Positive Polynomials, Sums of Squares and the Moment Problem

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Introduction

Real polynomials $f_1, \ldots, f_t$ in n variables $X_1, \ldots, X_n$ define a subset $\mathcal{S}$ of $\mathbb{R}^n$ by

$$\mathcal{S} := \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_t(x) \geq 0\}.$$

One is interested in finding an algebraic characterization of

$$\text{Pos}(\mathcal{S}) = \{f \in \mathbb{R}[X_1, \ldots, X_n] \mid f \geq 0 \text{ on } \mathcal{S}\},$$

the set of all polynomials that are nonnegative on $\mathcal{S}$. Obviously, any sum of squares of polynomials and all the f_i are nonnegative on $\mathcal{S}$, and adding and multiplying nonnegative polynomials gives a nonnegative polynomial again. The set of all polynomials we obtain in this way is called the *preordering* generated by the f_i. An important question is how this preordering relates to $\text{Pos}(\mathcal{S})$. In general, the preordering is smaller. For example, already in the case $n = 2, t = 1$ and $f_1 = 1$ (so $\mathcal{S} = \mathbb{R}^2$) equality fails; this is the fact that not every globally nonnegative polynomial in two variables is a sum of squares of polynomials.

However, in the case that $\mathcal{S}$ is compact, the preordering generated by the f_i at least contains every polynomial which is strictly positive on $\mathcal{S}$. This is Schmüdgen's famous theorem from 1991.

On the other hand, Scheiderer has proved that for any set $\mathcal{S}$, if its dimension is at least three, then there are always nonnegative polynomials that do not belong to the preordering.

In the two-dimensional case, there are preorderings that contain all nonnegative polynomials. This follows from different local-global principles developed by Scheiderer. Most of them only involve compact semi-algebraic sets. In the non-compact two-dimensional case, there is a surprising lack of examples of preorderings that contain all nonnegative polynomials. Only recently, Marshall was able to show that a strip in the plane can be described by a preordering that contains all nonnegative polynomials.

Another question arising in this context concerns the moment problem. One wants to characterize the linear functionals on $\mathbb{R}[X_1, \ldots, X_n]$ that are integration on $\mathcal{S}$. By Haviland's Theorem, these are precisely the functionals that are nonnegative on $\mathrm{Pos}(\mathcal{S})$. Now one can ask if being nonnegative on the preordering is sufficient for a functional to be integration. That translates to the problem of determining the closure of the preordering with respect to a suitable topology on the polynomial ring (the finest locally convex topology). If this closure equals $\mathrm{Pos}(\mathcal{S})$, then indeed all functionals nonnegative on the preordering have an integral representation. The moment problem for $\mathcal{S}$ is then said to be solved by the preordering. Schmüdgen's Theorem from 2003 gives a useful method to determine the closure of a preordering. One can often reduce the question to fibre preorderings that describe lower dimensional sets. A generalized and more elementary proof of his result is the content of Chapter 2 of this work.

Instead of looking at the closure of a preordering, one can consider the sequential closure only. A polynomial f belongs to it if and only if $f + \varepsilon q$ belongs to the preordering for a fixed polynomial q and all $\varepsilon > 0$. The notion was introduced and developed by Kuhlmann, Marshall and Schwartz. It helps dealing with the moment problem and allows to represent certain nonnegative polynomials in terms of the preordering. For different sorts of sets $\mathcal{S}$, it is known that every nonnegative polynomial belongs to the sequential closure of the preordering.

It was an open problem whether the closure and the sequential closure of a preordering always coincide, or at least in the case that the closure equals $\mathrm{Pos}(\mathcal{S})$. We solve this problem to the negative in Chapter 4. We also provide a theorem that allows, in the spirit of Schmüdgen's Fibre Theorem, to use a dimension reduction when dealing with the sequential closure.

A large class of preorderings that almost never solve the moment problem is given by *stable* preorderings. The notion stems from [PSc]. Roughly spoken, a stable preordering admits a bound on the degree of the sum of squares used in the representation of a polynomial. Stable preorderings are often closed and do not solve the moment problem. So stability is a useful property when examining preorderings. In Chapter 3 we develop a method to prove stability for preorderings under certain conditions. The conditions are either of geometric nature (as for example also in [PSc]) or of more combinatorial one. This in particular allows applications where the geometric criterion of [PSc] does not apply. In addition, all of the conditions, also the geometric ones, are very easy to check.

We conclude the work with a collection of explicit examples in Chapter 5.

Of course one can ask all the above questions in the more general context of a finitely generated $\mathbb{R}$-algebra instead of the polynomial ring. Even for arbitrary commutative $\mathbb{R}$-algebras this makes sense. And instead of the preordering generated by the elements f_i, one can consider the quadratic module defined by them. This quadratic module is much smaller in general. We try to prove all results in this more general context, whenever possible.

1 Preliminaries

In this chapter we introduce basic terminology to set the stage for the rest of this work. We begin with real vector spaces and proceed to commutative $\mathbb{R}$-algebras. Most of the results are commonly known and we only give some proofs for completeness.
We agree on $\mathbb{N} = \{0, 1, \ldots\}$ for the whole work. Also, if we use the word *positive*, we always mean *strictly* positive - we will say *nonnegative* if we allow for zero.

1.1 Real Vector Spaces

For the whole section we refer to [Schae] for exact and detailed proofs.
Let E be a real vector space. A *vector space topology* is a topology on E making the addition of vectors

$$E \times E \to E$$

and the scalar multiplication

$$\mathbb{R} \times E \to E$$

continuous. Such a topology is already uniquely defined by its neighborhoods of zero. A system $\mathcal{U}$ of subsets of E is a neighborhood base of zero of a vector space topology, if it fulfills the following conditions:

(i) For all $U, V \in \mathcal{U}$ there is some $W \in \mathcal{U}$ with $W \subseteq U \cap V$

(ii) For every $U \in \mathcal{U}$ there is some $V \in \mathcal{U}$ with $V + V \subseteq U$

(iii) All sets in $\mathcal{U}$ are *absorbing* and *circled*

Here, a set $U \subseteq E$ is called *absorbing*, if for every $x \in E$ there exists $\lambda_0 \geq 0$, such that $x \in \lambda U$ for all $\lambda \geq \lambda_0$; it is called *circled* if $\lambda U \subseteq U$ whenever $|\lambda| \leq 1$. Each vector space topology has a neighborhood base $\mathcal{U}$ of zero, fulfilling (i)-(iii).

A vector space topology is called *locally convex*, if it has a zero neighborhood base of *convex* sets, fulfilling (i)-(iii). Alternatively, if the topology is defined by a family of semi-norms, i.e. if the topology is the coarsest vector space topology making a given family of semi-norms continuous.

The collection of *all* convex, absorbing and circled subsets of E obviously fulfills the above conditions (i)-(iii) and is therefore a zero neighborhood base of a locally convex topology, called the *finest locally convex topology* on E. Alternatively, it is the coarsest vector space topology making all semi-norms on E continuous. Each linear functional on E is then continuous. Even more, any linear mapping to any vector space endowed with a locally convex topology is continuous. Every subspace of E is closed, and every finite dimensional subspace of E inherits the canonical topology. E is Hausdorff. From now on let E carry the finest locally convex topology.

Fix an algebraic basis $(e_i)_{i \in I}$ of E, i.e. I is a suitable index set and $E = \bigoplus_{i \in I} \mathbb{R} e_i$. For a family $\varepsilon = (\varepsilon_i)_{i \in I}$ of positive real numbers define the set

$$U_\varepsilon := \left\{ x \in E \mid x = \sum_{\text{finite}} \lambda_i e_i,\ |\lambda_i| \leq \varepsilon_i \right\}.$$

Each such set U_ε is convex, absorbing and circled, and therefore a neighborhood of zero. The following result is Exercise 7(b) in [Schae], Chapter II:

Lemma 1.1. *A sequence in E converges if and only if it lies in a finite dimensional subspace and converges there.*

Proof. The "if"-part if clear. Now let $(x_j)_{j \in \mathbb{N}}$ be a sequence, converging to zero without loss of generality. We have to show that it lies in a finite dimensional subspace of E. Write

$$x_j = \sum_{i \in I_j} \lambda_i^{(j)} e_i,$$

where I_j is a finite subset of I and all $\lambda_i^{(j)} \neq 0$. Suppose $\bigcup_{j\in\mathbb{N}} I_j$ is *not* finite. By induction on $j \in \mathbb{N}$ define for $i \in I_j \setminus \bigcup_{k<j} I_k$

$$\varepsilon_i := \frac{|\lambda_i^{(j)}|}{2}.$$

Complete these numbers to a positive family $\varepsilon = (\varepsilon_i)_{i\in I}$. As the union of all the I_j is not finite, there are arbitrary big indices j such that x_j does not belong to U_ε. This is a contradiction. □

So the *sequential closure* of some $B \subseteq E$ consists of all finite dimensional closures. We will denote this sequential closure by $B^\ddagger$, i.e. we have

$$B^\ddagger = \bigcup_W \overline{B \cap W},$$

where the union runs over all finite dimensional subspaces W of E. We denote by $\overline{B}$ the closure of B and observe

$$B \subseteq B^\ddagger \subseteq \overline{B}.$$

Now assume that E is *countable dimensional*, i.e. we can choose $I = \mathbb{N}$. In this case, the family of all U_ε defined above is a *basis* for the neighborhoods of zero. Indeed, any convex, absorbing and circled set U in E contains some U_ε. To see this, define

$$p(x) := \inf \{\lambda \geq 0 \mid x \in \lambda U\},$$

the so called *gauge* or *Minkowski functional* of U. It is a semi-norm on E. Now choose a positive sequence $\varepsilon = (\varepsilon_i)_{i\in\mathbb{N}}$ such that $\varepsilon_i \cdot p(e_i) < \left(\frac{1}{2}\right)^{i+1}$. If $x \in U_\varepsilon$, say $x = \sum \lambda_i e_i$, $|\lambda_i| \leq \varepsilon_i$ and the sum is finite, then

$$p(x) \leq \sum |\lambda_i| p(e_i) \leq \sum \varepsilon_i p(e_i) < 1.$$

This shows $x \in U$.

The next result says, that the finest locally convex topology on a countable dimensional space coincides with the topology of finitely open sets. A weaker version of it is Exercise 7(c) in [Schae], and it can also be found in [Bi].

Proposition 1.2. *If E has countable dimension, then a set in E is closed if and only if its intersection with every finite dimensional subspace is closed.*

Proof. Denote the set by B. The "only if"-part is clear. For the "if"-part define $W_n = \bigoplus_{i=0}^n \mathbb{R}e_i$. Then the increasing sequence of the finite dimensional subspaces W_n exhausts E. Now suppose $x \notin B$. Then $x \in W_n \setminus (B \cap W_n)$ for big enough n. As $B \cap W_n$ is closed in W_n, we can find a cube

$$C = [-\varepsilon_0, \varepsilon_0] \times \cdots \times [-\varepsilon_n, \varepsilon_n] \subseteq W_n,$$

all $\varepsilon_i > 0$, such that $x + C$ does not meet $B \cap W_n$. Then $(x+C)\times\{0\}$ does not meet the closed set $B\cap W_{n+1}$ in W_{n+1}. Due to compactness we find $\varepsilon_{n+1} > 0$, such that $x+(C \times [-\varepsilon_{n+1}, \varepsilon_{n+1}])$ does not meet $B \cap W_{n+1}$ in W_{n+1}. So inductively we find a positive sequence $\varepsilon = (\varepsilon_i)_{i\in\mathbb{N}}$ such that $(x + U_\varepsilon) \cap B = \emptyset$. So B is closed. □

This last result is *not* true without the assumption on the dimension, see [Bi] or [CMN] for examples.

Proposition 1.2 seems to suggest, that the closure of a set in E equals the union over the closures in all finite dimensional subspaces, i.e. the sequential closure. However, this is not true, as we will see later. But by Lemma 1.1 and Proposition 1.2, a set B in E is closed if and only if $B = B^\ddagger$ holds. So the (transfinite) sequence of iterated sequential closures of B terminates exactly at $\overline{B}$ (all in the case that E has countable dimension). It is an interesting question to determine *when* this sequence terminates. We will address this question for convex cones later in more detail.

We now drop the assumption on the dimension, i.e. we consider arbitrary real vector spaces E again. Let C be a a *convex cone* (or just *cone* for short) in E, i.e. a subset of E which is convex and closed under multiplication with nonnegative reals.

Alternatively, C is closed under addition and under multiplication with nonnegative reals. For such a convex cone let $C^\vee$ be the *dual cone*, i.e. the set of all linear functionals on E which are nonnegative on C. Then

$$C^{\vee\vee} := \{x \in E \mid L(x) \geq 0 \text{ for all } L \in C^\vee\}$$

is called the *double dual cone* of C. The following result can be found in [CMN]:

Proposition 1.3. *For convex cones C in E we have*

$$\overline{C} = C^{\vee\vee}$$

and

$$C^\ddagger = \{x \in E \mid \exists q \in E \ \forall \varepsilon > 0 \ x + \varepsilon q \in C\}.$$

Proof. $\overline{C} \subseteq C^{\vee\vee}$ holds, as all functionals are continuous. The other inclusion comes from the Hahn-Banach Theorem, see [Schae], Chapter II, Section 9.

