

# DETERMINANTAL REPRESENTATIONS AND THE HERMITE MATRIX

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ABSTRACT. We consider the problem of writing real polynomials as determinants of symmetric linear matrix polynomials. This problem of algebraic geometry, whose roots go back to the nineteenth century, has recently received new attention from the viewpoint of convex optimization. We relate the question to sums of squares decompositions of a certain Hermite matrix. If some power of a polynomial admits a definite determinantal representation, then its Hermite matrix is a sum of squares. Conversely, we show how a determinantal representation can sometimes be constructed from a sums-of-squares decomposition of the Hermite matrix. We finally show that definite determinantal representations always exist, if one allows for denominators.

## INTRODUCTION

A polynomial  $p \in \mathbb{R}[x]$  in  $n$  variables  $x = (x_1, \dots, x_n)$  with  $p(0) = 1$  is called a *real-zero polynomial* if  $p$  has only real zeros along every line through the origin. The terms *hyperbolic* or *real stable* polynomial are also common and mean essentially the same, but usually for homogeneous polynomials. The typical example is a polynomial given by a *definite* (linear symmetric) *determinantal representation*

$$p = \det(I + A_1x_1 + \dots + A_nx_n),$$

where  $A_1, \dots, A_n$  are real symmetric matrices and  $I$  is the identity. A representation of this form is a certificate for being a real-zero polynomial. In other words, the fact that  $p$  is a real-zero polynomial is apparent from the representation. A definite determinantal representation also provides a description of the *rigidly convex region* of  $p$ . This is the closed connected component of the origin in the complement of the zero-set of  $p$ . It is always convex, and given a definite determinantal representation of  $p$ , it coincides with the set of points where the matrix polynomial  $I + A_1x_1 + \dots + A_nx_n$  is positive semidefinite.

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In recent years, real-zero polynomials and their determinantal representations have been studied mostly with a view towards convex optimization, specifically semidefinite and hyperbolic programming. In general, one would like to answer the following questions:

- (1) Under what conditions does a real-zero polynomial have a definite determinantal representation?
- (2) If such a representation exists, what is the minimal matrix dimension and how can the representation be computed effectively?
- (3) If no such representation exists, what other certificates for being a real-zero polynomial are available?

Question (1) is the most immediate and has consequently received the most attention. It ties in with the theory of determinantal hypersurfaces in complex algebraic geometry, whose roots go back to the nineteenth century. Arguably the most important modern results are the Helton-Vinnikov theorem in [7], which gives a positive answer for  $n = 2$ , and Brändén's negative results in higher dimensions in [4]. Since there are various subtle variations of the question, it is not always easy to figure out what is known and what is not; we give a very brief overview after the introduction below.

Question (2), which should be of interest for practical purposes, has not been studied very systematically so far. Even in the case  $n = 2$ , the classical approach of Dixon for constructing determinantal representations is quite algorithmic in nature but hard to carry out in practice (see [5], or [14] for a more modern presentation.)

One approach to Question (3) is to study the determinantal representability of a suitable power or multiple of  $p$  if no representation for  $p$  exists. This is motivated by the Generalized Lax Conjecture, as described below. On the other hand, the real-zero property does not have to be expressed by a determinantal representation. That a polynomial  $p$  in one variable has only real roots is equivalent to its *Hermite matrix* being positive semidefinite. This is a symmetric real matrix associated with  $p$ , which provides one of the classical methods for root counting. To treat the multivariate case, we use a parametrized version of the Hermite matrix with polynomial entries. In a typical sums-of-squares-relaxation approach common in polynomial optimization, we then ask for the parametrized Hermite matrix  $\mathcal{H}(p)$  to be a sum of squares, which means that there exists a matrix  $\mathcal{Q}$  such that  $\mathcal{H}(p) = \mathcal{Q}^T \mathcal{Q}$ . (This is called a sum of squares rather than a square, because  $\mathcal{Q}$  is allowed to be rectangular of any size). This approach has been used before by Henrion in [9] and by Parrilo (unpublished) as a relaxation for the real-zero property, which is exact in the two-dimensional case.

While the Hermite matrix provides a practical way of certifying the real-zero property, having a definite determinantal representation of  $p$  is clearly much more desirable, since it

also yields a description of the rigidly convex region by a linear matrix inequality. And even if one is only interested in the real-zero property, the multivariate Hermite matrix is a fairly unwieldy object compared to the original polynomial, and a sum-of-squares decomposition even more so.

Our main goal is therefore to use a sum-of-squares decomposition of the parametrized Hermite matrix of a polynomial  $p$  to construct, as explicitly as possible, a definite determinantal representation of  $p$ , or at least of some multiple of  $p$ . We first show in Section 1 that a definite determinantal representation of some power of  $p$  of the correct size always yields a sum-of-squares decomposition of  $\mathcal{H}(p)$  (Thm. 1.6). In Section 2, we make an attempt at the converse. This is partly motivated by our experimental finding that the Hermite matrix of the Vámos polynomial, which is the counterexample of Brändén, is not a sum of squares (Example 1.9). Note also that in the case  $n = 2$ , where every real-zero polynomial possesses a definite determinantal representation by the Helton-Vinnikov theorem, the parametrized Hermite matrix can be reduced to the univariate case. It is therefore a sum of squares if and only if it is positive semidefinite, by a result of Jakubovič [10]. Given a decomposition  $\mathcal{H}(p) = \mathcal{Q}^T \mathcal{Q}$ , we show that a definite determinantal representation of a multiple of  $p$  can be found if a certain extension problem for linear maps on free graded modules derived from  $\mathcal{Q}$  has a solution (Thm. 2.5). Given  $\mathcal{Q}$ , the search for such a solution amounts only to solving a system of linear equations. This method can in principle also be applied if the sums of squares decomposition uses denominators. Finally, we show that by allowing a sum-of-squares decomposition with denominators, which exists whenever  $\mathcal{H}(p)$  is positive semidefinite, one can always obtain a determinantal representation with denominators:

**Theorem.** *Let  $p$  be a square-free real-zero polynomial with  $p(0) = 1$ . There exists a symmetric matrix  $\mathcal{M}$  whose entries are real homogeneous rational functions of degree 1 such that  $p = \det(I + \mathcal{M})$ .*

The precise statement is given in Thm. 3.1.

