AN ELEMENTARY PROOF OF SCHMÜDGENS THEOREM ON THE MOMENT PROBLEM OF CLOSED SEMI-ALGEBRAIC SETS

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Abstract. We prove the main result from Schmüdgen's 2003 article [S3] in a more elementary way. The result states, that the question whether a finitely generated preordering has the so called *strong moment property* can be reduced to the same question for preorderings corresponding to fiber sets of bounded polynomials.

1. Introduction

Preorderings in the ring of real polynomials are of great importance in Real Algebra and Real Algebraic Geometry. They correspond to semi-algebraic sets in a similar fashion as ideals correspond to algebraic sets.

Starting with Hilbert's question whether every nonnegative real polynomial in several variables is a sum of squares of real rational functions, many questions arose in this field and many interesting results are known. Different kinds of Positiv- and Nullstellensätze give representations of polynomials with certain properties on semi-algebraic sets. For example, for a compact basic closed semi-algebraic set, every strictly positive polynomial belongs to the corresponding finitely generated preordering (this is the main result in [S2]). If the set is not compact, then this result fails in general. See for example [PD] for a thorough treatment of the field.

Another important question concerns the so called *moment problem*. When is it true that a linear form on the polynomial ring which is nonnegative on a finitely generated preordering is integration with respect to a measure? This leads to the problem of determining the closure of the preordering with respect to the finest locally convex topology on the vector space of polynomials. Indeed, if this closure consists of all polynomials which are nonnegative on the semi-algebraic set, then, by Haviland's Theorem (Theorem 2.1), every linear form nonnegative on the preordering is given by a measure on the corresponding semi-algebraic set. We say that the preordering has the *strong moment property* in this case.

In his 2003 article [S3], Schmüdgen proves a very strong criterion for a preordering to have this property. Indeed, if every preordering from a certain family of preorderings constructed from the original one has the property, then the original preordering has it as well. This is in fact a necessary and sufficient condition. Geometrically, the constructed preorderings correspond to lower dimensional semi-algebraic sets, namely fiber sets of bounded polynomials. As the low-dimensional

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case is often better understood (see for example [KM] and [KMS]), the Schmüdgen criterion can be applied in many cases successfully.

The proof in [S3] uses deep results from functional analysis. A direct integral decomposition of a GNS representation is used to decompose a linear form on the polynomial ring into an integral of linear forms. The applied methods are from [S1] and [D]. In this work we give a more elementary proof of the theorem. We also decompose a linear form into an integral of linear forms. These linear forms are constructed from functions emerging from the Radon-Nikodym Theorem.

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During the work on this article, Prof. Murray Marshall from the University of Saskatchewan found a similar approach to the problem independently.

2. Notations and Preliminaries

For $n \in \mathbb{N}$ consider the real polynomial ring $\mathbb{R}[X_1, \ldots, X_n]$, which we will denote by $\mathbb{R}[X]$ in the following. Later we will also use the polynomial ring $\mathbb{R}[Y] = \mathbb{R}[Y_1, \ldots, Y_s]$ for some $s \in \mathbb{N}$. To distinguish elements, we will use capitals for elements from $\mathbb{R}[Y]$ and lowercase letters for elements from $\mathbb{R}[X]$.

For finitely many polynomials $f_1, \ldots, f_r \in \mathbb{R}[X]$ we write $T = T(f_1, \ldots, f_r)$ for the *preordering* generated by these polynomials, which consists of all finite sums of elements of the form

$$\sigma f_1^{e_1} \cdots f_r^{e_r},$$

where σ is a sum of squares of real polynomials and all $e_i \in \{0, 1\}$. On the geometric side, we have the so called *basic closed semi-algebraic set* $W = W(f_1, \ldots, f_r)$ corresponding to $f_1, \ldots, f_r$, defined as

$$W = \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_r(x) \geq 0\}.$$

We then consider the *saturation* of T, denoted by T^{sat}, consisting of all real polynomials nonnegative on W. Note that T^{sat} depends only on W, whereas T can be different for different sets of generators defining the same set W.

Obviously $T \subseteq T^{\text{sat}}$ and T^{sat} is again a preordering, i.e. T^{sat} is closed under addition, multiplication and contains all squares of polynomials. The relation between T and T^{sat} is an important object of study in Real Algebraic Geometry.