If $x + \varepsilon q \in C$ for all $\varepsilon > 0$, then x lies in the sequential closure of C. So suppose conversely there is a sequence $(x_i)_i$ of elements from C, converging to x. By Lemma 1.1, the elements x_i span a finite dimensional subspace of E, so let $y_1, \ldots, y_N \in C$ be an algebraic basis of this subspace. Write $x_i = \sum_{j=1}^N r_{ij} y_j$ and $x = \sum_{j=1}^N r_j y_j$ with real numbers r_j, r_{ij}. Then $\lim_{i\to\infty} r_{ij} = r_j$. Define $q = \sum_{j=1}^N y_j$. For any $\varepsilon > 0$ we have $r_{ij} \leq r_j + \varepsilon$ for large enough i and all j. So

$$\begin{aligned} x + \varepsilon q &= \sum_j (r_j + \varepsilon) y_j \\ &= \sum_j r_{ij} y_j + \sum_j (r_j + \varepsilon - r_{ij}) y_j \\ &= x_i + \sum_j (r_j + \varepsilon - r_{ij}) y_j \in C. \end{aligned}$$

□

The proof shows that the element q can always be picked from C. It is also clear that $\overline{C}$ as well as $C^{\ddagger}$ are again convex cones. We will later be interested in quadratic modules or preorderings in $\mathbb{R}$-algebras. These are in particular convex cones, and we consider their closures and sequential closures. It turns out that these closures are again quadratic modules or preorderings.

1.2 $\mathbb{R}$-Algebras

Most of the following notions and results are standard knowledge from Real Algebra and Real Algebraic Geometry. We refer to [BCR, M1, PD] for details and omitted proofs.

When we talk about an $\mathbb{R}$-algebra (or simply an algebra) A in this work, we *always* mean a commutative $\mathbb{R}$-algebra with unit 1. Morphisms between algebras are always assumed to be unitary, i.e. to map 1 to 1. We will always equip A with the finest locally convex topology, and the topological notions and result from the previous section apply. In case A is finitely generated as an $\mathbb{R}$-algebra, it is countable dimensional as a vector space. Indeed, if $x_1, \ldots, x_n$ generate A as an $\mathbb{R}$-algebra, then the countably many elements $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, with $\alpha \in \mathbb{N}^n$, generate A as a vector space. However, A can be countable dimensional without being finitely generated. In general we will not assume that A is finitely generated or countable dimensional, neither that A is reduced or even a domain. We will state additional assumptions on A whenever needed. The following easy result will be used throughout this work.

Lemma 1.4. *Let $L\colon A \to \mathbb{R}$ be a linear functional such that $L(a^2) \geq 0$ for all $a \in A$. Then for all $a, b \in A$*

$$L(ab)^2 \leq L(a^2)L(b^2).$$

In particular, if $L(1) = 0$, then $L \equiv 0$.

Proof. For all $r \in \mathbb{R}$ we have

$$0 \leq L\left((a+rb)^2\right) = L(a^2) + 2rL(ab) + r^2L(b^2).$$

From this we get

$$4L(ab)^2 - 4L(a^2)L(b^2) \leq 0,$$

which implies $L(ab)^2 \leq L(a^2)L(b^2)$.
If $L(1) = 0$ then for all a

$$L(a)^2 \leq L(1)L(a^2) = 0,$$

so $L(a) = 0$.

□

To A there corresponds a variety $\mathcal{V}_A$ (or just $\mathcal{V}$), defined as the set of all $\mathbb{R}$-algebra homomorphisms from A to $\mathbb{C}$. Its set of real points, denoted by $\mathcal{V}_A(\mathbb{R})$ (or just $\mathcal{V}(\mathbb{R})$), are the homomorphisms to $\mathbb{R}$. We will mostly be using the set of real points $\mathcal{V}(\mathbb{R})$.
Elements from A can be used as functions on $\mathcal{V}$ and $\mathcal{V}(\mathbb{R})$ by

$$\hat{a}(\alpha) := \alpha(a),$$

for $a \in A$ and $\alpha \in \mathcal{V}$. We equip $\mathcal{V}(\mathbb{R})$ with with the coarsest topology making all the elements from A continuous functions on $\mathcal{V}(\mathbb{R})$. We call this topology the *strong topology.* We have an algebra homomorphism

$$\hat{}\colon A \to C(\mathcal{V}(\mathbb{R}), \mathbb{R}); \quad a \mapsto \hat{a}.$$

The elements $\hat{a}$ separate points of $\mathcal{V}(\mathbb{R})$, i.e. for two distinct points in $\mathcal{V}(\mathbb{R})$ there is some $a \in A$, such that $\hat{a}$ takes different values in these two points. In particular, $\mathcal{V}(\mathbb{R})$ is hausdorff.
If A is finitely generated as an $\mathbb{R}$-algebra, and one chooses a set of generators $x_1, \ldots, x_n$, the variety can be embedded into affine space. Indeed, look at the algebra homomorphism

$$\mathbb{R}[X_1, \ldots, X_n] \twoheadrightarrow A$$

from the polynomial ring in n variables to A, which sends X_i to x_i. Denote its kernel by I. I is an ideal and

$$\mathbb{R}[X_1, \ldots, X_n]/I \cong A.$$

$\mathcal{V}$ can then be identified with the zero set of I in $\mathbb{C}^n$ by

$$\alpha \mapsto (\alpha(x_1), \ldots, \alpha(x_n)) .$$

$\mathcal{V}(\mathbb{R})$ is identified with the zero set of I in $\mathbb{R}^n$ under this mapping. The use of elements from A as polynomial functions on these embedded varieties coincides with the above defined use as functions. The topology on $\mathcal{V}(\mathbb{R})$ is exactly the topology inherited from the canonical topology on $\mathbb{R}^n$. However, we will often not choose generators of A and view elements from $\mathcal{V}$ as algebra homomorphisms instead.

Sometimes we also equip $\mathcal{V}$ with the *Zariski topology* defined by A. This is the topology having the sets

$$Z(J) := \{\alpha \in \mathcal{V} \mid \alpha(J) = \{0\}\}$$

for ideals J of A as its closed sets. The same definition applies to $\mathcal{V}(\mathbb{R})$. We can restrict ourself to real radical ideals J when defining the topology on $\mathcal{V}(\mathbb{R})$. However, we will not use this topology much, and state it explicitly whenever we do. So unless otherwise mentioned, the varieties are *always* equipped with the strong topology defined above. All notions as closed, compact, continuous... refer to it.

A *quadratic module* of an $\mathbb{R}$-algebra A is a subset M of A such that

$$M + M \subseteq M, \sum A^2 \cdot M \subseteq M \text{ and } 1 \in M.$$

Here, $\sum A^2$ denotes the set of sums of squares in A. Note that any quadratic module in A is a convex cone, so the results and notions for cones from the previous section apply.

For $a_1, \dots, a_t \in A$, the smallest quadratic module containing $a_1, \dots, a_t$ consists of all elements of the form

$$\sigma_0 + \sigma_1 a_1 + \cdots + \sigma_s a_t,$$

where $\sigma_i \in \sum A^2$. It is called the quadratic module *generated by* $a_1, \dots, a_t$ and most often denoted by $\mathrm{QM}(a_1, \dots, a_t)$. A quadratic module is called *finitely generated*, if it is of such a form. For any quadratic module M, $M \cap -M$ is called the *support* of M, also denoted by $\mathrm{supp}(M)$. It is an ideal of A.

A *preordering* is a quadratic module which is closed under multiplication. We will denote preorderings by P, whenever possible. For finitely many elements $a_1, \dots, a_t \in A$, the smallest preordering containing $a_1, \dots, a_t$ is the quadratic module generated by the 2^t products

$$a^e := a_1^{e_1} \cdots a_t^{e_t}; \quad e \in \{0, 1\}^t.$$

It is called the preordering *generated* by $a_1, \dots, a_t$ and denoted by $\mathrm{PO}(a_1, \dots, a_t)$. A preordering is called *finitely generated* if it is of such a form.

An *ordering* of A is a preordering P, such that $P \cap -P$ is a prime ideal and $P \cup -P = A$ holds. The set of all orderings is denoted by $\mathrm{Sper}(A)$, and for a quadratic module M, the set of all orderings containing M by $\mathcal{X}_M$. Elements $\alpha \in \mathcal{V}(\mathbb{R})$ define orderings

$$P_\alpha := \{a \in A \mid \hat{a}(\alpha) \geq 0\} = \alpha^{-1}([0, \infty))$$

on A, but in general not all orderings are of this form.

Subsets M of A (in particular quadratic modules) define closed subsets of $\mathcal{V}(\mathbb{R})$ by

$$\begin{aligned} \mathcal{S}(M) : &= \{\alpha \in \mathcal{V}(\mathbb{R}) \mid \hat{m}(\alpha) \geq 0 \text{ for all } m \in M\} \\ &= \{\alpha \in \mathcal{V}(\mathbb{R}) \mid \alpha(M) \subseteq [0, \infty)\} \\ &= \{\alpha \in \mathcal{V}(\mathbb{R}) \mid M \subseteq P_\alpha\}. \end{aligned}$$

So for $\alpha \in \mathcal{S}(M)$, P_α belongs to $\mathcal{X}_M$. If M is a finitely generated quadratic module, generated by $a_1, \ldots, a_t$, then

$$\mathcal{S}(M) = \{\alpha \in \mathcal{V}(\mathbb{R}) \mid \hat{a}_1(\alpha) \geq 0, \ldots, \hat{a}_t(\alpha) \geq 0\}$$

is called *basic closed semi-algebraic.*

Conversely, subsets S of $\mathcal{V}(\mathbb{R})$ define preorderings of A by

$$\operatorname{Pos}(S) = \{a \in A \mid \hat{a} \geq 0 \text{ on } S\} = \bigcap_{\alpha \in S} P_\alpha.$$

For a quadratic module M of A we call

$$M^{\text{sat}} := \operatorname{Pos}(\mathcal{S}(M))$$

the *saturation* of M, i.e. the saturation of M consists of all elements of A which are nonnegative as functions on $\mathcal{S}(M)$. The relation between M and M^{sat} is an important object of study in Real Algebra and Real Algebraic Geometry.

If both A and M are finitely generated, an important result from Real Algebra, based on the Tarski-Seidenberg Transfer-Principle, says

$$M^{\text{sat}} = \bigcap_{P \in \mathcal{X}_M} P. \tag{1}$$

To avoid confusion, note that the saturation of a quadratic module is often defined in terms of this last equation in the existing literature. In the non-finitely generated case, that definition might differ from the one we use in this work.

It is used for example in the proof of the following proposition, which we will apply extensively in Chapter 3:

Proposition 1.5. *Let A be a finitely generated $\mathbb{R}$-algebra and $M \subseteq A$ a finitely generated quadratic module. If $\mathcal{S}(M)$ is Zariski dense in $\mathcal{V}(\mathbb{R})$, then*

$$M \cap -M \subseteq \sqrt[rr]{\{0\}}.$$

If M is a finitely generated preordering, then the other implication is also true.

Here, $\sqrt[rr]{\{0\}}$ denotes the real radical of $\{0\}$. Although the proof is typical for Real Algebraic Geometry, we include it for completeness.

Proof. Suppose $a \in M \cap -M$. Then $\hat{a} = 0$ on $\mathcal{S}(M)$, so by the Zariski denseness also $\hat{a} = 0$ on $\mathcal{V}(\mathbb{R}) = \mathcal{S}\left(\sum A^2\right)$. So by (1),

$$-a, a \in \bigcap_{P \in \mathrm{Sper}(A)} P.$$

The abstract real Nullstellensatz ([PD], Theorem 4.2.5) yields

$$a^{2e} + \sigma = 0$$

for some $e \in \mathbb{N}$ and $\sigma \in \sum A^2$, so $a \in \sqrt[rr]{\{0\}}$.

Now suppose M is a preordering and $M \cap -M \subseteq \sqrt[rr]{\{0\}}$. Then

$$\sqrt[rr]{\{0\}} = \sqrt[rr]{M \cap -M} = \bigcap_{P \in \mathcal{X}_M} P \cap -P,$$

where the last equality uses the real Nullstellensatz again, see for example [Sc3] 1.3.12. Now suppose $a \in A \setminus \sqrt[rr]{\{0\}}$. We show there is some $\alpha \in \mathcal{S}(M)$ such that $\alpha(a) \neq 0$. There is some $P \in \mathcal{X}_M$ such that $a \notin P$, without loss of generality. So by (1), $a \notin M^{\mathrm{sat}}$, which implies the claim.

So whenever $\mathcal{S}(M) \subseteq Z(I)$ for some ideal I of A, then $I \subseteq \sqrt[rr]{\{0\}}$. But then obviously $Z(I) = \mathcal{V}(\mathbb{R})$, which shows the desired denseness.

□

We can also consider ℝ-algebra homomorphisms

$$\varphi\colon A \to B,$$

between arbitrary ℝ-algebras A and B. We have a corresponding map

$$\varphi^*\colon \mathcal{V}_B(\mathbb{R}) \to \mathcal{V}_A(\mathbb{R})$$

sending β to $\beta \circ \varphi$. φ^* is continuous with respect to both defined topologies. Indeed, for a set $M \subseteq A$ we have

$$(\varphi^*)^{-1}(\mathcal{S}(M)) = \mathcal{S}(\varphi(M)).$$

In particular, whenever $S \subseteq \mathcal{V}_A(\mathbb{R})$ is basic closed semi-algebraic, then so is $(\varphi^*)^{-1}(S) \subseteq \mathcal{V}_B(\mathbb{R})$.

For an arbitrary quadratic module M in an arbitrary $\mathbb{R}$-algebra A, we have the obvious relations

$$M \subseteq M^{\ddagger} \subseteq \overline{M} = M^{\vee\vee} \subseteq M^{\text{sat}}.$$

The last inclusion uses that each $\mathbb{R}$-algebra homomorphism from A to $\mathbb{R}$ is a linear functional.