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#### KNOWN RESULTS

- For  $n = 2$ , every real-zero polynomial of degree  $d$  has a real definite determinantal representation of matrix size  $d$  by the Helton-Vinnikov theorem [7].
- For  $n \geq 3$  and  $d$  sufficiently large, a simple count of parameters shows that only an exceptional set of polynomials can have a real determinantal representation of

size  $d$ . The question whether every real-zero polynomial has a definite determinantal representation of *any* size became known as the generalized Lax conjecture.

- The generalized Lax conjecture was disproven by Brändén who even showed the existence of real-zero polynomials  $p$  such that no power  $p^r$  has a determinantal representation of any size [4]. His smallest counterexample, the so-called *Vámos polynomial*, is of degree 4 in 8 variables (see 1.9 below).
- Netzer and Thom [11] have proved that only an exceptional set of polynomials can have a determinantal representation, even if one allows for matrices of arbitrary size. This is true for  $n \geq 3$  and  $d$  sufficiently large, or  $d \geq 4$  and  $n$  sufficiently large. They also show that if  $p$  is a real-zero polynomial of degree 2, then there exists  $r \geq 1$  such that  $p^r$  has a determinantal representation. On the other hand, there exists such  $p$  where one cannot take  $r = 1$ .
- Another result of Helton, McCullough and Vinnikov [8] (see also Quarez [13]) says that every real polynomial has a real symmetric determinantal representation, though not necessarily a definite one. This means that the constant term in the matrix polynomial cannot be chosen to be the identity matrix in their result.
- The most general form of the Lax conjecture says that every rigidly convex set is a spectrahedron. In terms of determinantal representations, this amounts to the following: For every real-zero polynomial  $p$  there exists another real-zero polynomial  $q$  such  $pq$  has a real definite determinantal representation and such that  $q$  is non-negative on the rigidly convex set of  $p$ . This conjecture is still wide open, even without the additional positivity condition on  $q$ . Note that if  $pq$  has a definite determinantal representation, then  $q$  is automatically a real-zero polynomial.

## 1. THE HERMITE MATRIX

In this section we introduce the parametrized Hermite matrix  $\mathcal{H}(p)$  of a polynomial. It is positive semidefinite at each point if and only if  $p$  is a real zero polynomial. If some power of  $p$  admits a determinantal representation of the correct size, then  $\mathcal{H}(p)$  even turns out to be a sum of squares of polynomial matrices.

Let  $p = t^d + p_1 t^{d-1} + \dots + p_{d-1} t + p_d \in \mathbb{R}[t]$  be a monic univariate polynomial of degree  $d$  and let  $\lambda_1, \dots, \lambda_d$  be the complex zeros of  $p$ . Then

$$N_k(p) = \sum_{i=1}^d \lambda_i^k$$

is called the  $k$ -th *Newton sum* of  $p$ . The Newton sums are symmetric functions in the roots, and can thus be expressed as polynomials in the coefficients  $p_i$  of  $p$ . The *Hermite matrix* of  $p$  is the symmetric  $d \times d$  matrix

$$H(p) := (N_{i+j-2}(p))_{i,j=1,\dots,d}.$$

It is a Hankel matrix whose entries are polynomial expressions in the coefficients of  $p$ . Note that  $H(p) = V^T V$ , where  $V$  is the Vandermonde matrix with coefficients  $\lambda_1, \dots, \lambda_d$ .

The following well-known fact goes back to Hermite. For a proof, see for example Theorem 4.59 in Basu, Pollack and Roy [1].

**Theorem 1.1.** *Let  $p \in \mathbb{R}[t]$  be a monic polynomial. The rank of  $H(p)$  is equal to the number of distinct zeros of  $p$  in  $\mathbb{C}$ . The signature of the Hermite matrix  $H(p)$  is equal to the number of distinct real zeros of  $p$ .*

*In particular,  $H(p)$  is positive definite if and only if all zeros of  $p$  are real and distinct, and  $H(p)$  is positive semidefinite if and only if all zeros are real.*  $\square$

Now let  $p \in \mathbb{R}[x]$  be a polynomial of degree  $d$  in  $n$  variables  $x = (x_1, \dots, x_n)$ . The polynomial  $p$  is called a *real-zero polynomial* (with respect to the origin) if  $p(0) = 1$  and for every  $a \in \mathbb{R}^n$ , the univariate polynomial  $p(ta) \in \mathbb{R}[t]$  has only real zeros. We want to express this condition in terms of a Hermite matrix. Write  $p = \sum_{i=0}^d p_i$  with  $p_i$  homogeneous of degree  $i$ , and let  $P(x, t) = \sum_{i=0}^d p_i t^{d-i}$  be the homogenization of  $p$  with respect to an additional variable  $t$ . We consider  $P$  as a monic univariate polynomial in  $t$  and call the Hermite matrix  $H(P)$  the *parametrized Hermite matrix* of  $p$ , denoted  $\mathcal{H}(p)$ . Its entries are polynomials in the homogeneous parts  $p_i$  of  $p$ . The  $(i, j)$ -entry is a homogeneous polynomial in  $x$  of degree  $i + j - 2$ .

