$S = \left\{ x \in \mathbb{R}^n \mid \exists z \in \mathbb{R}^m \colon A + \sum_{i=1}^n x_i B_i + \sum_{j=1}^m z_j C_j \succeq 0 \right\},$$

with suitable real symmetric $k \times k$-matrices A, B_i, C_j. Then with

$$C := \{x \in \mathbb{R}^n \mid \exists \lambda, r \in \mathbb{R}, z \in \mathbb{R}^m \colon \lambda A + \sum_{i=1}^n x_i B_i + \sum_{j=1}^m z_j C_j \succeq 0 \wedge \bigwedge_{i=1}^n \begin{pmatrix} \lambda & x_i \\ x_i & r \end{pmatrix} \succeq 0\}$$

we have $C = \mathrm{cc}(S)$ (note that C is a projection of a spectrahedron, since the conjunction can be eliminated, using block matrices).

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To see "$\subseteq$" let some x fulfill all the conditions from C, first with some $\lambda > 0$. Then $a := \frac{1}{\lambda} \cdot x$ belongs to S, using the first condition only. Since $x = \lambda \cdot a$, $x \in \mathrm{cc}(S)$. If x fulfills the conditions with $\lambda = 0$, then $x = 0$, by the last n conditions in the definition of C. So clearly also $x \in \mathrm{cc}(S)$.

For "$\supseteq$" take $x \in \mathrm{cc}(S)$. If $x \neq 0$ then there is some $\lambda > 0$ and $a \in S$ with $x = \lambda a$. Now there is some $z \in \mathbb{R}^m$ with $A + \sum_i a_i B_i + \sum_j z_j C_j \succeq 0$. Multiplying this equation with λ shows that x fulfills the first condition in the definition of C. But since $\lambda > 0$, the other conditions can clearly also be satisfied with some big enough r. So x belongs to C. Finally, $x = 0$ belongs to C, too. □

Remark 2.2. The additional n conditions in the definition of C avoid problems that could occur in the case $\lambda = 0$. This is the main difference to the approach of Helton and Nie in [3].

Corollary 2.3. *If $S_1, \dots, S_t \subseteq \mathbb{R}^n$ are projections of spectrahedra, then also the convex hull* $\mathrm{conv}(S_1 \cup \cdots \cup S_t)$ *is a projection of a spectrahedron.*

Proof. Consider $\widetilde{S}_i := S_i \times \{1\} \subseteq \mathbb{R}^{n+1}$, and let K_i denote the conic hull of $\widetilde{S}_i$ in $\mathbb{R}^{n+1}$. All $\widetilde{S}_i$ and therefore all K_i are projections of spectrahedra, and thus the Minkowski sum $K := K_1 + \cdots + K_t$ is also such a projection. Now one easily checks

$$\mathrm{conv}(S_1 \cup \cdots \cup S_t) = \{x \in \mathbb{R}^n \mid (x, 1) \in K\},$$

which proves the result. □

Example 2.4. Let $S_1 := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, xy \geq 1\}$ and $S_2 = \{(0, 0)\}$. Both subsets of $\mathbb{R}^2$ are spectrahedra, so the convex hull of their union,

$$\mathrm{conv}(S_1 \cup S_2) = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \cup \{(0, 0)\},$$

is a projection of a spectrahedron.

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