Lemma 1.6. *$M^{\ddagger}$ and $\overline{M}$ are again quadratic modules, even preorderings if M was a preordering.*

Proof. For $M^{\ddagger}$ this is clear, use for example the characterization from Proposition 1.3. $\overline{M}+\overline{M} \subseteq \overline{M}$ is also clear from Proposition 1.3. Now suppose $a \cdot M \subseteq \overline{M}$ for some a in A. The multiplication with a, denoted by φ_a, is a linear and therefore continuous map from A to A. So we have

$$a \cdot \overline{M} = \varphi_a(\overline{M}) \subseteq \overline{\varphi_a(M)} = \overline{a \cdot M} \subseteq \overline{M}.$$

This shows that $\overline{M}$ is closed under multiplication with squares and therefore a quadratic module. It also shows $M \cdot \overline{M} \subseteq \overline{M}$ if M is a preordering. Using $\overline{M} \cdot M = M \cdot \overline{M} \subseteq \overline{M}$ and applying the result one more time, we see that $\overline{M}$ is multiplicatively closed. □

Definition 1.7. (i) A quadratic module M is called *saturated*, if $M = M^{\text{sat}}$ holds. It is called *closed* if $M = \overline{M}$ holds.

(ii) We say that M has the *strong moment property* (SMP), if $\overline{M} = M^{\text{sat}}$ holds.

(iii) The strong moment problem is said to be *finitely solvable* for a set $S \subseteq \mathcal{V}(\mathbb{R})$, if there is a finitely generated quadratic module M in A, such that $\mathcal{S}(M) = S$ and M has (SMP).

(iv) M has the ‡-*property*, if $M^{\ddagger} = M^{\text{sat}}$ holds.

Saturatedness of finitely generated quadratic modules or preorderings is a rather rare phenomenon. In the one-dimensional case, it still occurs often, see for example [KM, KMS]. From dimension three upwards, it never occurs, see [Sc1]. In dimension two, several local-global principles from [Sc2, Sc5] give examples of finitely generated saturated preorderings. Most of the examples involve compact semi-algebraic sets. There are only few non-compact two-dimensional examples of finitely generated and saturated preorderings, see for example [M2].

To a great part, the interest in (SMP) comes from Haviland's Theorem. The original version from [H] applies to polynomial rings, but we state a much more general version here, which is taken from [M1].

Theorem 1.8. *Let A be an $\mathbb{R}$-algebra, X a Hausdorff space and suppose $\hat{}\colon A \to C(X,\mathbb{R})$ is an $\mathbb{R}$-algebra homomorphism. Assume that the following condition is fulfilled:*

($*$) *there is some $p \in A$ such that $\hat{p} \geq 0$ on X and for all $i \in \mathbb{N}$, the set $X_i = \{x \in X \mid \hat{p}(x) \leq i\}$ is compact.*

Then for every linear functional $L\colon A \to \mathbb{R}$ with

$$L\left(\{a \in A \mid \hat{a} \geq 0 \text{ on } X\}\right) \subseteq [0,\infty)$$

there exists a positive regular Borel measure μ on X such that

$$L(a) = \int_X \hat{a}\, d\mu$$

for all $a \in A$.

For a quadratic module M in A look at the morphism

$$\hat{}\colon A \to C(\mathcal{S}(M), \mathbb{R})$$

obtained by restricting the functions $\hat{a}\colon \mathcal{V}(\mathbb{R}) \to \mathbb{R}$ to $\mathcal{S}(M)$. Assume that the assumption $(*)$ from Theorem 1.8 is fulfilled, which is for example the case if A is finitely generated; one can take $p = x_1^2 + \cdots + x_n^2$, where $x_1, \ldots, x_n$ are generators of A; it also holds trivially if $\mathcal{S}(M)$ is compact. Now the set $\{a \in A \mid \hat{a} \geq 0 \text{ on } \mathcal{S}(M)\}$ is M^{sat}. So if M has the strong moment property, then every functional which is nonnegative on M is already nonnegative on M^{sat}, and therefore integration on $\mathcal{S}(M)$ by Haviland's Theorem. Nonnegativity on M is a priori a weaker condition than nonnegativity on M^{sat}. So (SMP) is a useful property.

There are many important and interesting works concerning the strong moment property and representations of positive polynomials. A ground-breaking result is the following:

Theorem 1.9 (Schmüdgen, [Sm2])**.** *If A is finitely generated, P a finitely generated preordering in A, and $\mathcal{S}(P)$ is compact, then every element from A which is strictly positive on $\mathcal{S}(P)$ belongs to P. In particular, P has (SMP).*

The first purely algebraic proof of the theorem can be found in [Wö]. Further, there is a wide range of generalizations of this important result. For example, one can ask if preorderings can be replaced by quadratic modules in Schmüdgen's Theorem. First work in that direction can be found in [P]. In full generality, the answer is due to Jacobi [J]. Also see [PD] or [M1] for a proof.

Theorem 1.10. *Let M be an archimedean quadratic module of the $\mathbb{R}$-algebra A. Then every element from A which is strictly positive on $\mathcal{S}(M)$ belongs to M. In particular, M has (SMP).*

Archimedean means, that for every $x \in A$ there is some $N \in \mathbb{N}$, such that $N - x$ belongs to M. If A is finitely generated, then $N \pm x_i \in M$ for generators $x_1, \ldots, x_n$ of A already implies that M is archimedean. For a finitely generated preordering P in a finitely generated $\mathbb{R}$-algebra, the compactness of $\mathcal{S}(P)$ implies that P is archimedean. So Theorem 1.10 is a generalization of Theorem 1.9.

There is a lot of literature concerning these last two results. Generalizations, alternative proofs and also quantitative versions can for example be found in [NiSw, Sw1, Sw2, Sw3].

The question whether a quadratic module is archimedean has been dealt with in [JP], using valuation theoretic arguments.

2 A Fibre Theorem for Closures

In this chapter we give a generalized version of Schmüdgen's Theorem from [Sm3]. It characterizes the closure of a preordering in terms of so called *fibre-preorderings*, constructed from bounded polynomials. The original proof uses deep results from functional analysis, taken from [D, Sm1]. Our proof is more elementary. It relies heavily on the Radon-Nikodym Theorem. The main ideas are published in [N]. Murray Marshall found a similar approach to the same problem independently. It appears in his book [M1].

We begin with a short section about closures and (SMP) on quotients. It comprehends some helpful remarks, most of them taken from [Sc4]. In Section 2.2 we prove a generalized fibre theorem for closures of quadratic modules. As we show in Section 2.3, it implies Schmüdgen's original result (Theorem 2.8). In the last section of this chapter we give some new applications of the fibre theorem.

2.1 Closures and Quotients

The results from this section are all contained in [Sc4], Section 4. We just state them for arbitrary algebras and modules whenever possible, not only for finitely generated ones. The proofs are mostly the same. So let A be an $\mathbb{R}$-algebra and M a quadratic module in A. Let I be an ideal of A, contained in the real radical of the support of M, and let

$$\pi \colon A \to A/I$$

be the canonical projection. An easy observation is

$$\pi(M)^{\text{sat}} = \pi(M^{\text{sat}}).$$

Lemma 2.1. and Corollary 3.12 from [Sc4] tell us that whenever $a \in I$, then $a + \varepsilon \in M$ for all $\varepsilon > 0$. In particular $I \subseteq \overline{M}$. So the following proposition is clear.

Proposition 2.1. *Let A be an $\mathbb{R}$-algebra and M a quadratic module in A. Let I be an ideal of A, contained in $\sqrt[rr]{M \cap -M}$, and $\pi\colon A \to A/I$ the canonical projection. Then*

$$\overline{\pi(M)} = \pi(\overline{M}).$$

So M has (SMP) in A if and only if $\pi(M)$ has (SMP) in A/I, which is the case if and only if $M + I$ has (SMP) in A.

If A is finitely generated, then (SMP) carries over from M to $M + I$ for arbitrary quadratic modules M and ideals I. For finitely generated quadratic modules, this is Proposition 4.8 in [Sc4]. However, the same proof works for arbitrary M. It shows that the fibre condition for (SMP) in Theorem 2.6 and Theorem 2.8 below is necessary for (SMP) to hold.

Proposition 2.2. *Let A be a finitely generated $\mathbb{R}$-algebra, M a quadratic module and I and ideal in A. If M has (SMP) in A, then so does $M + I$.*

Proof. If $L \in (M + I)^\vee$, then of course $L \in M^\vee$. So by Theorem 1.8, L is integration with respect to some measure μ on $\mathcal{S}(M) \subseteq \mathcal{V}(\mathbb{R})$. For $c \in I$ we have

$$0 = L(c) = \int_{\mathcal{S}(M)} \hat{c} d\mu.$$

This shows that $\mu(\mathcal{S}(M) \setminus Z(I)) = 0$ (a standard argument, using the fact that $\mathcal{S}(M) \setminus Z(I)$ is a countable union of compact sets).

So

$$L(a) = \int_{\mathcal{S}(M+I)} \hat{a} d\mu$$

for all $a \in A$, which shows that L is nonnegative on elements from $(M + I)^{\text{sat}}$. So $M + I$ has (SMP). □

2.2 The Main Theorem

Our goal in this section is to proof Theorem 2.5 below. It is a generalized fibre theorem for closures of quadratic modules. Before dealing with it, we describe the setup that we will use, explain some constructions, and give some helpful results. So let A be an $\mathbb{R}$-algebra. Let X be a Hausdorff topological space and

$$\hat{}: A \to C(X, \mathbb{R})$$

a morphism of $\mathbb{R}$-algebras. Denote the image of A in $C(X, \mathbb{R})$ by $\hat{A}$. As in Theorem 1.8 we assume condition $(*)$ to hold, i.e. there exists $p \in A$ such that $\hat{p} \geq 0$ on X and for all $i \in \mathbb{N}$, the set $X_i = \{x \in X \mid \hat{p}(x) \leq i\}$ is compact. This in particular implies that X is locally compact, as observed in [M1]. Note that in case X is compact, assumption $(*)$ is always fulfilled with $p = 1$, and if A is finitely generated by $x_1, \ldots, x_n$ and $X \subseteq \mathcal{V}(\mathbb{R})$ closed, we can choose $p = x_1^2 + \cdots + x_n^2$.

Replacing p by $p+1$ if necessary, we can assume without loss of generality that $\hat{p}$ is *strictly* positive on X. So $\frac{1}{\hat{p}}$ is a continuous positive function on X, vanishing at infinity. This means that it takes arbitrary small values outside of compact sets.

Now we make the additional assumption that the functions $\hat{a}$ which are *bounded* on X separate points. That means, for any two distinct points in X there is some $a \in A$, such that $\hat{a}$ is bounded on X and takes different values in the two points. This is for example fulfilled if X is a compact subset of $\mathcal{V}(\mathbb{R})$, as elements from A separate points of $\mathcal{V}(\mathbb{R})$. However, there are also non-compact examples, as we will see later.

We use the assumption to apply [Bu], Theorem 3, which says that the bounded functions from $\hat{A}$ lie dense in the set of bounded continuous functions on X, under the locally convex topology defined by the family of seminorms

$$||f||_\psi := \sup_{x \in X} |\psi(x) \cdot f(x)|,$$

where ψ is a continuous function vanishing at infinity. We will use the seminorm defined by $\frac{1}{\hat{p}}$. Note that in case X is compact, we could use the standard Stone-Weierstrass approximation instead of [Bu] in the following proofs.

We begin with a technical lemma.

Lemma 2.3. *Let ν be a positive regular Borel measure on X and $h\colon X \to \mathbb{R}$ a measurable function. Suppose for all $a \in A$ we have*

$$\int_X \hat{a}^2 |h| d\nu < \infty$$

as well as

$$0 \leq \int_X \hat{a}^2 h d\nu.$$

Then $h \geq 0$ on X, except on a ν-null set.

Proof. For $n = 1, 2, \ldots$ define

$$A_n := \{x \in X \mid -\frac{1}{n-1} < h(x) \leq -\frac{1}{n}\},$$

where $-\frac{1}{0} := -\infty$. Suppose $\nu(A_i) > 0$ for some i. Let χ be the characteristic function of A_i, so

$$\int_X \chi h d\nu \leq -\frac{1}{i}\nu(A_i) < 0.$$

Now choose a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions on X with values in $[0,1]$, that converges pointwise except on a ν-null set to χ. This can be done, using the regularity of ν as well as Urysohn's Lemma as stated for example in [Ru]. Using [Bu], Theorem 3, we find a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that

$$\sup_{x\in X}\left|\frac{1}{\hat{p}(x)}\left(\hat{a}_n(x) - \sqrt{f_n(x)}\right)\right| \leq \frac{1}{n}$$

for all n. So

$$|\hat{a}_n - \sqrt{f_n}| \leq \frac{1}{n}\hat{p}$$

on X. From this we get $|\hat{a}_n| \leq \hat{p}+1$ on X, as $\sqrt{f_n}$ takes on values only in $[0,1]$. So

$$\begin{aligned} |\hat{a}_n^2 - f_n| &= |\hat{a}_n - \sqrt{f_n}| \cdot |\hat{a}_n + \sqrt{f_n}| \\ &\leq \frac{1}{n}\hat{p}(\hat{p}+2) \end{aligned}$$

holds on X. Thus the sequence $|\left(\hat{a}_n^2 - f_n\right) \cdot h|$ converges to 0 pointwise on X. As $|\left(\hat{a}_n^2 - f_n\right) h| \leq \hat{p}(\hat{p}+2)|h| \leq ((\hat{p}+1)(\hat{p}+2))^2|h|$ on X and

$$\int_X ((\hat{p}+1)(\hat{p}+2))^2 \, |h| d\nu < \infty$$

by assumption, the Theorem of Majorized Convergence applies and yields

$$\left| \int_X \hat{a}_n^2 h d\nu - \int_X f_n h d\nu \right| \leq \int_X \left| (\hat{a}_n^2 - f_n) h \right| d\nu \stackrel{n\to\infty}{\longrightarrow} 0.$$

As $|f_n - \chi| \, |h|$ converges pointwise except on a zero set to 0, and is bounded from above by the function $|h|$ which has a finite integral, we get in the same way as above

$$\int_X f_n h d\nu \stackrel{n\to\infty}{\longrightarrow} \int_X \chi h d\nu < 0.$$

Combining these result we have

$$\int_X \hat{a}_n^2 h d\nu \stackrel{n\to\infty}{\longrightarrow} \int_X \chi h d\nu < 0,$$

which contradicts our assumption. So $\nu(A_i) = 0$ for all i, which proves the result. $\square$

Towards the main theorem, we need a second $\mathbb{R}$-algebra B and an algebra homomorphism $\varphi\colon A \to B$. So we have the following diagram:

$$\begin{array}{ccc} B & & \\ \uparrow{\scriptstyle\varphi} & & \\ A & \xrightarrow{\ \widehat{}\ } & C(X,\mathbb{R}) \end{array}$$

Suppose $M \subseteq B$ is a quadratic module. We want to describe the closure of M in terms of fibre-modules, indexed by elements from X. Namely, for $x \in X$, we denote by I_x the ideal in B generated by the set

$$\{\varphi(a) \mid a \in A,\ \hat{a}(x) = 0\}\,.$$

We call $M_x := M + I_x$ the *fibre-module* to x, and we want to prove

$$\overline{M} = \bigcap_{x\in X} \overline{M_x}.$$

For this we have to make more assumptions. Namely, suppose $\varphi(a) \in \overline{M}$ whenever $\hat{a} \geq 0$ on X. This assumption is fulfilled in a large class of examples, as we will see below.