As $\mathbb{R}[X]$ is a real vector space and T is a convex cone, we define

$$T^{\vee} := \{L \colon \mathbb{R}[X] \to \mathbb{R} \text{ linear } \mid L(T) \subseteq [0, \infty) \text{ and } L(1) = 1\},$$

the *algebraic dual cone* of T. As explained in [KM], $L(1) > 0$ holds for all $L \neq 0$ which are nonnegative on T. Therefore the condition $L(1) = 1$ in the definition of $T^{\vee}$ can be ensured for those forms by skaling with a positive real and it does not affect any of the following considerations.

We define the *double dual cone*

$$T^{\vee\vee} := \{p \in \mathbb{R}[X] \mid L(p) \geq 0 \text{ for all } L \in T^{\vee}\}.$$

$T^{\vee\vee}$ is the closure of T with respect to the finest locally convex topology on the vector space $\mathbb{R}[X]$, it is again a preordering and we have

$$T \subseteq T^{\vee\vee} \subseteq T^{\mathrm{sat}},$$

the first inclusion being obvious, the second one coming from the fact that evaluation in $x \in W$ defines an element from $T^{\vee}$. See [KM] and [KMS] for a more thorough discussion.

Now one is interested in the relation between $T^{\vee\vee}$ and T^{sat}. In particular, one wants to know whether $T^{\vee\vee} = T^{\mathrm{sat}}$ holds. Following the notation in [S3], we say that the preordering T has the *strong moment property* (SMP) in this case. The importance of this notion is obvious from the following classical theorem by Haviland [H]:

Theorem 2.1 (Haviland)**.** *Let K be a closed subset of $\mathbb{R}^n$ and $L\colon \mathbb{R}[X] \to \mathbb{R}$ be a linear form. Then L is given by a positive Borel measure ν on K, i.e. $L(f) = \int_K f d\nu$ for all $f \in \mathbb{R}[X]$, if and only if $L(f) \geq 0$ for all f nonnegative on K.*

So if T has (SMP), then every $L \in T^{\vee}$ is nonnegative on T^{sat} and is given, by Haviland's Theorem, by a positive Borel measure on W. As it is much easier to check nonnegativity on T than on T^{sat}, (SMP) is a useful property.

Schmüdgen proved in [S2] that T has (SMP) whenever W is compact, independent of the choice of generators. Indeed, he proved a bit more without bringing it up explicitly. Denote by $\mathcal{B}_W$ the ring of all polynomials which are bounded on W. Then we have, even if W is not compact:

Theorem 2.2 (Schmüdgen)**.** $\mathcal{B}_W \cap T^{\mathrm{sat}} \subseteq T^{\vee\vee}$.

If W is compact, then every polynomial is bounded on W and therefore the Theorem shows $T^{\vee\vee} = T^{\mathrm{sat}}$.

As this result is not stated explicitly in [S2], we give a short sketch of the proof.

Proof. For every $b \in \mathcal{B}_W$ and every $L \in T^{\vee}$ we have

$$L(b^2) \leq \| b \|_\infty^2, \tag{1}$$

where $\| b \|_\infty$ denotes the supremum of b on W. This is shown in the proof of Theorem 1 in [S2] or in the proof of Proposition 2 in [S3]. It uses the Positivstellensatz and the one dimensional Hamburger moment problem.

So if b is nonnegative on W in addition, for any $\delta > \| b \|_\infty$ we have

$$-\delta \leq b - \delta < 0$$

on W, and so

$$L((b-\delta)^2) \leq \delta^2,$$

using (1). This yields $0 \leq L(b^2) \leq 2\delta L(b)$, so $L(b) \geq 0$ since $\delta > 0$. This shows $b \in T^{\vee\vee}$. □

The non-compact case is investigated by Schmüdgen in [S3]. He reduces the question whether T has (SMP) to the same question for preorderings corresponding to lower dimensional semi-algebraic sets, namely fibers of polynomials bounded on W.

3. Main Theorem

For the whole section fix $f_1, \dots, f_r \in \mathbb{R}[X]$ and consider T, W and $\mathcal{B}_W$ as in the section before. In addition, we fix bounded polynomials $h_1, \dots, h_s \in \mathcal{B}_W$. We will often write h in the following if we refer to the s-tuple $h_1, \dots, h_s$. The set

$$h(W) := \{(h_1(x), \dots, h_s(x)) \mid x \in W\}$$

is a bounded subset of $\mathbb{R}^s$, and its closure with respect to the usual topology is denoted by $\overline{h(W)}$.