**Corollary 1.2.** *A polynomial  $p \in \mathbb{R}[x]$  with  $p(0) = 1$  is a real-zero polynomial if and only if the matrix  $\mathcal{H}(p)(a)$  is positive semidefinite for all  $a \in \mathbb{R}^n$ .*

*Proof.* By Theorem 1.1,  $\mathcal{H}(p)(a)$  is positive semidefinite for  $a \in \mathbb{R}^n$  if and only if the univariate polynomial  $t^d p(a_1 t^{-1}, \dots, a_n t^{-1})$  has only real zeros. Substituting  $t^{-1}$  for  $t$ , we see that this is equivalent to  $p(ta)$  having only real zeros.  $\square$

The following is Proposition 2.1 in Netzer and Thom [11].

**Proposition 1.3.** *Let  $\mathcal{M} = x_1 M_1 + \dots + x_n M_n$  be a symmetric linear matrix polynomial, and let  $p = \det(I - \mathcal{M})$ . Then for each  $a \in \mathbb{R}^n$ , the nonzero eigenvalues of  $\mathcal{M}(a)$  are in one to one correspondence with the zeros of the univariate polynomial  $p(ta)$ , counting multiplicities. The correspondence is given by the rule  $\lambda \mapsto \frac{1}{\lambda}$ .*  $\square$

**Lemma 1.4.** *Let  $p \in \mathbb{R}[x]$  be a real-zero polynomial of degree  $d$ , and assume that  $p^r = \det(I - \mathcal{M})$  is a symmetric determinantal representation of size  $k$ , for some  $r > 0$ . Then*

$$\mathcal{H}(p)_{i,j} = \frac{1}{r} \cdot \left( \operatorname{tr}(\mathcal{M}^{i+j-2}) \right),$$

except possibly for  $(i, j) = (1, 1)$ , where  $\mathcal{H}(p)_{1,1} = d$  and  $\operatorname{tr}(\mathcal{M}^0) = k$ .

*Proof.* For each  $a \in \mathbb{R}^n$ , the trace of  $\mathcal{M}(a)^s$  is the  $s$ -power sum of the nonzero eigenvalues of  $\mathcal{M}(a)$ . These eigenvalues are the inverses of the zeros of  $p(ta)$ , by Proposition 1.3, but each such zero gives rise to  $r$  many eigenvalues. Since the zeros of  $p(ta)$  correspond to the inverses of the zeros of  $t^d p(t^{-1}a)$ , the trace of  $\mathcal{M}(a)^s$  equals the  $s$ -power sum of the zeros of  $t^d p(t^{-1}a)$  multiplied with  $r$ . This proves the claim.  $\square$

**Definition 1.5.** Let  $\mathcal{H} \in \operatorname{Sym}_d(\mathbb{R}[x])$  be a symmetric matrix with polynomial entries.  $\mathcal{H}$  is a *sum of squares*, if there is a  $d' \times d$ -matrix  $\mathcal{Q}$  with polynomial entries, such that  $\mathcal{H} = \mathcal{Q}^T \mathcal{Q}$ . This is equivalent to the existence of  $d'$  many  $d$ -vectors  $\mathcal{Q}_i$  with polynomial entries, such that  $\mathcal{H} = \sum_{i=1}^{d'} \mathcal{Q}_i \mathcal{Q}_i^T$ .

**Theorem 1.6.** *Let  $p \in \mathbb{R}[x]$  be a real-zero polynomial of degree  $d$ . If a power  $p^r$  admits a definite determinantal representation of size  $r \cdot d$ , for some  $r > 0$ , then the parametrized Hermite matrix  $\mathcal{H}(p)$  is a sum of squares.*

*Proof.* Let  $p^r = \det(I - \mathcal{M})$  with  $\mathcal{M}$  of size  $k = rd$ , and denote by  $q_{\ell m}^{(s)}$  the  $(\ell, m)$ -entry of  $\mathcal{M}^s$ . Put  $\mathcal{Q}_{\ell m} = \left( q_{\ell m}^{(0)}, \dots, q_{\ell m}^{(d-1)} \right)^T \in \mathbb{R}[x]^d$ . Then we find

$$\sum_{\ell, m=1}^k \mathcal{Q}_{\ell m} \mathcal{Q}_{\ell m}^T = \left( \sum_{\ell, m=1}^k q_{\ell m}^{(i-1)} q_{\ell m}^{(j-1)} \right)_{i, j=1, \dots, d} = \left( \operatorname{tr}(\mathcal{M}^{i-1} \mathcal{M}^{j-1}) \right)_{i, j=1, \dots, d} = r \mathcal{H}(p),$$

by Lemma 1.4.  $\square$

*Remarks 1.7.* (1) If the determinantal representation of  $p^r$  is of size  $k > rd$ , then  $\mathcal{H}(p)$  becomes a sum of squares after increasing the  $(1, 1)$ -entry from  $d$  to  $k/r$ . This is clear from the above proof.

(2) It was shown in Netzer and Thom [11] that if a polynomial  $p$  admits a definite determinantal representation, then it admits one of size  $dn$ , where  $d$  is the degree of  $p$  and  $n$  is the number of variables. So if *any* power  $p^r$  admits a determinantal representation of *any* size, then  $\mathcal{H}(p)$  is a sum of squares, after increasing the  $(1, 1)$ -entry from  $d$  to  $dn$ . Note that this is independent of  $r$ .

(3) The determinant of  $\mathcal{H}(p)$  is the discriminant of  $t^d p(t^{-1}x)$  in  $t$ . If  $\mathcal{H}(p) = \mathcal{Q}^T \mathcal{Q}$ , it follows from the Cauchy-Binet formula that the determinant of  $\mathcal{H}(p)$  is a sum of squares in

$\mathbb{R}[x]$ . Thus, by the above theorem, the discriminant of  $\det(tI + \mathcal{M})$  in  $t$  is a sum of squares, a fact that has long been known, at least since Borchardt's work from 1846 [3].