Now take $L \in M^\vee$, i.e. L is a linear functional on B that maps M to $[0,\infty)$. For $b \in B$ we define a linear functional L_b on A by

$$L_b(a) := L\left(b \cdot \varphi(a)\right).$$

We can apply Haviland's Theorem (Theorem 1.8) to the functionals L_{b^2}. Indeed, whenever $\hat{a} \geq 0$ on X, then $\varphi(a) \in \overline{M}$, so also $b^2 \cdot \varphi(a) \in \overline{M}$, so

$$L_{b^2}(a) = L(b^2 \cdot \varphi(a)) \geq 0.$$

So we get positive regular Borel measures ν_b on X such that

$$L_{b^2}(a) = L(b^2 \cdot \varphi(a)) = \int_X \hat{a} d\nu_b$$

holds for all $a \in A$. As all considered measures are defined on X, we omit X under the integral sign from now on. The following result is a key ingredient for the proof of Theorem 2.5:

Proposition 2.4. *For all $b \in B$*

$$\nu_b \ll \nu_1,$$

that is, every ν_1-null set is also a ν_b-null set.

Proof. Let $b \in B$ be fixed and suppose $N \subseteq X$ is a Borel set with $\nu_1(N) = 0$. We have to show $\nu_b(N) = 0$. Denote the characteristic function of N by χ.

Choose a sequence of functions $(f_n)_{n\in\mathbb{N}}$ from $C(X, [0, 1])$ that converges pointwise to χ, except on a set that is a ν_1- and a ν_b-null set. This can be done, using the regularity of ν_1, ν_b and Urysohn's Lemma. So

$$\int f_n d\nu_1 \overset{n\to\infty}{\longrightarrow} \nu_1(N) = 0, \tag{2}$$

by the Theorem of Majorized Convergence.

Apply [Bu], Theorem 3, to obtain a sequence $(a_n)_{n\in\mathbb{N}}$ from A with

$$\left|\hat{a}_n - \sqrt{f_n}\right| \le \frac{1}{n}\hat{p}$$

on X for all n. Exactly as in the proof of Lemma 2.3, the sequences $|\hat{a}_n - \sqrt{f_n}|$ and $|\hat{a}_n^2 - f_n|$ converge pointwise on X to zero. $|\hat{a}_n^2 - f_n|$ is bounded from above by $\hat{p}(\hat{p} + 2)$ on X and

$$\int \hat{p}(\hat{p} + 2)d\nu_1 = L_1(p(p + 2)) < \infty,$$

so the Theorem of Majorized Convergence implies

$$\left|\int \hat{a}_n^2 d\nu_1 - \int f_n d\nu_1\right| \overset{n\to\infty}{\longrightarrow} 0.$$

Combined with (2) we get

$$\int \hat{a}_n^2 d\nu_1 \stackrel{n\to\infty}{\longrightarrow} 0. \tag{3}$$

Using the inequality from Lemma 1.4, we have

$$\begin{aligned}\left(\int \hat{a}_n d\nu_b\right)^2 &= L_{b^2}(a_n)^2\\ &= L(b^2 \cdot \varphi(a_n))^2\\ &\leq L(b^4)L(\varphi(a_n)^2)\\ &= L(b^4) \cdot L_1(a_n^2)\\ &= L(b^4)\int \hat{a}_n^2 d\nu_1.\end{aligned}$$

Together with (3) we find

$$\int \hat{a}_n d\nu_b \stackrel{n\to\infty}{\longrightarrow} 0. \tag{4}$$

As the sequence $|\hat{a}_n - \sqrt{f_n}|$ is bounded from above by $\hat{p}$ on X and

$$\int \hat{p}\ d\nu_b = L_b(p) < \infty,$$

we get, again by Majorized Convergence,

$$\left|\int \hat{a}_n d\nu_b - \int \sqrt{f_n} d\nu_b\right| \stackrel{n\to\infty}{\longrightarrow} 0. \tag{5}$$

The fact that $(\sqrt{f_n})_n$ converges pointwise except on a ν_b-null set to χ, combined with (4) and (5), finally yields

$$0 = \lim_{n\to\infty}\int \sqrt{f_n} d\nu_b = \int \chi d\nu_b = \nu_b(N).$$

$\square$

Proposition 2.4 allows us to apply the Radon-Nikodym Theorem (see [Ru]). For every $b \in B$ we get a ν_1-integrable function $\Phi_b\colon X \to [0, \infty)$ such that

$$L_{b^2}(a) = \int \hat{a}\; d\nu_b = \int \hat{a} \cdot \Phi_b\; d\nu_1 \text{ for all } a \in A.$$

If we define $\theta_b := \Phi_{\frac{b+1}{2}} - \Phi_{\frac{b-1}{2}}$ for $b \in B$, then all θ_b are ν_1-integrable and

$$L_b(a) = L_{\left(\frac{b+1}{2}\right)^2}(a) - L_{\left(\frac{b-1}{2}\right)^2}(a) = \int \hat{a} \cdot \theta_b\; d\nu_1$$

holds for all $a \in A$. Before stating and proving Theorem 2.5 below, we look at some properties of the functions θ_b.

For $b_1, b_2 \in B$, $r_1, r_2 \in \mathbb{R}$ and all $a \in A$ we have

$$\begin{aligned}\int \hat{a} \cdot \theta_{r_1 b_1 + r_2 b_2} d\nu_1 &= L_{r_1 b_1 + r_2 b_2}(a) \\ &= r_1 L_{b_1}(a) + r_2 L_{b_2}(a) \\ &= \int \hat{a} \cdot (r_1 \theta_{b_1} + r_2 \theta_{b_2})\, d\nu_1.\end{aligned}$$

We apply Lemma 2.3 to the functions $h = \theta_{r_1 b_1 + r_2 b_2} - r_1\theta_{b_1} - r_2\theta_{b_2}$ and $-h$. The condition $\int \hat{a}^2 |h| d\nu_1 < \infty$ for all $a \in A$ is obtained by reducing to $\int \hat{a}^2 |\Phi_b| d\nu_1 = L_{b^2}(a^2) < \infty$ for all b. So we get

$$\theta_{r_1 b_1 + r_2 b_2} = r_1\theta_{b_1} + r_2\theta_{b_2}$$

except on a ν_1-null set that depends on b_1, b_2, r_1, r_2.

For $m \in M$ and any $a \in A$ we have

$$0 \le L(m \cdot \varphi(a)^2) = L_m(a^2) = \int \hat{a}^2 \theta_m d\nu_1,$$

so again by Lemma 2.3,

$$\theta_m \ge 0,$$

except on a ν_1-null set that depends on m.

Last, for $a, c \in A$ and $b \in B$ we have

$$\begin{aligned}\int \hat{c} \cdot \theta_{b\cdot\varphi(a)} d\nu_1 &= L_{b\cdot\varphi(a)}(c) \\ &= L(b \cdot \varphi(a)\varphi(c)) \\ &= L_b(ac) \\ &= \int \hat{c} \cdot \hat{a}\theta_b \, d\nu_1.\end{aligned}$$

So $\theta_{b\cdot\varphi(a)} = \hat{a} \cdot \theta_b$, except on a ν_1-null set depending on a and b.

We are now prepared for the main theorem.

Theorem 2.5. *Let A, B be $\mathbb{R}$-algebras of countable vector space dimension, and let $\varphi\colon A \to B$ be an $\mathbb{R}$-algebra homomorphism. Let X be a Hausdorff space,*

$$\hat{}\colon A \to C(X, \mathbb{R})$$

an $\mathbb{R}$-algebra homomorphism fulfilling $(*)$ *(see Theorem 1.8) and suppose the set*

$$\{\hat{a} \mid a \in A,\ \hat{a} \textit{ bounded on } X\}$$

separates points of X. Further suppose M is a quadratic module in B and $\varphi(a) \in \overline{M}$ whenever $\hat{a} \geq 0$ on X. For $x \in X$ denote by I_x the ideal in B generated by $\{\varphi(a) \mid a \in A,\ \hat{a}(x) = 0\}$. Then

$$\overline{M} = \bigcap_{x\in X} \overline{M + I_x}.$$

Proof. One inclusion is obvious. For the other one fix $q \in \bigcap_{x\in X} \overline{M + I_x}$ and $L \in M^\vee$. We have to show $L(q) \geq 0$. From L we construct all the functions θ_b as explained above.

Let $B' \subseteq B$ and $A' \subseteq A$ be a countable linear basis of B and A, respectively. Using the fact that each element in B is a difference of two squares, we can assume that B' consist only of squares.

Denote the $\mathbb{Q}$-subspace of A spanned by A' by $\mathcal{A}$. Let $\mathcal{B}$ be the $\mathbb{Q}$-subalgebra of B generated by

$$B' \cup \varphi(A') \cup \{q\}\,.$$

$\mathcal{B}$ is a countable set and $\varphi(\mathcal{A}) \subseteq \mathcal{B}$. For each element $b \in B$ we have a unique representation as a finite sum $b = \sum r_i \cdot b_i$, where all $b_i \in B'$ and all $r_i \in \mathbb{R}$. Using the above demonstrated properties of the functions θ_b, we can find one single ν_1-null set $N \subseteq X$ such that for all $x \in X \setminus N$ the following conditions hold:

(i) $\theta_b(x) = \sum r_i \theta_{b_i}(x)$ for all $b \in \mathcal{B}$

(ii) $\theta_m(x) \geq 0$ for all $m \in M \cap \mathcal{B}$

(iii) $\theta_{b\cdot\varphi(a)}(x) = \hat{a}(x) \cdot \theta_b(x)$ for all $b \in \mathcal{B}$ and all $a \in \mathcal{A}$.

Because $\mathcal{A}$ and $\mathcal{B}$ are countable sets, this can indeed be ensured with one single null set N.

For $x \in X \setminus N$ we get linear functionals L_x on B by defining them on the basis B':

$$L_x(b) := \theta_b(x) \text{ for } b \in B'.$$

For $b \in \mathcal{B}$ with $b = \sum r_i \cdot b_i$ as above we have

$$L_x(b) = \sum r_i L_x(b_i) = \sum r_i \theta_{b_i}(x) = \theta_b(x),$$

where the last equality uses (i). So for $m \in M \cap \mathcal{B}$

$$L_x(m) = \theta_m(x) \geq 0$$

holds, using (ii). Now let $b \in M$ be arbitrary, i.e. b is not necessarily from $\mathcal{B}$. Write $b = \sum r_i \cdot b_i$ with $r_i \in \mathbb{R}$ and $b_i \in B'$. As all b_i are squares, $b + \sum t_i \cdot b_i \in M$ whenever all $t_i \geq 0$. So b can be approximated in a finite dimensional $\mathbb{R}$-subspace of B by elements from $M \cap \mathcal{B}$. Just choose $t_i > 0$ arbitrary small such that $r_i + t_i \in \mathbb{Q}$. This shows

$$L_x(M) \subseteq [0, \infty).$$

For $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have $b \cdot \varphi(a) \in \mathcal{B}$ and therefore

$$L_x(b \cdot \varphi(a)) = \theta_{b \cdot \varphi(a)}(x) = \hat{a}(x) \cdot \theta_b(x),$$

using (iii). Now take $a \in A$ with $\hat{a}(x) = 0$. Approximate a by a sequence of elements a_n from $\mathcal{A}$ in the same way as above, so $\hat{a}_n(x) \stackrel{n \to \infty}{\longrightarrow} 0$. So for $b \in \mathcal{B}$,

$$L_x(b \cdot \varphi(a)) = \lim_{n \to \infty} L_x(b \cdot \varphi(a_n)) = \lim_{n \to \infty} \hat{a}_n(x) \theta_b(x) = 0.$$

By approximating arbitrary $b \in B$ by elements of $\mathcal{B}$ we finally get

$$L_x(I_x) = \{0\}.$$

Defining

$$L_x \equiv 0 \text{ for } x \in N$$

we have

$$L_x \in (M + I_x)^{\vee} \text{ for all } x \in X. \tag{6}$$

Now

$$L(q) = L_q(1) = \int \theta_q(x) \; d\nu_1(x) = \int L_x(q) \; d\nu_1(x),$$

using the fact that $q \in \mathcal{B}$ and N is a ν_1-null set. By (6) and our assumption on q, the function $x \mapsto L_x(q)$ is nonnegative on X, so $L(q) \geq 0$. □

In the following section, we show how an algebra A and a topological space X can be constructed for a given quadratic module M in B, in a way that allows to deduce Schmüdgen's Theorem from Theorem 2.5.