For any $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbb{R}^s$ we consider the preordering

$$T_\lambda := T + I_\lambda,$$

where $I_\lambda := (h_1 - \lambda_1, \dots, h_s - \lambda_s)$ is the ideal generated by the $h_i - \lambda_i$. T_λ is again a finitely generated preordering, as every polynomial is a difference of two squares of polynomials it is indeed generated by $f_1, \dots, f_r, \pm(h_1 - \lambda_1), \dots, \pm(h_s - \lambda_s)$. This shows that the basic closed semi-algebraic set W_λ corresponding to T_λ is just W intersected with the zero set of I_λ.

Now Schmüdgen proved the remarkable theorem, that if T_λ has (SMP) for all $\lambda \in h(W)$, then so does T ([S3], see Corollary 3.3 below). He uses direct integral decompositions of $*$-represenations of $*$-algebras. Indeed, he proved the following slightly stronger theorem, which we will prove here in an more elementary way:

Theorem 3.1.

$$T^{\vee\vee} = \bigcap_{\lambda \in h(W)} T_\lambda^{\vee\vee}$$

Before we begin with the proof, we explain some constructions that we will use. We fix $L \in T^\vee$ and define linear forms L_p on $\mathbb{R}[Y] = \mathbb{R}[Y_1, \dots, Y_s]$ for any $p \in \mathbb{R}[X]$ in the following way:

$$L_p \colon \mathbb{R}[Y] \to \mathbb{R}; \quad F \mapsto L(F(h)p).$$

Here, $F(h)$ is an abbreviation for $F(h_1, \dots, h_s)$, where $F \in \mathbb{R}[Y]$. Since $p = \left(\frac{p+1}{2}\right)^2 - \left(\frac{p-1}{2}\right)^2$ for any $p \in \mathbb{R}[X]$, we have

$$L_p = L_{(\frac{p+1}{2})^2} - L_{(\frac{p-1}{2})^2}. \tag{2}$$

Now for $p \in \mathbb{R}[X]$, the linear form L_{p^2} fulfills the condition of Haviland's theorem. Indeed, if $F \in \mathbb{R}[Y]$ is nonnegative on $\overline{h(W)}$, then $F(h)$ is nonnegative on W. As all h_i are bounded on W, so is $F(h)$. By Theorem 2.2 we get $F(h) \in T^{\vee\vee}$. Being the closure of T in the finest locally convex topology on $\mathbb{R}[X]$, $T^{\vee\vee}$ is again a preordering, and so $F(h)p^2 \in T^{\vee\vee}$, which yields $0 \le L(F(h)p^2) = L_{p^2}(F)$.

Applying Theorem 2.1, there is a positive Borel measure ν_p on $\overline{h(W)}$ with

$$L_{p^2}(F) = L(F(h)p^2) = \int_{\overline{h(W)}} F d\nu_p \quad \text{for all } F \in \mathbb{R}[Y].$$

These measures are obviously finite and therefore regular. As all considered measures are defined on $\overline{h(W)}$, we omit the subscipts under the integral signs from now on.

The following Lemma is an important ingredient in the proof of Theorem 3.1.

Lemma 3.2. *For all* $p \in \mathbb{R}[X]$,

$$\nu_p \ll \nu_1,$$

that is, every ν_1*-null set is also a* ν_p*-null set.*

Proof. Take $p \in \mathbb{R}[X]$. For any $F \in \mathbb{R}[Y]$ and any $t \in \mathbb{R}$ we have

$$0 \leq L\left((F(h) + tp^2)^2\right) = L\left(F(h)^2\right) + 2tL\left(F(h)p^2\right) + t^2 L\left(p^4\right),$$

which implies

$$4L\left(F(h)p^2\right)^2 - 4L\left(F(h)^2\right) L\left(p^4\right) \leq 0$$

and so

$$L\left(F(h)p^2\right)^2 \leq L\left(F(h)^2\right) L\left(p^4\right). \tag{3}$$

Now let $A \subseteq \overline{h(W)}$ be a ν_1-null set.

We choose a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions on $\overline{h(W)}$, which take on values in the interval $[0,1]$ and which converge pointwise except on a set N which is a ν_1- and a ν_p-null set, to the characteristic function χ_A of A. This can be done using the fact that both measures are regular and by applying Urysohn's Lemma.