(4) The sums-of-squares decomposition of  $\mathcal{H}(p)$  obtained by Thm. 1.6 from a determinantal representation  $p^r = \det(I - \mathcal{M})$  is extremely special. In principle, it is possible to characterize the decompositions of  $\mathcal{H}(p)$  coming from a determinantal representation by a recurrence relation that they must satisfy. But this does not appear to be a promising approach for finding determinantal representations.

*Example 1.8.* It was shown in Netzer and Thom [11] that if  $p$  is quadratic, a high enough power admits a definite determinantal representation of the correct size. Thus  $\mathcal{H}(p)$  is a sum of squares in this case. This can also be shown directly. Write

$$p = x^T A x + b^T x + 1$$

with  $A \in \text{Sym}_n(\mathbb{R})$  and  $b \in \mathbb{R}^n$ . Then  $p$  is a real-zero polynomial if and only if  $bb^T - 4A \succeq 0$ , as is easily checked. We find  $t^2 p(t^{-1}x) = x^T A x + b^T x \cdot t + t^2$ , and so we compute

$$\mathcal{H}(p) = \begin{pmatrix} 2 & -b^T x \\ -b^T x & x^T (bb^T - 2A)x \end{pmatrix}.$$

Write  $bb^T - 4A = \sum_{i=1}^n v_i v_i^T$  as a sum of squares of column vectors  $v_i \in \mathbb{R}^n$ . Set

$$Q = \begin{pmatrix} 1 & -\frac{1}{2}b^T x \\ 0 & \frac{1}{2}v_1^T x \\ \vdots & \vdots \\ 0 & \frac{1}{2}v_n^T x \end{pmatrix}.$$

Then  $\mathcal{H}(p) = 2 \cdot Q^T Q$ . □

*Example 1.9.* We consider Brändén's example from [4]. It is constructed from the Vámos cube as shown in Figure 1. Its set of bases  $\mathcal{B}$  consists of all four element subsets of  $\{1, \dots, 8\}$  that do not lie in one of the five affine hyperplanes. Define

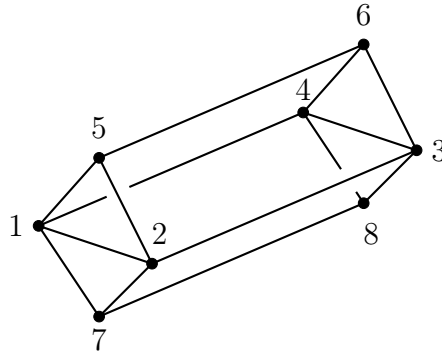
$$q := \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i,$$

a degree four polynomial in  $\mathbb{R}[x_1, \dots, x_8]$ . It contains as its terms the product of any choice of four pairwise different variables, except for the following five:

$$x_1 x_4 x_5 x_6, x_2 x_3 x_5 x_6, x_2 x_3 x_7 x_8, x_1 x_4 x_7 x_8, x_1 x_2 x_3 x_4.$$

Now  $p = q(x_1 + 1, \dots, x_8 + 1)$  turns out to be a real-zero polynomial, of which Brändén has shown that no power has a determinantal representation.

FIGURE 1. The Vámos Cube



We can apply the sums-of-squares-test to the Hermite matrix  $\mathcal{H}(p)$  here. Unfortunately, the matrix is too complicated to do the computations by hand. When using a numerical sums-of-squares-plugin for matlab, such as Yalmip, the result however indicates that  $\mathcal{H}(p)$  is not a sum of squares. In view of Theorem 1.6 this shows again that no power of  $p$  admits a determinantal representation. Note that if some power  $p^r$  has a determinantal representation, then it has one of size  $4r$ . This was proven by Brändén or follows more generally from Netzer and Thom, Theorem 2.7 [11].

Finally, we can apply the sums-of-squares-test also to small perturbations of Brändén’s polynomial. For example,  $p$  can be approximated as closely as desired by real-zero polynomials, which have only simple roots on each line through the origin (in other words, the Hermite matrix is positive definite at each point  $a \neq 0$ ). Such a smoothing procedure is for example describe in Nuij [12]. Still, Yalmip reports that the Hermite matrix is not a sum of squares, if the approximation is close enough. This is exactly what one expects, since the cone of sums of squares of polynomial matrices is closed, and the Hermite matrix depends continuously on the polynomial.

## 2. A GENERAL CONSTRUCTION METHOD

In this section we are interested in the converse of the above result. Namely, can a sums-of-squares decomposition of  $\mathcal{H}(p)$  be used to produce a definite determinantal representation of  $p$  or some multiple? We describe a method to do this, which amounts to only solving a system of linear equations.

Let  $p = 1 + p_1 + \cdots + p_d \in \mathbb{R}[x]$  be a real-zero polynomial of degree  $d$ . Since the matrix  $\mathcal{H}(p)$  is everywhere positive semidefinite, it can be expressed as a sum of squares if one allows denominators in  $\mathbb{R}[x]$ . This generalization of Artin’s solution to Hilbert’s 17th problem was

first proved by Gondard and Ribenboim in [6]. We need to make a slight adjustment to our situation.

**Lemma 2.1.** *There exist a matrix polynomial  $\mathcal{Q} \in \text{Mat}_{k \times d}(\mathbb{R}[x])$ , for some  $k > 0$ , and a homogeneous non-zero polynomial  $q \in \mathbb{R}[x]$  such that*

$$q^2 \mathcal{H}(p) = \mathcal{Q}^T \mathcal{Q}.$$

*Proof.* By the original result of Gondard and Ribenboim [6] there is some non-zero polynomial  $q \in \mathbb{R}[x]$  such that  $q^2 \mathcal{H}(p) = \mathcal{Q}^T \mathcal{Q}$  for some  $\mathcal{Q} \in \text{Mat}_{k \times d}(\mathbb{R}[x])$ . We want to make  $q$  homogeneous.