2.3 Schmüdgen's Result

Let B be an $\mathbb{R}$-algebra of countable vector space dimension. Let M be a quadratic module in B. We take finitely many elements $b_1, \dots, b_s \in B$ such that $C_i - b_i, b_i - c_i \in \overline{M}$ for some real numbers $C_i \geq c_i$. Consider the subalgebra $A = \mathbb{R}[b_1, \dots, b_s]$ of B generated by these elements, and the quadratic module $\widetilde{M}$ in A generated by $C_i - b_i, b_i - c_i$ $(i = 1, \dots, s)$. It is archimedean. The role of φ from Theorem 2.5 is played by the canonical inclusion

$$\iota \colon A \hookrightarrow B$$

and we have $\iota(\widetilde{M}) \subseteq \overline{M}$.

Define $X := \mathcal{S}(\widetilde{M}) \subseteq \mathcal{V}_A(\mathbb{R})$, so we have the usual morphism

$$\hat{\cdot} \colon A \to C(X, \mathbb{R}).$$

Obviously X is compact, so the separating points condition from Theorem 2.5 and the condition $(*)$ are fulfilled. Now if some $\hat{a} \geq 0$ on X, then $a + \varepsilon \in \widetilde{M}$ for all $\varepsilon > 0$ by Theorem 1.10, so $\iota(a) \in \overline{M}$.

For $\alpha \in X$ one checks that I_α is the ideal in B generated by

$$b_1 - \alpha(b_1), \dots, b_s - \alpha(b_s),$$

and $(\alpha(b_1), \dots, \alpha(b_s)) \in \prod_{i=1}^s [c_i, C_i]$. So we get the following result:

Theorem 2.6. *Let B be an $\mathbb{R}$-algebra of countable vector space dimension and let $M \subseteq B$ be a quadratic module. Suppose $b_1, \dots, b_s \in B$ are such that*

$$C_1 - b_1, b_1 - c_1, \dots, C_s - b_s, b_s - c_s \in \overline{M}$$

for some real numbers $C_i \geq c_i$ $(i = 1, \dots, s)$. Then

$$\overline{M} = \bigcap_{\lambda \in \Lambda} \overline{M + (b_1 - \lambda_1, \dots, b_s - \lambda_s)},$$

where $\Lambda = \prod_{i=1}^s [c_i, C_i]$. In particular, if each quadratic module $M + (b_1 - \lambda_1 \dots, b_s - \lambda_s)$ has (SMP), then so does M.

Proof. Apply Theorem 2.5 to the above explained setup to get the equality. For the remark concerning (SMP), write

$$M_\lambda := M + (b_1 - \lambda_1, \ldots, b_s - \lambda_s)\,.$$

Note that obviously

$$\mathcal{S}(M) = \bigcup_{\lambda \in \Lambda} \mathcal{S}(M_\lambda)$$

and therefore

$$M^{\mathrm{sat}} = \bigcap_{\lambda \in \Lambda} (M_\lambda)^{\mathrm{sat}}$$

holds. So if all M_λ have (SMP), then

$$\overline{M} = \bigcap_{\lambda \in \Lambda} \overline{M_\lambda} = \bigcap_{\lambda \in \Lambda} (M_\lambda)^{\mathrm{sat}} = M^{\mathrm{sat}}.$$

□

The assumption $C_i - b_i, b_i - c_i \in \overline{M}$ for all i is indeed necessary for the theorem to hold, as we will later see. However, Schmüdgen's original result from [Sm3] is stated only for finitely generated preorderings and does not need it. To obtain this, we need another result. It is implicitly already proven in [Sm2] and again in [Sm3], but without bringing it up explicitly. We think it is interesting for itself, so we state it as a proposition.

Proposition 2.7. *Let B be a finitely generated $\mathbb{R}$-algebra and $P \subseteq B$ a finitely generated preordering. Let*

$$\hat{}: B \to C(\mathcal{S}(P), \mathbb{R})$$

be the canonical morphism. For any $b \in B$, whenever $\hat{b}$ is bounded and nonnegative on $\mathcal{S}(P)$, then

$$b \in \overline{P}.$$

Proof. Let $L \in P^\vee$. As shown in [Sm2], Theorem 1, and [Sm3], Proposition 2, for any bounded $\hat{b}$ we have

$$L(b^2) \leq || \hat{b} ||_\infty^2.$$

This result uses the Positivstellensatz and the one dimensional Hamburger moment problem. It also uses that P is closed under multiplication. It is proven for polynomial rings, but carries over directly to finitely generated algebras.

So let $\hat{b}$ be nonnegative on $\mathcal{S}(P)$ in addition. We have to show $L(b) \geq 0$. Using Lemma 1.4, we can assume $L(1) = 1$, maybe after scaling with a positive real. For any $\delta > || \hat{b} ||_\infty$ we have

$$-\delta \leq \hat{b} - \delta < 0$$

on $\mathcal{S}(P)$, and so

$$L\left((b-\delta)^2\right) \leq \delta^2.$$

This implies $0 \leq L(b^2) \leq 2\delta L(b)$, so $L(b) \geq 0$, as $\delta > 0$. □

The proposition implies that P has (SMP), whenever $\mathcal{S}(P)$ is compact. This is one half of the important result from [Sm2]. And finally we get Schmüdgen's main result from [Sm3] with it:

Theorem 2.8 (Schmüdgen, [Sm3])**.** *Let B be a finitely generated $\mathbb{R}$-algebra and $P \subseteq B$ a finitely generated preordering. Let $b_1, \ldots, b_s \in B$ be bounded as functions on $\mathcal{S}(P)$. Then*

$$\overline{P} = \bigcap_{\beta \in \mathcal{S}(P)} \overline{P + (b_1 - \beta(b_1), \ldots, b_s - \beta(b_s))}.$$

In particular, if each $P + (b_1 - \beta(b_1), \ldots, b_s - \beta(b_s))$ has (SMP), then so does P.

Proof. Choose real numbers $C_i \geq c_i$ such that $C_i - b_i, b_i - c_i$ are nonnegative as functions on $\mathcal{S}(P)$. By Proposition 2.7, they belong to $\overline{P}$. Now Theorem 2.6 yields

$$\overline{P} = \bigcap_{\lambda \in \Lambda} \overline{P + (b_1 + \lambda_1, \ldots, b_s + \lambda_s)}, \tag{7}$$

where $\Lambda = \prod_{i=1}^{s}[c_i, C_i]$. Now if the semi-algebraic set S_λ in $\mathcal{V}_B(\mathbb{R})$ corresponding to the finitely generated preordering $P + (b_1 + \lambda_1, \ldots, b_s + \lambda_s)$ for some $\lambda \in \Lambda$ is not empty, then $\lambda = (\beta(b_1), \ldots, \beta(b_s))$ for any $\beta \in S_\lambda \subseteq \mathcal{S}(P)$. If S_λ is empty, then

$$P + (b_1 + \lambda_1, \ldots, b_s + \lambda_s) = A,$$

see for example [PD], Remark 4.2.13. This shows that we can let the intersection in (7) run over $\beta \in \mathcal{S}(P)$ instead of $\lambda \in \Lambda$.

The remark about (SMP) is proven similar to the one in 2.6. □

Note that by Proposition 2.2, the preorderings

$$P_\beta := P + (b_1 - \beta(b_1), \ldots, b_s - \beta(b_s))$$

have (SMP) if P has. So it is necessary and sufficient for all P_β to have (SMP), for P to have (SMP).

The bigger the fibre-preorderings appearing in Theorem 2.8 are, the more is generally known about them. For example, if they describe one dimensional sets, the works [KM, KMS, P2, Sc2] allow to solve the questions for (SMP), closedness etc. in almost all cases.

So if the whole ring of functions bounded on $\mathcal{S}(P)$ is finitely generated, Theorem 2.8 applies in the best possible way. See [P2] for a discussion of the question, when this ring is finitely generated. Note that the ring of bounded polynomials, sometimes also called the *Real Holomorphy Ring*, is also of interest beside its use in Schmüdgen's Fibre Theorem; see for example [BP, Sw2].

2.4 Applications

We conclude this chapter with some applications of the fibre theorems from above. The first is a slight generalization of the fact, that every finitely generated preordering describing a compact set has (SMP).

Corollary 2.9. *Let B be a finitely generated $\mathbb{R}$-algebra and $S \subseteq \mathcal{V}_B(\mathbb{R})$ a basic closed semi-algebraic set. Suppose there are $b_1, \ldots, b_s \in B$ that are bounded as functions on S and separate its points. Then every finitely generated preordering P in B describing S has (SMP).*

Proof. Apply Theorem 2.8 and note that all the preorderings

$$P + (b_1 - \alpha(b_1), \ldots, b_s - \alpha(b_s))$$

describe singletons. So they all have (SMP), by the remark following Proposition 2.7. □

The following corollary can for example be applied to actions of compact algebraic groups on affine varieties. At the end of Section 3.3, that setup is explained in more detail.

Corollary 2.10. *Let A, B be finitely generated $\mathbb{R}$-algebras and $\varphi\colon A \to B$ a morphism, such that the induced map*

$$\varphi^*\colon \mathcal{V}_B(\mathbb{R}) \to \mathcal{V}_A(\mathbb{R})$$

has compact fibres. Let $S \subseteq \mathcal{V}_A(\mathbb{R})$ be basic closed semi-algebraic and suppose there are finitely many elements from A which are bounded as functions on S and separate points of S. Then for $(\varphi^*)^{-1}(S)$, the strong moment problem is finitely solvable.

Proof. Take a finitely generated preordering P in A that describes S. P has (SMP) in A by Corollary 2.9, and the finitely generated preordering P' generated by $\varphi(P)$ in B describes $(\varphi^*)^{-1}(S)$. So we can apply Theorem 2.5 to this setup.

The semi-algebraic set defined by $P' + I_x$ for $x \in S$ equals

$$(\varphi^*)^{-1}(x)$$

and is therefore compact. So all the fibre-preorderings have (SMP). The usual argument shows that also P' has (SMP). $\square$

The last corollary allows to pass from semi-algebraic sets to subsets, and obtain (SMP) under certain conditions.

Corollary 2.11. *Let B be a finitely generated $\mathbb{R}$-algebra and suppose M is a quadratic module in B that has (SMP). If $b \in B$ is bounded from above as a function on $\mathcal{S}(M)$, then*

$$M' := M + b \cdot \sum B^2$$

has (SMP) as well.

Proof. We have $N - b \in M^{\text{sat}} = \overline{M} \subseteq \overline{M'}$ for some big enough $N \geq 0$. As also $b \in \overline{M'}$, we can apply Theorem 2.6 to M'. For $\lambda \in [0, N]$, $M' + (b - \lambda) = M + (b - \lambda)$, so it has (SMP) by Proposition 2.2. So also M' has (SMP). $\square$

3 Stability

The notion of a *stable* quadratic module is first explicitly used in [PSc]. The authors show, that stable quadratic modules are closed under certain conditions, and they give a geometric criterion for stability. Implicitly, similar notions have been used in the proofs of [KM], Theorem 3.5, [PD], Proposition 6.4.5 and, as pointed out by the authors of [PSc], also in [Sm1]. All these authors use stability to show closedness of certain quadratic modules.

In [Sc4], stability is linked to the moment problem, generalizing an idea by Prestel and Berg. Indeed, stability often excludes (SMP). We start by defining stability as in [PSc]. Therefore let A be an $\mathbb{R}$-algebra, $a_1, \ldots, a_s \in A$ and W a linear subspace of A. Let

$$\sum (W; a_1, \ldots, a_s)$$

denote the set of all elements of A of the form

$$\sigma_0 + \sigma_1 a_1 + \cdots + \sigma_s a_s,$$

where all σ_i are sums of squares of elements from W, $\sigma_i \in \sum W^2$ for short. We obviously have

$$\mathrm{QM}(a_1, \ldots, a_s) = \bigcup_W \sum (W; a_1, \ldots, a_s),$$

where the union runs over all finite dimensional subspaces of A. If W is finite dimensional, then $\sum(W; a_1, \ldots, a_s)$ is contained in some finite dimensional subspace of A. The authors of [PSc] show, that such a set $\sum(W; a_1, \ldots, a_s)$ is closed if A is finitely generated, reduced, and $\mathcal{S}(a_1, \ldots, a_s)$ is Zariski dense in $\mathcal{V}(\mathbb{R})$. The following definition is Definition 2.10 in [PSc]:

Definition 3.1. Let A be an $\mathbb{R}$-algebra and $M = \mathrm{QM}(a_1, \ldots, a_s)$ a finitely generated quadratic module in A. M is called *stable*, if for every finite dimensional subspace U of A there is another finite dimensional subspace W of A such that

$$M \cap U \subseteq \sum(W; a_1, \dots, a_s).$$

We call a map that assigns to each finite dimensional subspace U such a finite dimensional subspace W a *stability map for* $a_1, \dots, a_s$.

In polynomial rings, stability just means that we can represent each element a from M with sums of squares of a degree that is bounded by a function of the degree of a.

Of course one has to show that the notion of stability does not depend on the specific choice of generators of M. This is done in [PSc], Lemma 2.9, and can also be found in our next section (Lemma 3.8).