Due to the Theorem of Majorized Convergence, we get

$$\int f_n d\nu_1 \longrightarrow \int \chi_A d\nu_1 = 0. \tag{4}$$

Now take a sequence of polynomials $(Q_n)_{n\in\mathbb{N}} \subset \mathbb{R}[Y]$ such that

$$\| Q_n - \sqrt{f_n} \|_\infty \stackrel{n\to\infty}{\longrightarrow} 0, \tag{5}$$

where $\| \cdot \|_\infty$ denotes the supremum on the compact set $\overline{h(W)}$. So

$$\left| \int Q_n^2 d\nu_1 - \int f_n d\nu_1 \right| \leq \| Q_n^2 - f_n \|_\infty \cdot \nu_1\left(\overline{h(W)}\right) \stackrel{n\to\infty}{\longrightarrow} 0. \tag{6}$$

Further we get

$$\begin{aligned} \left(\int Q_n d\nu_p\right)^2 &= L\left(Q_n(h)p^2\right)^2 \\ &\leq L\left(Q_n(h)^2\right) L\left(p^4\right) \\ &= L\left(p^4\right) \int Q_n^2 d\nu_1, \end{aligned}$$

where the inequality uses (3). So (4) combined with (6) yields

$$\int Q_n d\nu_p \stackrel{n\to\infty}{\longrightarrow} 0.$$

As the sequence $(\sqrt{f_n})_{n\in\mathbb{N}}$ obviously also converges pointwise except on N to χ_A, so does the sequence $(Q_n)_{n\in\mathbb{N}}$ by (5). As N is also a ν_p-null set, again by the Theorem of Majorized Convergence,

$$\int Q_n d\nu_p \longrightarrow \int \chi_A d\nu_p = \nu_p(A).$$

So $\nu_p(A) = 0$, what was to be shown. □

Lemma 3.2 allows us to apply the Radon-Nikodym Theorem. For every $p \in \mathbb{R}[X]$ we get a ν_1-integrable function

$$\Phi_p \colon \overline{h(W)} \to [0,\infty),$$

such that

$$L_{p^2}(F) = \int F d\nu_p = \int F \Phi_p d\nu_1 \quad \text{for all } F \in \mathbb{R}[Y].$$

If we define $\theta_p := \Phi_{\frac{p+1}{2}} - \Phi_{\frac{p-1}{2}}$ for $p \in \mathbb{R}[X]$, then all θ_p are ν_1-integrable and

$$L_p(F) = L(F(h)p) = \int F\theta_p d\nu_1 \tag{7}$$

holds for every $F \in \mathbb{R}[Y]$ by (2).

Before we start with the proof of Theorem 3.1, we look at some of the properties of the θ_p.

For $g_1, g_2 \in \mathbb{R}[X]$, $t_1, t_2 \in \mathbb{R}$ and any $F \in \mathbb{R}[Y]$ we have

$$\begin{aligned} \int F\theta_{t_1g_1+t_2g_2} d\nu_1 &= L\left(F(h)(t_1g_1+t_2g_2)\right) \\ &= L\left(F(h)t_1g_1\right) + L\left(F(h)t_2g_2\right) \\ &= \int Ft_1\theta_{g_1} d\nu_1 + \int Ft_2\theta_{g_2} d\nu_1 \\ &= \int F\left(t_1\theta_{g_1} + t_2\theta_{g_2}\right) d\nu_1. \end{aligned}$$

By Lemma 4.1 from the Appendix, this implies

$$\theta_{t_1g_1+t_2g_2} = t_1\theta_{g_1} + t_2\theta_{g_2} \tag{8}$$

except on a ν_1-null set which depends on g_1, g_2, t_1, t_2.

Further, for any $Q \in \mathbb{R}[Y]$ and $t \in T$,

$$0 \leq L\left(Q(h)^2 t\right) = \int Q^2 \theta_t d\nu_1$$

holds, so again by Lemma 4.1,

$$\theta_t \geq 0, \tag{9}$$

except on a ν_1-null set depending on t.

Last, for $p \in \mathbb{R}[X], Q \in \mathbb{R}[Y]$ and any $F \in \mathbb{R}[Y]$, we have

$$\begin{aligned} \int F\theta_{Q(h)p} d\nu_1 &= L\left(F(h)Q(h)p\right) \\ &= \int FQ\theta_p d\nu_1. \end{aligned}$$

So by Lemma 4.1,

$$\theta_{Q(h)p} = Q \cdot \theta_p, \tag{10}$$

except on a ν_1-null set depending on p and Q.

With all these constructions in mind, we prove the main theorem.