Write  $q = q_r + q_{r+1} + \dots + q_R$ , where each  $q_i$  is homogeneous of degree  $i$ , and  $q_r \neq 0$ ,  $q_R \neq 0$ . Since the  $i$ -th diagonal entry in  $\mathcal{H}(p)$  is homogeneous of degree  $2(i-1)$ , each entry in the  $i$ -th column of  $\mathcal{Q}$  has homogeneous parts of degree between  $r+i-1$  and  $R+i-1$ . Let  $\mathcal{Q}_{\min}$  be the matrix one obtains from  $\mathcal{Q}$  by choosing only the homogeneous part of degree  $r+i-1$  of each entry in each  $i$ -th column. Put  $\tilde{\mathcal{Q}} = \mathcal{Q} - \mathcal{Q}_{\min}$  and note that all entries in the  $i$ -th column of  $\tilde{\mathcal{Q}}$  have non-zero homogeneous parts only in degrees at least  $r+i$ . We now compute  $q^2 \mathcal{H}(p) = \mathcal{Q}_{\min}^T \mathcal{Q}_{\min} + \mathcal{Q}_{\min}^T \tilde{\mathcal{Q}} + \tilde{\mathcal{Q}}^T \mathcal{Q}_{\min} + \tilde{\mathcal{Q}}^T \tilde{\mathcal{Q}}$ , compare degrees on both sides, and find  $q_r^2 \mathcal{H}(p) = \mathcal{Q}_{\min}^T \mathcal{Q}_{\min}$ , as desired.  $\square$

We will now describe the setup that we are going to use for the rest of this section. We fix a representation of  $q^2 \mathcal{H}(p) = \mathcal{Q}^T \mathcal{Q}$  as in Lemma 2.1. As before, let  $P = t^d \cdot p(t^{-1}x) = t^d + p_1 t^{d-1} + \dots + p_d \in \mathbb{R}[x, t]$ , and consider the free  $\mathbb{R}[x]$ -module

$$A = \mathbb{R}[x, t]/(P) \cong \bigoplus_{i=0}^{d-1} \mathbb{R}[x] \cdot t^i \cong \mathbb{R}[x]^d.$$

Since  $P$  is homogeneous, the standard grading induces a grading on  $A$ . We shift this grading by  $r$ , the degree of  $q$ , and obtain a grading with  $\deg(t^i) = r+i$  for  $i = 0, \dots, d-1$ . This turns  $A$  into a graded  $\mathbb{R}[x]$ -module, where  $\mathbb{R}[x]$  is equipped with the standard grading. Furthermore, we equip  $A$  with a symmetric  $\mathbb{R}[x]$ -bilinear and  $\mathbb{R}[x]$ -valued map  $\langle \cdot, \cdot \rangle_p$  defined by

$$\langle f, g \rangle_p := f^T (q^2 \mathcal{H}(p)) g,$$

for  $f = (f_1, \dots, f_d)^T$  and  $g = (g_1, \dots, g_d)^T$  in  $A$ .

Next, consider the map  $\mathcal{L}_t: A \rightarrow A$  given by multiplication with  $t$ . This is an  $\mathbb{R}[x]$ -linear map which we can compute with respect to our chosen basis:

$$\mathcal{L}_t: (f_1, \dots, f_d)^T \mapsto (-p_d f_d, f_1 - p_{d-1} f_d, \dots, f_{d-1} - p_1 f_d)^T.$$

Note that  $\mathcal{L}_t$  is of degree 1 with respect to the grading, i.e.  $\deg(\mathcal{L}_t(f)) = \deg(f) + 1$ . We identify  $\mathcal{L}_t$  with the matrix that represents it, so that

$$\mathcal{L}_t = \begin{pmatrix} 0 & 0 & 0 & -p_d \\ 1 & 0 & 0 & -p_{d-1} \\ 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & -p_1 \end{pmatrix},$$

which is exactly the *companion matrix* of  $P$ , viewed as a univariate polynomial in  $t$ . It is well known and easy to see that  $P$  is the characteristic polynomial of  $\mathcal{L}_t$ , so that

$$\det(I - \mathcal{L}_t) = p$$

**Lemma 2.2.** *The linear map  $\mathcal{L}_t$  is self-adjoint with respect to  $\langle \cdot, \cdot \rangle_p$ , i.e.*

$$\langle \mathcal{L}_t f, g \rangle_p = \langle f, \mathcal{L}_t g \rangle_p$$

holds for all  $f, g \in A$ .

*Proof.* We may divide by  $q^2$  on both sides and hence assume that  $q = 1$ . It is enough to show  $\langle \mathcal{L}_t e_i, e_j \rangle_p = \langle e_i, \mathcal{L}_t e_j \rangle_p$  for all  $i, j$ , where  $e_i$  is the  $i$ -th unit vector. For  $i, j < d$ , this follows from the fact that  $\mathcal{H}(p)$  is a Hankel matrix. For  $i = j = d$ , it is clear from symmetry. So assume  $j < i = d$ . We find

$$\langle \mathcal{L}_t e_d, e_j \rangle_p = - \sum_{i=1}^d p_{d-i+1} e_i \mathcal{H}(p) e_j = - \sum_{i=1}^d p_{d-i+1} N_{i+j-2},$$

where  $N_k$  is the  $k$ -th Newton sum of  $P$ . On the other hand, we compute  $\langle e_d, \mathcal{L}_t e_j \rangle_p = \langle e_d, e_{j+1} \rangle_p = N_{d+j-1}$ . In conclusion, we have to show that

$$\sum_{i=0}^d p_{d-i} N_{i+j-1} = 0,$$

where we have set  $p_0 = 1$ . This statement is equivalent to  $\sum_{i=0}^d p_i N_{k-i} = 0$ , where  $k = d + j - 1 \geq d$ . This last equation, however, follows immediately from the Newton identity  $k p_k + \sum_{i=0}^{k-1} p_i N_{k-i} = 0$ , where we let  $p_k = 0$  for  $k > d$ .  $\square$