The interest in stability comes from several facts. One of them is Theorem 3.17 from [Sc4], which generalizes Corollary 2.11 from [PSc]:

Theorem 3.2. *Let M be a finitely generated quadratic module in the finitely generated $\mathbb{R}$-algebra A. If M is stable, then*

$$\overline{M} = M + \sqrt{M \cap -M}.$$

Here, $\sqrt{M \cap -M}$ denotes the radical of the ideal $M \cap -M$. If for example A is reduced and $M \cap -M = \{0\}$, then M is closed. This is in particular the case for stable quadratic modules in polynomials rings, whose semi-algebraic set has nonempty interior.

Another fact making stability so interesting is [Sc4], Theorem 5.4, which we state in a slightly weaker version:

Theorem 3.3. *Let M be a finitely generated quadratic module in the polynomial ring $\mathbb{R}[X_1, \dots, X_n]$. If M is stable and the semi-algebraic set $\mathcal{S}(M) \subseteq \mathbb{R}^n$ has dimension at least 2, then M does not have (SMP).*

Further, stability solves the so called *Membership-Problem* for quadratic modules (see [Au]), and allows to use semi-definite optimization to obtain explicit representations of polynomials.

So stability is a very important notion when dealing with quadratic modules. This directs one's attention to the question how to find out, whether a finitely generated quadratic module is stable. In the proof of Theorem 3.5 in [KM], the authors show that a finitely generated quadratic module in $\mathbb{R}[X_1, \ldots, X_n]$ is stable, if its semi-algebraic set in $\mathbb{R}^n$ contains a full dimensional cone (without explicitly using the notion of stability). Theorem 2.14 in [PSc], that appeared at the same time, is a stronger version of that:

Theorem 3.4. *For a finitely generated* $\mathbb{R}$*-algebra* A *suppose that the variety* $V = \mathrm{Spec}(A)$ *is normal. Let* P *be a finitely generated preordering in* A*. Assume* V *has an open embedding into a normal complete* $\mathbb{R}$*-variety* $\overline{V}$ *such that the following is true: For any irreducible component* Z *of* $\overline{V} \setminus V$*, the subset* $\overline{\mathcal{S}(P)} \cap Z(\mathbb{R})$ *is Zariski dense in* Z*, where* $\overline{\mathcal{S}(P)}$ *denotes the closure of* $\mathcal{S}(P)$ *in* $\overline{V}(\mathbb{R})$*. Then* P *is stable and closed.*

See [P1, P2] for a thorough discussion and applications of this result. Our approach is to generalize the idea from the proof of Theorem 3.5 in [KM]. We develop tools for the analysis of cancelling of highest degree terms of polynomials. It turns out that this produces very easy to check conditions for stability. These conditions can be of geometric nature (as in the theorem above), or of more combinatorial one. So it also allows applications to quadratic modules to which Theorem 3.4 does not apply. In addition, the geometric and combinatorial methods can be mixed. On the other hand, our method mostly applies to real domains only. Geometrically, that limits the focus to irreducible varieties. A lot of the results even work in polynomial rings only. However, the ease of application makes up for that to some part.

3.1 Generalized Definition of Stability

During the whole rest of the chapter, let A be a finitely generated $\mathbb{R}$-algebra which is a real domain. That means A does not contain zero divisors and a sum of squares $a_1^2 + \cdots + a_s^2$ is only zero if all a_i are zero.

Let $(\Gamma, \leq)$ be an ordered Abelian group, i.e. an Abelian group Γ with a linear ordering, such that $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$ holds for any $\alpha, \beta, \gamma \in \Gamma$.

Definition 3.5. A *filtration* of A is a family $\{U_\gamma\}_{\gamma\in\Gamma}$ of linear subspaces of A, such that for all $\gamma, \gamma' \in \Gamma$

$$\gamma \leq \gamma' \Rightarrow U_\gamma \subseteq U_{\gamma'},$$

$$U_\gamma \cdot U_{\gamma'} \subseteq U_{\gamma+\gamma'},$$

$$\bigcup_{\gamma\in\Gamma} U_\gamma = A \text{ and}$$

$$1 \in U_0$$

holds.

Definition 3.6. A *grading* of A is a decomposition of the vector space A into a direct sum of linear subspaces:

$$A = \bigoplus_{\gamma\in\Gamma} A_\gamma,$$

such that $A_\gamma \cdot A_{\gamma'} \subseteq A_{\gamma+\gamma'}$ holds for all $\gamma, \gamma' \in \Gamma$.

Any element $0 \neq a \in A$ can then be written in a unique way as

$$a = a_{\gamma_1} + \cdots + a_{\gamma_d}$$

for some $d \in \mathbb{N}$ and $0 \neq a_{\gamma_i} \in A_{\gamma_i}$, where $\gamma_1 < \gamma_2 < \cdots < \gamma_d$. Then $\deg(a) := \gamma_d$ is called the *degree of* a, and $a^{\max} := a_{\gamma_d}$ is called the *highest degree part of* a. Elements from A_γ are called *homogeneous of degree* γ. The degree of 0 is $-\infty$. One easily checks that $1 \in A_0$.

The following are some easy observations: If $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ is a grading, then

$$U_\tau := \bigoplus_{\gamma\leq\tau} A_\gamma$$

defines a filtration $\{U_\tau\}_{\tau\in\Gamma}$ of A. If $\nu\colon K \to \Gamma \cup \{\infty\}$ is a valuation of the quotient field K of A which is trivial on $\mathbb{R}$, then

$$U_\gamma := \{a \in A \mid \nu(a) \geq -\gamma\}$$

defines a filtration $\{U_\gamma\}_{\gamma\in\Gamma}$ of A. If $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ is a grading, then

$$\nu\left(\frac{f}{g}\right) := \deg(g) - \deg(f)$$

defines a valuation on the quotient field K, trivial on $\mathbb{R}$. This valuation induces the same filtration on A as the grading. For any grading and all $a, b \in A$ we have $\deg(a \cdot b) = \deg(a) + \deg(b)$ and

$$\deg(a^2 + b^2) = \max\{\deg(a^2), \deg(b^2)\} = 2\max\{\deg(a), \deg(b)\}.$$

This uses that A is a real domain. We now define stability relative to a filtration.

Definition 3.7. Let $\{U_\gamma\}_{\gamma\in\Gamma}$ be a filtration of A and $a_1, \ldots, a_s$ generators of the quadratic module M. We set $a_0 = 1$.

(1) $a_1, \ldots, a_s$ are called *stable generators of* M *with respect to the filtration*, if there is a monotonically increasing map $\varrho\colon \Gamma \to \Gamma$, such that

$$M \cap U_\gamma \subseteq \sum \left(U_{\varrho(\gamma)}; a_1, \ldots, a_s\right)$$

holds for all $\gamma \in \Gamma$.

(2) $a_1, \ldots, a_s$ are called *strongly stable generators of* M *with respect to the filtration*, if there is a monotonically increasing

map $\varrho\colon \Gamma \to \Gamma$, such that for all sums of squares $\sigma_0, \ldots, \sigma_s$, where $\sigma_i = f_{i,1}^2 + \cdots + f_{i,k_i}^2$, we have

$$\sum_{i=0}^{s} \sigma_i a_i \in U_\gamma \Rightarrow f_{i,j} \in U_{\varrho(\gamma)} \text{ for all } i, j.$$

Obviously, strongly stable generators of M are stable generators of M. The notion of strong stability has also been introduced in [P1], but under a different name. The following Lemma is essentially the same as [PSc], Lemma 2.9.

Lemma 3.8. *If M has stable generators with respect to a given filtration, then any finitely many generators of M are stable generators with respect to that filtration.*

Proof. Suppose $a_1, \ldots, a_s$ are stable generators of M with stability map ϱ as in Definition 3.7 (1). Let $b_1, \ldots, b_t$ be arbitrary generators of M. Then we find representations

$$a_i = \sum_{j=0}^{t} \sigma_j^{(i)} b_j,$$

where all $\sigma_j^{(i)} \in \sum (U_\tau)^2$ for some big enough $\tau \in \Gamma$. Now take $f \in M \cap U_\gamma$ for some γ and find a representation $f = \sum_{i=0}^{s} \sigma_i a_i$ with $\sigma_i \in \sum \left(U_{\varrho(\gamma)}\right)^2$ for all i. Then

$$f = \sum_i \sigma_i a_i = \sum_i \sigma_i \sum_j \sigma_j^{(i)} b_j = \sum_j \left(\sum_i \sigma_i \sigma_j^{(i)} \right) b_j,$$

and all $\sum_i \sigma_i \sigma_j^{(i)}$ are in $\sum \left(U_{\varrho(\gamma)+\tau}\right)^2$. This shows that $b_1, \ldots, b_t$ are stable generators of M with stability map $\gamma \mapsto \varrho(\gamma) + \tau$. □

So it makes sense to talk about stability of a finitely generated quadratic module with respect to a filtration, without mentioning the generators. However, the stability map ϱ may depend on the generators in general. Note also that M is stable in the usual sense (defined in the previous section), if and only if it is stable with respect to a filtration consisting of finite dimensional subspaces U_γ.

Now suppose we are given a grading on A. We will talk about stable generators, strongly stable generators and stable quadratic modules *with respect to the grading*, and always mean these notions with respect to the induced filtration. However, things become easier to handle in this case.

Lemma 3.9. *Let $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ be a grading and let M be a finitely generated quadratic module in A. Then M has strongly stable generators with respect to the grading if and only if there is a monotonically increasing map $\psi\colon \Gamma \to \Gamma$, such that for all $f, g \in M$*

$$\deg(f), \deg(g) \leq \psi\left(\deg(f+g)\right)$$

holds. In particular, if M has strongly stable generators, then any finitely many generators are strongly stable generators.

Proof. Suppose $a_1, \ldots, a_s$ are strongly stable generators of M with stability map ϱ. Take f, g from M with representations $f = \sum_i \sigma_i a_i, g = \sum_i \tau_i a_i$. Then for all j

$$\begin{aligned}\deg\left(\sigma_j a_j\right) &= \deg\left(\sigma_j\right) + \deg\left(a_j\right)\\ &\leq \deg\left(\sigma_j + \tau_j\right) + \deg(a_j)\\ &= \deg(\left(\sigma_j + \tau_j\right) a_j)\\ &\leq \psi\left(\deg\left(\sum_i \left(\sigma_i + \tau_i\right) a_i\right)\right)\\ &= \psi\left(\deg\left(f+g\right)\right),\end{aligned}$$

where the last inequality is fulfilled with

$$\psi(\gamma) := 2\varrho(\gamma) + \max_i \deg(a_i),$$

by the strong stability of the a_i. So $\deg(f) \leq \psi(\deg(f+g))$ holds, and the same is true for g. Note that ψ is monotonically increasing, as ϱ was.

So now suppose $\deg(f), \deg(g) \leq \psi(\deg(f+g))$ for some suitable map ψ and all $f, g \in M$. Take any finitely many (nonzero) generators $a_1, \ldots, a_s$ and sums of squares $\sigma_0, \ldots, \sigma_s$, where $\sigma_j = f_{j,1}^2 + \cdots + f_{j,k_j}^2$. Set $a_0 = 1$. Then

$$\deg(\sigma_j a_j) \leq \psi\left(\deg\left(\sum_i \sigma_i a_i\right)\right)$$

for all j. Thus for all j, l,

$$2\deg(f_{j,l}) \leq \psi\left(\deg\left(\sum_i \sigma_i a_i\right)\right) - \min_i \deg(a_i).$$

So

$$\deg(f_{j,l}) \leq \max\left\{0, \psi\left(\deg\left(\sum_i \sigma_i a_i\right)\right) - \min_i \deg(a_i)\right\}$$

holds. Now $\varrho(\tau) := \max\{0, \psi(\tau) - \min_i \deg(a_i)\}$ defines a monotonically increasing map, and whenever

$$f = \sum_i \sigma_i a_i \in \bigoplus_{\gamma \leq \tau} A_\gamma \text{ for some } \tau,$$

then $\deg(f_{j,l}) \leq \varrho(\deg(f)) \leq \varrho(\tau)$, which shows the strong stability of the $a_1, \ldots, a_s$. The proof indeed shows that any finitely many generators are strongly stable generators in this case. $\square$

So we can talk about strong stability of a quadratic module with respect to a grading, without mentioning the generators. A very special case of strong stability is the following, which will have a nice characterization below.

Definition 3.10. Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a grading and let $M \subseteq A$ be a finitely generated quadratic module. M is *totally stable with respect to the grading*, if

$$\deg(f), \deg(g) \leq \deg(f+g)$$

holds for all $f, g \in M$. The proof of Lemma 3.9 shows that this is equivalent to the fact, that there are generators $a_1, \ldots, a_s$ of M such that

$$\deg(\sigma_j a_j) \leq \deg\left(\sum_i \sigma_i a_i\right) \text{ for all } j$$

holds for all $\sigma_j \in \sum A^2$. All finitely many generators of M fulfill this condition, then.

Note that a quadratic module M in A which is totally stable with respect to any grading has trivial support. Indeed if $f, -f \in M$, then $\deg(f) \leq \deg(f-f) = \deg(0) = -\infty$, so $f = 0$.

If $\nu\colon K \to \Gamma \cup \{\infty\}$ is the valuation corresponding to a given grading, then the notion of total stability is equivalent to saying that for any $f, g \in M$,

$$\nu(f+g) = \min\{\nu(f), \nu(g)\}$$

holds. This is usually called *weak compatibility of* ν *and* M.