Proof of Theorem 3.1. One of the inclusions is obvious, for the other one fix $f \in \bigcap_{\lambda \in h(W)} T_\lambda^{\vee\vee}$. Take $L \in T^\vee$. We have to show $L(f) \geq 0$.

Define

$$A := \mathbb{Q}[X_1, \dots, X_n, f_1, \dots, f_r, h_1, \dots, h_s, f].$$

(Remember that the f_i where the polynomials defining W and T, whereas the h_i where the bounded polynomials we fixed.) Any $a \in A$ can be written as a real polynomial in the form

$$a = \sum a_\alpha X^\alpha,$$

where the a_α a real, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and the sum is finite. We will use this representation for a in the following.

From L we construct all the functions θ_p introduced above. Using (8), (9) and (10), we find a single ν_1-null set $N \subseteq \overline{h(W)}$, such that the following conditions are true for all $\lambda \notin N$:

$$\text{For all } a \in A \text{ we have } \theta_a(\lambda) = \sum a_\alpha \theta_{X^\alpha}(\lambda) \tag{11}$$

$$\text{For all } t \in A \cap T \text{ we have } \theta_t(\lambda) \geq 0 \tag{12}$$

$$\text{For all } p \in \mathbb{Q}[X], Q \in \mathbb{Q}[Y] \text{ we have } \theta_{Q(h)p}(\lambda) = Q(\lambda)\theta_p(\lambda) \tag{13}$$

As A is countable, we can indeed insure all this with one single null set N.

Now for $\lambda \in \overline{h(W)} \setminus N$ we define a linear form L_λ on $\mathbb{R}[X]$ by

$$L_\lambda\left(X^\alpha\right) := \theta_{X^\alpha}(\lambda) \quad \text{for all } \alpha \in \mathbb{N}^n.$$

These linear forms fulfill

$$L_\lambda(a) = \theta_a(\lambda) \tag{14}$$

for all $a \in A$ by (11). So for any $t \in A \cap T$ we have

$$L_\lambda(t) = \theta_t(\lambda) \geq 0,$$

using (12). As we can approximate every element from T coefficientwise by a sequence of elements from $A \cap T$ of bounded degree,

$$L_\lambda(T) \subseteq [0, \infty) \tag{15}$$

holds.

Using (14) and (13), for any $p \in \mathbb{Q}[X]$, $\xi \in \mathbb{Q}$ and $i \in \{1, \dots, s\}$

$$\begin{aligned} L_\lambda\left((h_i - \xi)p\right) &= \theta_{(h_i - \xi)p}(\lambda) \\ &= (Y_i - \xi)(\lambda) \cdot \theta_p(\lambda) \\ &= (\lambda_i - \xi)\theta_p(\lambda). \end{aligned}$$

By approximating first λ_i by elements ξ from $\mathbb{Q}$ and then arbitrary $p \in \mathbb{R}[X]$ by elements from $\mathbb{Q}[X]$ of bounded degree, this shows

$$L_\lambda(I_\lambda) = \{0\}.$$

Combined with (15) this implies

$$L_\lambda(T_\lambda) \subseteq [0, \infty). \tag{16}$$

For $\lambda \in N$ we define $L_\lambda \equiv 0$. As f is in A, we get

$$\int_{\overline{h(W)}} L_\lambda(f) d\nu_1(\lambda) = \int_{\overline{h(W)}} \theta_f(\lambda) d\nu_1(\lambda) = L_f(1) = L(f), \tag{17}$$

using (14) and $\nu_1(N) = 0$ for the first equality.

Now the rest is straighforward. For any $\lambda \in N$ we have $L_\lambda(f) = 0$. For $\lambda \in h(W) \setminus N$ we get $L_\lambda(f) \geq 0$ from the assumption, as L_λ is either the zero form or an element in $T_\lambda^\vee$ by (16), maybe after scaling with a positive real. Finally, if $\lambda \in \overline{h(W)} \setminus h(W)$, this means that the semi-algebraic set W_λ corresponding to T_λ is empty. It is a corollary of the Positivstellensatz that T_λ equals $\mathbb{R}[X]$ in this case (see for example [PD], Remark 4.2.13). So again by (16), $L_\lambda \equiv 0$. Thus

$$L_\lambda(f) \geq 0$$

holds for all $\lambda \in \overline{h(W)}$, so (17) yields $L(f) \geq 0$. This completes the proof. □

The most important corollary of Theorem 3.1 is the following, which is Theorem 1 in [S3].