Let  $B = \mathbb{R}[x]^k$ . The  $k \times d$ -matrix  $\mathcal{Q}$  in the decomposition of  $\mathcal{H}(p)$  describes an  $\mathbb{R}[x]$ -linear map  $A = \mathbb{R}[x]^d \rightarrow B$ ,  $f \mapsto \mathcal{Q}f$ . From the degree structure of  $\mathcal{H}(p)$ , we see that each entry in the  $i$ -th column of  $\mathcal{Q}$  is homogeneous of degree  $r + i - 1$ . So  $\mathcal{Q}$  is of degree 0 with respect to the canonical grading on  $B$ .

**Lemma 2.3.**

(1) If  $p$  is square-free, then  $\mathcal{Q}: A \rightarrow B$  is injective.

(2) We have

$$\langle f, g \rangle_p = \langle \mathcal{Q}f, \mathcal{Q}g \rangle$$

for all  $f, g \in A$ . In other words,  $\mathcal{Q}$  is an isometry, taking  $\langle \cdot, \cdot \rangle_p$  to the canonical bilinear form  $\langle \cdot, \cdot \rangle$  on  $B$ .

*Proof.* (2) is immediate from the fact that  $q^2\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$ . (1) If  $\mathcal{Q}f = 0$ , then

$$0 = \langle \mathcal{Q}f, \mathcal{Q}f \rangle = \langle f, f \rangle_p = q^2 \cdot f^T \mathcal{H}(p) f.$$

For each  $a \in \mathbb{R}^n$  for which  $p(ta)$  has only distinct roots, the matrix  $\mathcal{H}(p)(a)$  is positive definite. So  $f(a) = 0$  for generic  $a$ , and thus  $f = 0$ .  $\square$

Time for a brief summary of what we have done so far.

**Setup 2.4.**

- Let  $p \in \mathbb{R}[x]$  be a real-zero polynomial of degree  $d$  with  $p(0) = 1$ , and let  $\mathcal{H}(p)$  be its parametrized Hermite matrix. Fix a decomposition  $q^2\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$ , where  $q$  is homogeneous of degree  $r$  and  $\mathcal{Q}$  is a matrix of size  $k \times d$  with entries in  $\mathbb{R}[x]$ .
- We have equipped the free module  $A = \mathbb{R}[x]^d$  with a particular grading and with a bilinear form  $\langle \cdot, \cdot \rangle_p: A \times A \rightarrow \mathbb{R}[x]$ .
- Let  $B = \mathbb{R}[x]^k$  be equipped with the canonical bilinear form and the canonical grading.
- The map  $\mathcal{Q}: A \rightarrow B$  is an isometry and of degree 0.
- Let  $\mathcal{L}_t$  be the companion matrix of  $t^d p(t^{-1}x)$  with respect to  $t$ , so that

$$\det(I - \mathcal{L}_t) = p.$$

The map  $\mathcal{L}_t: A \rightarrow A$  is self-adjoint with respect to  $\langle \cdot, \cdot \rangle_p$  and of degree 1.

The following is our main result.

**Theorem 2.5.** *Let  $p \in \mathbb{R}[x]$  be a square-free real-zero polynomial of degree  $d$  with  $p(0) = 1$ . Assume that there exists a homogeneous symmetric linear matrix polynomial  $\mathcal{M}$  of size  $k \times k$  such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{R}[x]^d = A & \xrightarrow{\mathcal{Q}} & B = \mathbb{R}[x]^k \\ \downarrow \mathcal{L}_t & & \downarrow \mathcal{M} \\ \mathbb{R}[x]^d = A & \xrightarrow{\mathcal{Q}} & B = \mathbb{R}[x]^k \end{array}$$

*Then  $p$  divides  $\det(I - \mathcal{M})$ .*

*Remark 2.6.* Note that the above described setup exactly means that we can hope for such a linear symmetric  $\mathcal{M}$  to exist. Indeed the "strange" symmetry of  $\mathcal{L}_t$  is transformed into the standard symmetry by  $\mathcal{Q}$ , and the "strange" grading is translated to the standard grading.

*Proof.* For generic  $a \in \mathbb{R}^n$ , the map  $\mathcal{Q}(a)$  is injective by Lemma 2.3. Therefore, all eigenvalues of  $\mathcal{L}_t(a)$  are also eigenvalues of  $\mathcal{M}(a)$ . The eigenvalues of  $\mathcal{L}_t(a)$  are precisely the zeros of  $P(t, a)$ , i.e. the inverses of the zeros of  $p(ta)$ . So  $q = \det(I - \mathcal{M})$  vanishes on the zero set of  $p$ , by Proposition 1.3. Since  $p$  is a square-free real zero polynomial, the ideal  $(p)$  generated by  $p$  in  $\mathbb{R}[x]$  is real-radical (see Bochnak, Coste and Roy [2], Theorem 4.5.1(v)). It follows that  $q$  is contained in  $(p)$ , in other words  $p$  divides  $q$ .  $\square$

*Remark 2.7.* Whether there exists such  $\mathcal{M}$  can be decided by solving a system of linear equations. Indeed, set  $\mathcal{M} = x_1 M_1 + \dots + x_n M_n$ , where the  $M_i$  are symmetric matrices with indeterminate entries. The equation  $\mathcal{M}\mathcal{Q} = \mathcal{Q}\mathcal{L}_t$  of matrix polynomials can be considered entrywise, and comparison of the coefficients in  $x$  gives rise to a system of linear equations in the entries of the  $M_i$ .