3.2 Conditions for Stability

Total stability with respect to a grading turns out to be well accessible. First, when checking total stability of a finitely generated quadratic module, one can apply a reduction result, to obtain possibly smaller quadratic modules. Therefore take generators $a_1, \ldots, a_s$ of M, define an equivalence relation on the generators by saying

$$a_i \equiv a_j :\Leftrightarrow \deg(a_i) \equiv \deg(a_j) \mod 2\Gamma,$$

and group them into equivalence classes

$$\{a_{i1}, \ldots, a_{is_i}\} \ \ (i = 1, \ldots, r).$$

Then total stability reduces to total stability of the quadratic modules generated by these equivalence classes:

Proposition 3.11. *M is totally stable with respect to the given grading if and only if all the quadratic modules*

$$M_i := \mathrm{QM}(a_{i1}, \ldots, a_{is_i})$$

are totally stable.

Proof. The "only if"-part it obvious. For the "if"-part take $f, g \in M$ with representations $f = \sigma_0 + \sigma_1 a_1 + \cdots + \sigma_s a_s$ and $g = \tau_0 + \tau_1 a_1 + \cdots + \tau_s a_s$. By grouping the terms with respect to the equivalence relation and using the total stability of the modules M_i, we get decompositions

$$f = f_1 + \cdots + f_r, \quad g = g_1 + \cdots + g_r$$

with $f_i, g_i \in M_i$ and all the f_i (as well as the g_i) have a different degree modulo 2Γ. So if f and g have the same degree and $\deg(f) = \deg(f_k), \deg(g) = \deg(g_l)$, then $k = l$ and the highest degree parts of f and g cannot cancel out, due to the total stability of M_k. □

Now total stability has the following easy characterization:

Proposition 3.12. *Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a grading and let M be a finitely generated quadratic module in A. Let $a_1, \ldots, a_s$ be generators of M. Then*

$$M \textit{ is totally stable} \Leftrightarrow \mathrm{supp}\,(\mathrm{QM}(a_1^{\max}, \ldots, a_s^{\max})) = \{0\}.$$

Proof. First suppose $\operatorname{supp}(\mathrm{QM}(a_1^{\max},\dots,a_s^{\max}))\neq\{0\}$. So there are sums of squares $\sigma_0,\dots,\sigma_s$, not all zero, such that $\sum_{i=0}^s \sigma_i a_i^{\max}=0$. Now

$$\begin{aligned}\deg\left(\sum_{i=0}^s \sigma_i a_i\right) &= \deg\left(\sum_{i=0}^s \sigma_i(a_i-a_i^{\max})\right)\\ &\leq \max_i\left\{\deg(\sigma_i(a_i-a_i^{\max})\right\}\\ &< \max_i\left\{\deg(\sigma_i a_i)\right\},\end{aligned}$$

so M is not totally stable. Conversely, for any sum of squares σ_j, the highest degree part of $\sigma_j a_j$ lies in $\mathrm{QM}(a_1^{\max},\dots,a_s^{\max})$. So when adding elements of the form $\sigma_i a_i$, the highest degree parts cannot cancel out, if $\operatorname{supp}(\mathrm{QM}(a_1^{\max},\dots,a_s^{\max}))=\{0\}$. So M is totally stable. □

The good thing about Proposition 3.12 is, that it allows to link total stability to a geometric condition, via Proposition 1.5:

Theorem 3.13. *Let $A=\bigoplus_{\gamma\in\Gamma}A_\gamma$ be a grading and M a finitely generated quadratic module in A. If for a set of generators $a_1,\dots,a_s$ of M, the set*

$$\mathcal{S}(a_1^{\max},\dots,a_s^{\max})\subseteq\mathcal{V}(\mathbb{R})$$

is Zariski dense, then M is totally stable with respect to the grading. If M is closed under multiplication, then total stability implies the Zariski denseness for any finite set of generators of M.

Proof. If $\mathcal{S}(a_1^{\max},\dots,a_s^{\max})$ is Zariski dense, then

$$\operatorname{supp}(\mathrm{QM}(a_1^{\max},\dots,a_s^{\max}))=\{0\},$$

by Proposition 1.5 (note that A is real). So Proposition 3.12 yields the total stability of M. If M is a preordering, generated by $a_1,\dots,a_s$ as a quadratic module, and totally stable, then $\mathrm{QM}(a_1^{\max},\dots,a_s^{\max})$ is also a preordering. So Propositions 3.12 and 1.5 imply the denseness of $\mathcal{S}(a_1^{\max},\dots,a_s^{\max})$ in $\mathcal{V}(\mathbb{R})$. □

Note that if M is a finitely generated quadratic module which is closed under multiplication, and $b_1, \dots, b_t$ generate M as a *preordering*, then the products $b_e := b_1^{e_1} \cdots b_t^{e_t}$ $(e \in \{0,1\}^t)$ generate M as a quadratic module, and

$$\mathcal{S}(b_1^{\max}, \dots, b_t^{\max}) = \mathcal{S}\left(b_e^{\max} \mid e \in \{0,1\}^t\right).$$

In the next section we will consider different kinds of gradings on the polynomial ring $A = \mathbb{R}[X_1, \dots, X_n]$. The denseness condition from Theorem 3.13 will be translated into a geometric condition on the original set $\mathcal{S}(M)$.

Recall that we are mostly interested in stability of a finitely generated quadratic module in the sense of [PSc] (see the previous section), that is, stability with respect to a filtration of finite dimensional subspaces. Many of the later considered gradings do *not* induce such finite dimensional filtrations. Our goal is then to find stability with respect to enough different gradings, so that in the end the desired stability is still obtained. Therefore we consider the following setup:

Let $\Gamma, \Gamma_1, \dots, \Gamma_m$ be ordered Abelian groups and let

$$\{W_\gamma\}_{\gamma\in\Gamma}, \left\{U_\gamma^{(j)}\right\}_{\gamma\in\Gamma_j} (j = 1, \dots, m)$$

be filtrations of A.

Definition 3.14. The filtration $\{W_\gamma\}_{\gamma\in\Gamma}$ is *covered* by the filtrations

$$\left\{U_\gamma^{(j)}\right\}_{\gamma\in\Gamma_j} (j = 1, \dots, m),$$

if there are monotonically increasing maps

$$\eta\colon \Gamma_1 \times \cdots \times \Gamma_m \to \Gamma,\ \eta_j\colon \Gamma \to \Gamma_j\ (j = 1, \dots, m),$$

such that for all $\gamma \in \Gamma, \gamma_j \in \Gamma_j$ $(j = 1, \dots, m)$, the following holds:

$$W_\gamma \subseteq \bigcap_{j=1}^{m} U_{\eta_j(\gamma)}^{(j)} \quad \text{and}$$

$$\bigcap_{j=1}^{m} U^{(j)}_{\gamma_j} \subseteq W_{\eta(\gamma_1,\dots,\gamma_m)}.$$

For η, *monotonically increasing* refers to the *partial* ordering on the product group obtained by the componentwise orderings of the factors.

We will speak about covering of/by *gradings*, and mean the notion from Definition 3.14 applied to the induced filtrations. The next theorem makes clear why we are interested in coverings.

Theorem 3.15. *Suppose a quadratic module M in A has generators $a_1, \dots, a_s$, which are strongly stable generators with respect to all the filtrations*

$$\left\{U^{(j)}_\gamma\right\}_{\gamma\in\Gamma_j} (j = 1, \dots, m).$$

Then $a_1, \dots, a_s$ are also strongly stable generators of M with respect to any filtration $\{W_\gamma\}_{\gamma\in\Gamma}$ which is covered by these filtrations.

Proof. For every $j = 1, \dots, m$, take a stability map ϱ_j for the generators with respect to the filtration $\left\{U^{(j)}_\gamma\right\}_{\gamma\in\Gamma_j}$ (remember Definition 3.7(2)). As in Definition 3.14 , the covering maps are denoted by η and η_j.

Take sums of squares $\sigma_0, \dots, \sigma_s$, where $\sigma_i = f^2_{i,1} + \cdots + f^2_{i,k_i}$ and suppose $\sum_{i=0}^{s} \sigma_i a_i \in W_\gamma$ for some $\gamma \in \Gamma$. Then $\sum_{i=0}^{s} \sigma_i a_i \in U^{(j)}_{\eta_j(\gamma)}$ for all j. So by strong stability,

$$f_{i,l} \in U^{(j)}_{\varrho_j(\eta_j(\gamma))} \text{ for all } j, i, l.$$

But then

$$f_{i,l} \in W_{\eta(\varrho_1(\eta_1(\gamma)),\dots,\varrho_m(\eta_m(\gamma)))} \text{ for all } i, l,$$

which shows the strong stability with respect to $\{W_\gamma\}_{\gamma\in\Gamma}$. □

Before we apply these results, we conclude this section with an easy observation. It will be helpful in generalizing an interesting result from [CKS] later on. For this we drop the assumption that all algebras are real domains. We consider arbitrary $\mathbb{R}$-algebras A, B and an $\mathbb{R}$-algebra homomorphism $\varphi\colon A \to B$. Of course, the definitions of stability still apply.

Proposition 3.16. *Let $S \subseteq \mathcal{V}_A(\mathbb{R})$ be basic closed semi-algebraic. Suppose every finitely generated quadratic module in B describing $(\varphi^*)^{-1}(S)$ has only strongly stable generators with respect to a fixed filtration $(U_\gamma)_{\gamma\in\Gamma}$ on B. Then every finitely generated quadratic module in A describing S has only strongly stable generators with respect to the induced filtration $\big(\varphi^{-1}(U_\gamma)\big)_{\gamma\in\Gamma}$.*

Proof. Let $a_1, \ldots, a_s$ be generators of a quadratic module M in A with $\mathcal{S}(M) = S$. Take sums of squares $\sigma_0, \ldots, \sigma_s$ with $\sigma_i = f_{i,1}^2 + \cdots + f_{i,k_i}^2$ and assume

$$\sigma_0 + \sigma_1 a_1 + \cdots + \sigma_s a_s \in \varphi^{-1}(U_\gamma) \tag{8}$$

for some $\gamma \in \Gamma$. The elements $\varphi(a_1), \ldots, \varphi(a_s)$ generate a quadratic module in B that describes $(\varphi^*)^{-1}(S)$. So they are strongly stable generators and we denote the corresponding stability map by ϱ. Applying φ to (8) and using the strong stability yields $\varphi(f_{i,j}) \in U_{\varrho(\gamma)}$ for all i, j, and therefore all $f_{i,j} \in \varphi^{-1}(U_{\varrho(\gamma)})$. This shows that $a_1, \ldots, a_s$ are strongly stable generators (with stability map ϱ). □

We will apply the result at the end of the following section.

3.3 Applications

In this section we apply the results from the previous section, mostly in the polynomial ring $A = \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \ldots, X_n]$. We identify $\mathcal{V}_A(\mathbb{R})$ with $\mathbb{R}^n$ in the usual way. We start by defining a class of useful gradings.

For $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ we write

$$\underline{X}^\delta := X_1^{\delta_1} \cdots X_n^{\delta_n}$$

and

$$z \circ \delta := z_1\delta_1 + \cdots + z_n\delta_n.$$

For $d \in \mathbb{Z}$ define

$$A_d^{(z)} := \left\{ \sum_{\delta \in \mathbb{N}^n,\ z\circ\delta = d} c_\delta \underline{X}^\delta \mid c_\delta \in \mathbb{R} \right\}.$$

Then

$$A = \bigoplus_{d \in \mathbb{Z}} A_d^{(z)}$$

is a grading indexed in the ordered group $(\mathbb{Z}, \leq)$, to which we will refer to as the *z-grading*. For example, $z = (1, \ldots, 1)$ gives rise to the usual degree-grading on A, whereas $z = (1, 0, \ldots, 0)$ defines the grading with respect to the usual degree in X_1. Note that the filtration induced by such a z-grading consists of finite dimensional linear subspaces of A if and only if all entries of z are positive.

We want to characterize the denseness condition from Theorem 3.13 for these z-gradings. For a compact set K in $\mathbb{R}^n$ with nonempty interior, we define the *tentacle* in direction of z in the following way:

$$T_{K,z} := \{(\lambda^{z_1}x_1, \ldots, \lambda^{z_n}x_n) \mid \lambda \geq 1,\ x = (x_1, \ldots, x_n) \in K\}.$$

For $z = (1, \ldots, 1)$, such a set is just a full dimensional cone in $\mathbb{R}^n$. For $z = (1, 0, \ldots, 0)$ it is a full dimensional cylinder going to infinity in the direction of x_1. For $z = (1, -1) \in \mathbb{Z}^2$, something like the set defined by $xy \leq 2, xy \geq 1$ and $x \geq 1$ would be such a set.

Proposition 3.17. *Let $a_1, \ldots, a_s$ be polynomials in the graded polynomial ring $\mathbb{R}[\underline{X}] = \bigoplus_{d \in \mathbb{Z}} A_d^{(z)}$, where $z \in \mathbb{Z}^n$. Then the set*

$$\mathcal{S}(a_1^{\max}, \ldots, a_s^{\max}) \subseteq \mathbb{R}^n$$

is Zariski-dense in $\mathbb{R}^n$, if and only if the set

$$\mathcal{S}(a_1, \ldots, a_s) \subseteq \mathbb{R}^n$$

contains a set $T_{K,z}$ for some compact $K \subseteq \mathbb{R}^n$ with nonempty interior.