Corollary 3.3. *If T_λ has (SMP) for all $\lambda \in h(W)$, then so does T.*

Proof. Obviously $T^{\text{sat}} = \bigcap_{\lambda\in h(W)} T_\lambda^{\text{sat}}$. So by Theorem 3.1,

$$T^{\vee\vee} = \bigcap_{\lambda\in h(W)} T_\lambda^{\vee\vee} = \bigcap_{\lambda\in h(W)} T_\lambda^{\text{sat}} = T^{\text{sat}}.$$

□

The conditions in Corollary 3.3 are also necessary. If T has (SMP) then so do all the T_λ. Indeed, for *any* ideal I, $T + I$ has (SMP). This is Proposition 4.8 in [SCH].

There are are lot of interesting examples in [S3] illustrating the use of Corollary 3.3. So we just add one more.

Example 3.4. As in [KMS], we consider the preordering

$$T = T(X, Y, 1 - XY) \subseteq \mathbb{R}[X, Y].$$

The polynomial XY is bounded on W and all the T_λ have (SMP), and so does T.

If we change the set of defining polynomials suitably, e.g. consider

$$T' := T(X, Y^n, 1 - XY)$$

for some odd $n \geq 3$, the semi-algebraic set stays the same. But T' does not have (SMP) any more. Indeed, Y is nonnegative on W. If Y was in $T'^{\vee\vee}$ then by evaluating in $X = 0$, we get $Y \in T(Y^n)^{\vee\vee} \subseteq \mathbb{R}[Y]$. By [KM], Theorem 3.5, $T(Y^n)^{\vee\vee} = T(Y^n)$ and one checks that this preordering does not contain Y, a contadiction.

On the other hand, if we change the defining polynomials to

$$\tilde{T} := T(X, Y, (1 - XY)^n)$$

for some odd $n \geq 3$, $\tilde{T}$ has (SMP). This can again be obtained by Corollary 3.3, just as for the first preordering T.

Corollary 3.3 yields a general easy result about changing the defining polynomials. If T, defined by $f_1, \ldots, f_r$, has (SMP) and for example f_1 is bounded on W, then replacing f_1 by some odd power of f_1 preserves (SMP). Indeed $T + (f_1 - \lambda)$ has (SMP) for all $\lambda \in f_1(W)$ by the mentioned result in [SCH]. As all those λ are nonnegative, $T(f_1^n, f_2, \ldots, f_r) + (f_1 - \lambda)$ contains f_1 and is therefore equal to $T + (f_1 - \lambda)$. So Corollary 3.3 shows that also $T(f_1^n, f_2, \ldots, f_r)$ has (SMP).

4. Appendix

Lemma 4.1. *Let K be a compact subset of $\mathbb{R}^s$ and ν be a regular Borel measure on K. Let $h\colon K \to \mathbb{R}$ be ν-integrable and suppose*

$$\int_K F^2 h d\nu \geq 0 \quad \text{for all } F \in \mathbb{R}[Y_1, \ldots, Y_s].$$

Then there is a ν-null set $N \subseteq K$, such that $h \geq 0$ except on N.

Proof. For $n = 1, 2, \ldots$ define

$$A_n := \left\{ x \in K \mid -\frac{1}{n-1} < h(x) \leq -\frac{1}{n} \right\},$$

where $\frac{1}{0} := \infty$. Suppose $\nu(A_n) > 0$ for some n and let χ be the characteristic function of A_n. Then

$$\int \chi h d\nu \leq -\frac{1}{n}\nu(A_n) < 0.$$

Choose a sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$ on K which take on values in $[0,1]$, such that $\int f_n h d\nu \overset{n\to\infty}{\longrightarrow} \int \chi h d\nu$. This can be done, using the regularity of ν as well as Urysohn's Lemma and the Theorem of Majorized Convergence. Now take $P_n \in \mathbb{R}[Y]$ such that $\| P_n^2 - f_n \|_{\infty,K} \overset{n\to\infty}{\longrightarrow} 0$. As

$$\left| \int P_n^2 h d\nu - \int f_n h d\nu \right| \leq \| P_n^2 - f_n \|_{\infty,K} \cdot \int |h| d\nu,$$

we get $\int P_n^2 h d\nu \longrightarrow \int \chi h d\nu < 0$, a contradiction.

So $\nu(A_n) = 0$ for all n, so $\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 0$, which proves the result. □

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