*Example 2.8.* Let  $p \in \mathbb{R}[x]$  be quadratic. Write  $p = x^T A x + b^T x + 1$  with  $A \in \text{Sym}_n(\mathbb{R})$  and  $b \in \mathbb{R}^n$ . We have seen in Example 1.8 that  $\mathcal{H}(p)$  admits a sums of squares decomposition if  $p$  is a real-zero polynomial, given by the matrix

$$\mathcal{Q} = \sqrt{2} \cdot \begin{pmatrix} 1 & -\frac{1}{2}b^T x \\ 0 & \frac{1}{2}v_1^T x \\ \vdots & \vdots \\ 0 & \frac{1}{2}v_n^T x \end{pmatrix}$$

if  $bb^T - 4A = \sum_{i=1}^n v_i v_i^T$ . It is now easy to find a homogeneous linear matrix polynomial  $\mathcal{M}$  that makes the diagram in Theorem 2.5 commute, namely we can take

$$\mathcal{M} = \frac{1}{2} \cdot \begin{pmatrix} -b^T x & v_1^T x & \dots & v_n^T x \\ v_1^T x & -b^T x & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ v_n^T x & 0 & 0 & -b^T x \end{pmatrix}.$$

The resulting determinantal representation is

$$\det(I - \mathcal{M}) = \left(1 + \frac{1}{2} \cdot b^T x\right)^{n-1} \cdot p.$$

To give an explicit example, consider  $p = (x_1 + \sqrt{2})^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2$ , which itself does not admit a determinantal representation (by Netzer and Thom [11]). The procedure

just described now gives rise to the linear matrix polynomial

$$\mathcal{M} = \begin{pmatrix} -\sqrt{2}x_1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & -\sqrt{2}x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & -\sqrt{2}x_1 & 0 & 0 & 0 \\ x_3 & 0 & 0 & -\sqrt{2}x_1 & 0 & 0 \\ x_4 & 0 & 0 & 0 & -\sqrt{2}x_1 & 0 \\ x_5 & 0 & 0 & 0 & 0 & -\sqrt{2}x_1 \end{pmatrix}$$

and finally

$$\det(I - \mathcal{M}) = (1 + \sqrt{2}x_1)^4 \cdot p.$$

*Example 2.9.* There are also examples where no suitable  $\mathcal{M}$  exists. We are grateful to Rainer Sinn and Cynthia Vinzant for helping us find this example. Consider the plane cubic  $p = (x_1 - 1)^2(x_1 + 1) - x_2^2$ . One computes

$$\mathcal{H}(p) = \begin{pmatrix} 3 & x_1 & 3x_1^2 + 2x_2^2 \\ x_1 & 3x_1^2 + 2x_2^2 & x_1^3 + 3x_1x_2^2 \\ 3x_1^2 + 2x_2^2 & x_1^3 + 3x_1x_2^2 & 3x_1^4 + 8x_1^2x_2^2 + 2x_2^4 \end{pmatrix} = \mathcal{Q}^T \mathcal{Q},$$

where

$$\mathcal{Q} = \begin{pmatrix} 0 & x_2 & ax_1x_2 \\ 0 & -x_2 & bx_1x_2 \\ \sqrt{2} & \sqrt{2}x_1 & \sqrt{2}(x_1^2 + x_2^2) \\ 1 & -x_1 & x_1^2 \end{pmatrix}$$

and  $a = \frac{1}{2}(\sqrt{7} + 1)$ ,  $b = \frac{1}{2}(\sqrt{7} - 1)$ . The equation  $\mathcal{M}\mathcal{Q} = \mathcal{Q}\mathcal{L}_t$  has 12 entries, each of which gives rise to several linear equations by comparing coefficients in  $x$ . One can check that already the equations obtained from the first two rows of  $\mathcal{M}\mathcal{Q} = \mathcal{Q}\mathcal{L}_t$  are unsolvable.

### 3. RATIONAL REPRESENTATIONS OF DEGREE ONE

There is always a way to make the diagram from the last section commute, if one allows for *rational* linear matrix polynomials. This will lead to rational determinantal representations, as described now.

Let  $p$  be a square-free real-zero polynomial. Since the parametrized Hermite matrix  $\mathcal{H}(p)$  evaluated at a point  $a \in \mathbb{R}^n$  is positive definite for generic  $a$ , the matrix polynomial  $\mathcal{H}(p)$  is invertible over the function field  $\mathbb{R}(x)$ . Recall that the degree of a rational function  $f/g \in \mathbb{R}(x)$  is defined as  $\deg(f) - \deg(g)$ . Furthermore, we say that  $f/g$  is homogeneous if both  $f$  and  $g$  are homogeneous, not necessarily of the same degree. Equivalently,  $f/g$  is homogeneous of degree  $d$  if and only if  $(f/g)(\lambda a) = \lambda^d(f/g)(a)$  holds for all  $a \in \mathbb{R}^n$  with  $g(a) \neq 0$ .

**Theorem 3.1.** *Let  $p$  be a square-free real-zero polynomial. Write  $q^2\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$  with  $q$  homogeneous as in Lemma 2.1 and let*

$$\mathcal{M} := q^{-2}\mathcal{Q}\mathcal{L}_t\mathcal{H}(p)^{-1}\mathcal{Q}^T.$$

*The matrix  $\mathcal{M}$  is symmetric with entries in  $\mathbb{R}(x)$  homogeneous of degree 1, and satisfies*

$$\det(I - \mathcal{M}) = p.$$

*Proof.* Abbreviate  $\mathcal{H}(p)$  by  $\mathcal{H}$  and  $\mathcal{L}_t$  by  $\mathcal{L}$ . By Sylvester's determinant theorem, we have  $\det(I_k - \mathcal{A}\mathcal{B}) = \det(I_d - \mathcal{B}\mathcal{A})$  for any matrix polynomials  $\mathcal{A}$  of size  $k \times d$  and  $\mathcal{B}$  of size  $d \times k$ . In our situation, this yields

$$\det(I_k - \mathcal{M}) = \det(I_k - q^{-2}\mathcal{Q}\mathcal{L}\mathcal{H}^{-1}\mathcal{Q}^T) = \det(I_d - q^{-2}\mathcal{L}\mathcal{H}^{-1}\mathcal{Q}^T\mathcal{Q}) = \det(I_d - \mathcal{L}) = p.$$

We find

$$\mathcal{M}^T = q^{-2}\mathcal{Q}(\mathcal{H}^{-1})^T\mathcal{L}^T\mathcal{Q}^T = q^{-2}\mathcal{Q}\mathcal{L}\mathcal{H}^{-1}\mathcal{Q}^T = \mathcal{M},$$

where we have used  $\mathcal{L}^T\mathcal{H} = \mathcal{H}^T\mathcal{L}$ , which is Lemma 2.2. Thus  $\mathcal{M}$  is symmetric.

Let  $r$  be the degree of  $q$ . By examining the degree structure of  $q^2\mathcal{H}$ , we find

$$\begin{aligned} \mathcal{Q}(\lambda a) &= \mathcal{Q}(a) \cdot \text{diag}(\lambda^r, \lambda^{r+1}, \dots, \lambda^{r+d-1}) \\ \mathcal{H}(\lambda a) &= \text{diag}(\lambda^0, \dots, \lambda^{d-1}) \cdot \mathcal{H}(a) \cdot \text{diag}(\lambda^0, \dots, \lambda^{d-1}) \\ \mathcal{L}(\lambda a) &= \text{diag}(\lambda^d, \dots, \lambda^1) \cdot \mathcal{L}(a) \cdot \text{diag}(\lambda^{-d+1}, \lambda^{-d+2}, \dots, \lambda^0) \end{aligned}$$

for all  $a \in \mathbb{R}^n$  and  $\lambda \neq 0$ . Hence for all  $a \in \mathbb{R}^n$  for which  $\mathcal{H}(a)$  is invertible and  $q(a) \neq 0$ , and all  $\lambda \neq 0$ , we have

$$\begin{aligned} \mathcal{M}(\lambda a) &= \lambda^{-2r}q(a)^{-2}\mathcal{Q}(a) \cdot \text{diag}(\lambda^r, \dots, \lambda^{r+d-1}) \cdot \text{diag}(\lambda^d, \dots, \lambda) \mathcal{L}(a) \cdot \text{diag}(\lambda^{-d+1}, \dots, \lambda^0) \\ &\quad \cdot \text{diag}(\lambda^0, \dots, \lambda^{-d+1}) \cdot \mathcal{H}^{-1}(a) \cdot \text{diag}(\lambda^0, \dots, \lambda^{-d+1}) \cdot \text{diag}(\lambda^r, \dots, \lambda^{r+d-1}) \cdot \mathcal{Q}^T(a) \\ &= \lambda^{-2r}q(a)^{-2}\mathcal{Q}(a)\lambda^{r+d}\mathcal{L}(a)\lambda^{-d+1}\mathcal{H}^{-1}(a)\lambda^r\mathcal{Q}^T(a) \\ &= \lambda \cdot \mathcal{M}(a). \end{aligned}$$

□

*Remark 3.2.* Note that a representation  $p = \det(I - \mathcal{M})$  as in Theorem 3.1 gives an algebraic certificate for  $p$  being a real-zero polynomial. Since  $p(ta) = \det(I - t\mathcal{M}(a))$ , using homogeneity, the zeros of  $p(ta)$  are just the inverses of the eigenvalues of  $\mathcal{M}(a)$ . Since  $\mathcal{M}$  is symmetric, all of these zeros are real. Theorem 3.1 now states that such an algebraic certificate exists for *each* real-zero polynomial  $p$ .

*Example 3.3.* Consider the quadratic polynomial  $p = (x_1 + 1)^2 - x_2^2 - x_3^2 - x_4^2$ . We have

$$\mathcal{H} = \begin{pmatrix} 2 & -2x_1 \\ -2x_1 & 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) \end{pmatrix} = \mathcal{Q}^T \mathcal{Q} \quad \text{with} \quad \mathcal{Q}^T = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ -\sqrt{2}x_1 & \sqrt{2}x_2 & \sqrt{2}x_3 & \sqrt{2}x_4 \end{pmatrix},$$

which results in

$$\mathcal{M} = \begin{pmatrix} -x_1 & x_2 & x_3 & x_4 \\ x_2 & -\frac{x_1 x_2^2}{x_2^2 + x_3^2 + x_4^2} & -\frac{x_1 x_2 x_3}{x_2^2 + x_3^2 + x_4^2} & -\frac{x_1 x_2 x_4}{x_2^2 + x_3^2 + x_4^2} \\ x_3 & -\frac{x_1 x_2 x_3}{x_2^2 + x_3^2 + x_4^2} & -\frac{x_1 x_3^2}{x_2^2 + x_3^2 + x_4^2} & -\frac{x_1 x_3 x_4}{x_2^2 + x_3^2 + x_4^2} \\ x_4 & -\frac{x_1 x_2 x_4}{x_2^2 + x_3^2 + x_4^2} & -\frac{x_1 x_3 x_4}{x_2^2 + x_3^2 + x_4^2} & -\frac{x_1 x_4^2}{x_2^2 + x_3^2 + x_4^2} \end{pmatrix}.$$

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