Proof. First suppose $\mathcal{S}(a_1^{\max}, \ldots, a_s^{\max})$ is Zariski-dense, which is equivalent to saying that there is a compact set K with nonempty interior, on which all $a_i^{\max}$ are positive. Write each a_i as a sum of homogeneous elements (with respect to the z-grading), for example

$$a_1 = a_{d_1} + \ldots + a_{d_t},$$

where $d_1 < \ldots < d_t$ and $0 \neq a_{d_i} \in A_{d_i}^{(z)}$. Then for $x \in \mathbb{R}^n$ and $\lambda > 0$

$$a_1(\lambda^{z_1} x_1, \ldots, \lambda^{z_n} x_n) = \lambda^{d_1} a_{d_1}(x) + \ldots + \lambda^{d_t} a_{d_t}(x).$$

As $a_{d_t}(x) = a_1^{\max}(x) > 0$ if x is taken from K, the expression is positive for $\lambda \geq N$ with N big enough. Thereby N can be chosen to depend only on the size of the coefficients $a_{d_i}(x)$. So N can be chosen big enough to make $a_i(\lambda^{z_1} x_1, \ldots, \lambda^{z_n} x_n)$ positive for all $\lambda \geq N$, $x \in K$ and all $i = 1, \ldots, s$. Replacing K by

$$K' := \{(N^{z_1} x_1, \ldots, N^{z_n} x_n) \mid x = (x_1, \ldots, x_n) \in K\}$$

we find $T_{K',z} \subseteq \mathcal{S}(a_1, \ldots, a_s)$.

Conversely, suppose $\mathcal{S}(a_1, \ldots, a_s)$ contains a set $T_{K,z}$. Then all the highest degree parts of the a_i must be nonnegative on K, with the same argument as above. So $\mathcal{S}(a_1^{\max}, \ldots, a_s^{\max})$ contains K and is therefore Zariski-dense in $\mathbb{R}^n$. □

Combined with Theorem 3.13 we get:

Theorem 3.18. *Let $a_1, \ldots, a_s$ be polynomials in the graded polynomial ring $\mathbb{R}[\underline{X}] = \bigoplus_{d \in \mathbb{Z}} A_d^{(z)}$, where $z \in \mathbb{Z}^n$. If the set*

$$\mathcal{S}(a_1, \ldots, a_s) \subseteq \mathbb{R}^n$$

contains some tentacle $T_{K,z}$ (K compact with nonempty interior), then the quadratic module $\mathrm{QM}(a_1, \ldots, a_s)$ is totally stable. If $\mathrm{QM}(a_1, \ldots, a_s)$ is a preordering and totally stable, then $\mathcal{S}(a_1, \ldots, a_s)$ contains such a tentacle.

For the z-gradings, we can also settle the questions of coverings:

Proposition 3.19. *Let $z, z^{(1)}, \ldots, z^{(m)} \in \mathbb{Z}^n$ and assume there exist numbers $r_1, \ldots, r_m, t_1, \ldots, t_m \in \mathbb{N}$, such that the following conditions hold (where $v \succeq w$ means $\geq$ in each component of the vectors v, w in $\mathbb{Z}^n$):*

$$r_1 z^{(1)} + \cdots + r_m z^{(m)} \succeq z \text{ and}$$

$$t_j z \succeq z^{(j)} \text{ for } j = 1, \ldots, m.$$

Then the z-grading on $\mathbb{R}[\underline{X}]$ is covered by the $z^{(j)}$-gradings.

Proof. We denote by $\deg(f)$ and $\deg^{(j)}(f)$ the degree of a polynomial f with respect to the z- and the $z^{(j)}$-grading, respectively. First take a polynomial f and suppose $\deg(f) \leq d$ for $d \in \mathbb{Z}$. So for every monomial $c\underline{X}^\delta$ occurring in f we have $z \circ \delta \leq d$. Now for every $j = 1, \ldots, m$,

$$z^{(j)} \circ \delta \leq t_j \, (z \circ \delta) \leq t_j d,$$

so $\deg^{(j)}(f) \leq t_j d$. Thus $\psi_j \colon \mathbb{Z} \to \mathbb{Z}; d \mapsto t_j d$ fulfills the condition from Definition 3.14.

Now suppose $\deg^{(j)}(f) \leq d_j$ for $d_j \in \mathbb{Z}$ and $j = 1, \ldots, m$. Now for every monomial $c\underline{X}^\delta$ occurring in f,

$$z \circ \delta \leq r_1 \left(z^{(1)} \circ \delta\right) + \cdots + r_m \left(z^{(m)} \circ \delta\right) \leq r_1 d_1 + \cdots + r_m d_m$$

holds. So $\psi\colon \mathbb{Z}^m \to \mathbb{Z}; (d_1, \ldots, d_m) \mapsto r_1 d_1 + \cdots + r_m d_m$ fulfills the other condition from Definition 3.14. □

For example, the usual grading ($z = (1, \ldots, 1)$) is covered by the gradings defined by

$$z^{(1)} = (1, 0, \ldots, 0), z^{(2)} = (0, 1, 0, \ldots, 0), \ldots, z^{(n)} = (0, \ldots, 0, 1).$$

For $n = 2$, the two gradings defined by

$$z^{(1)} = (0, 1), z^{(2)} = (1, -1)$$

also cover the usual grading.

Combining Proposition 3.19, Theorem 3.15 and Theorem 3.18, we get geometric conditions for stability in the sense of [PSc]. Indeed, take a covering of the usual grading by some z-gradings. For all the z-gradings we have a geometric interpretation of total stability (Theorem 3.18). So Theorem 3.15 yields a geometric condition for (strong) stability with respect to the usual grading:

Theorem 3.20. *Let $S \subseteq \mathbb{R}^n$ be a basic closed semi-algebraic set that contains sets $T_{K_j, z^{(j)}}$, where K_j is compact with nonempty interior and $z^{(j)} \in \mathbb{Z}^n$ ($j = 1, \ldots, m$). If there exist $r_1, \ldots, r_m \in \mathbb{N}$ such that*

$$r_1 z^{(1)} + \cdots + r_m z^{(m)} \succ 0,$$

then any finitely generated quadratic module describing S is stable and closed. If $n \geq 2$, (SMP) is not finitely solvable for S. Such natural numbers r_i exist, if and only if the only polynomial functions bounded on

$$\bigcup_{j=1}^{m} T_{K_j, z^{(j)}}$$

are the reals.

Proof. The first part of the theorem is clear from the above results. We only have to prove the part concerning the bounded polynomial functions. Note that a polynomial f is bounded on a set $T_{K,z}$, if and only if it has degree less or equal to 0 with respect to the z-grading. This follows easily, using the ideas from the proof of Proposition 3.17, and the fact that K is compact and has nonempty interior. So in case there are natural numbers $r_1, \ldots, r_m \in \mathbb{N}$ with

$$r_1 z^{(1)} + \cdots + r_m z^{(m)} \succ 0,$$

there is no nontrivial monomial $\underline{X}^\delta$ that has degree less or equal to 0 with respect to all the $z^{(j)}$-gradings. As all the monomials are homogeneous elements, there can be no nontrivial polynomial bounded on $\bigcup_{j=1}^m T_{K_j,z^{(j)}}$.

Conversely, assume there do *not* exists suitable numbers r_i. Then, by a Theorem of the Alternative (see for example [A], Lemma 1.2), there must be $\delta \in \mathbb{N}^n \setminus \{0\}$, such that

$$\delta \circ z^{(j)} \leq 0$$

for all j. But this means that the (nontrivial) monomial $\underline{X}^\delta$ is bounded on $\bigcup_{j=1}^m T_{K_j,z^{(j)}}$. □

Another class of gradings on the polynomial ring A is given by term-orders. A term order is defined to be a linear ordering $\leq$ on $\mathbb{N}^n$ which fulfills

$$\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$$

for all $\alpha, \beta, \gamma \in \mathbb{N}^n$. Such a term order extends in a canonical way to an ordering of the Abelian group $\mathbb{Z}^n$. Indeed write $\gamma \in \mathbb{Z}^n$ as a difference $\alpha - \beta$ of elements from $\mathbb{N}^n$; then define $\gamma \geq 0$ if and only if $\alpha \geq \beta$.

We have a grading

$$A = \bigoplus_{\gamma \in \mathbb{Z}^n} A_\gamma^{(\leq)},$$

where $A_\gamma^{(\leq)} := \mathbb{R} \cdot \underline{X}^\gamma$ if $\gamma \in \mathbb{N}^n$ and $A_\gamma^{(\leq)} := \{0\}$ otherwise. We refer to this grading as the $\leq$-grading. The decomposition of a polynomial $f \in \mathbb{R}[\underline{X}]$ is

$$f = c_{\gamma_1}\underline{X}^{\gamma_1} + \cdots + c_{\gamma_t}\underline{X}^{\gamma_t},$$

where $c_{\gamma_i} \neq 0$ are the coefficients of f and $\gamma_1 < \cdots < \gamma_t$ with respect to the term order. The degree of f is γ_t then, and the highest degree part is the monomial $c_{\gamma_t}\underline{X}^{\gamma_t}$. Now for these term order gradings, the question of total stability is easy to solve. First we apply the reduction result from Proposition 3.11 to the generators of the quadratic module. So we can assume that all the generators have the same degree mod $2\mathbb{Z}^n$. The highest degree parts of the generators are then monomials $c_\gamma\underline{X}^\gamma$, where all the γ are congruent modulo $2\mathbb{Z}^n$. So obviously the quadratic module is totally stable if and only if all the occurring coefficients c_γ have the same sign, and are positive in case the γ are congruent 0 modulo $2\mathbb{Z}^n$. So this gives an easy to apply method to decide total stability of a quadratic module with respect to a term order grading.

Note that not all of these $\leq$-gradings induce filtrations with finite dimensional linear subspaces. For example, a lexicographical ordering on $\mathbb{N}^n$ does not. However, if we first sort by the norm $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and then lexicographically, the subspaces are finite dimensional.

These term order gradings can show stability of quadratic modules, where the purely geometric conditions derived above and in [PSc] do not apply. So they allow to take into account the difference between quadratic modules and preorderings. See Chapter 5 for examples.

For algebras A other than the polynomial ring, it is not so obvious how to get gradings. However, if an ideal I in the graded $\mathbb{R}$-algebra $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ is generated by homogeneous elements, the factor algebra A/I carries a grading

$$A/I = \bigoplus_{\gamma\in\Gamma} \overline{A}_\gamma,$$

where $\overline{A}_\gamma$ consists of the residue classes of the elements from A_γ.

For example, the polynomial $1-X_1X_2$ is homogeneous with respect to the $(1,-1)$- as well as the $(-1,1)$-grading on $\mathbb{R}[X_1,X_2]$, defined above. So these two gradings push down to

$$A = \mathbb{R}[X_1,X_2]/(1-X_1X_2),$$

and they cover the usual finite dimensional filtration, obtained by pushing down the canonical filtration (by degree) of $\mathbb{R}[X_1,X_2]$. Note also that A is a real domain.

We conclude this section with an application of Proposition 3.16, namely a generalization of Theorem 6.23 from [CKS]. We briefly recall the setup of that article and refer to it for more detailed information. Consider a finitely generated and reduced $\mathbb{R}$-algebra B with affine $\mathbb{R}$-variety $\mathcal{V}_B$ and its set of real points $\mathcal{V}_B(\mathbb{R})$. Then B equals $\mathbb{R}[\mathcal{V}_B]$, the algebra of real regular functions on $\mathcal{V}_B$. Let $\mathcal{G}$ be a linear algebraic group defined over $\mathbb{R}$, acting on $\mathcal{V}_B$ by means of $\mathbb{R}$-morphisms. Then $\mathcal{G}(\mathbb{R})$ acts canonically on $B=\mathbb{R}[\mathcal{V}_B]$, and if $\mathcal{G}(\mathbb{R})$ is compact, the set of invariant regular functions, denoted by $A=\mathbb{R}[\mathcal{V}_B]^{\mathcal{G}}$, is a finitely generated $\mathbb{R}$-algebra. So it corresponds to an affine $\mathbb{R}$-variety $\mathcal{V}_A$ and the inclusion $\iota\colon A=\mathbb{R}[\mathcal{V}_B]^{\mathcal{G}} \hookrightarrow \mathbb{R}[\mathcal{V}_B]=B$ corresponds to a morphism $\mathcal{V}_B\to\mathcal{V}_A$. The restricted morphism $\iota^*\colon \mathcal{V}_B(\mathbb{R})\to\mathcal{V}_A(\mathbb{R})$ can be seen as the orbit map of the group action, by a Theorem by Procesi, Schwarz and Bröcker. Indeed, the nonempty fibers are precisely the $\mathcal{G}(\mathbb{R})$-orbits. Furthermore, for any basic closed semi-algebraic set S in $\mathcal{V}_B(\mathbb{R})$, the set $\iota^*(S)$ is basic closed semi-algebraic in $\mathcal{V}_A(\mathbb{R})$. The affine variety $\mathcal{V}_A$ is denoted by $\mathcal{V}_B//\mathcal{G}$.

Now suppose $S\subseteq\mathcal{V}_B(\mathbb{R})$ is $\mathcal{G}$-invariant. Then one can look at the invariant moment problem for S. That is, one wants to find

a finitely generated quadratic module $M \subseteq \mathbb{R}[\mathcal{V}_B]^{\mathcal{G}}$, such that every linear functional L on B, which is invariant under the action of $\mathcal{G}(\mathbb{R})$, and which is nonnegative on M, is integration with respect to a measure on S. One of the main results from [CKS] concerning the invariant moment problem is, that this is possible if and only if M defines $\iota^*(S)$ in $\mathcal{V}_A(\mathbb{R})$ and has (SMP) in A (Lemma 6.9 in [CKS]). The situation in $\mathcal{V}_A(\mathbb{R})$ is often simpler than the one in $\mathcal{V}_B(\mathbb{R})$, and so the invariant moment problem can be solved in cases where the strong moment problem cannot.
However, Theorem 6.23 in [CKS] yields a negative result about the invariant moment problem. Roughly spoken, it says that if (SMP) is not finitely solvable due to some geometric conditions on S, then the invariant moment problem is not solvable either. The result is proven for finite groups $\mathcal{G}$ and irreducible varieties only. The following result holds for arbitrary compact groups.

Theorem 3.21. *Let the compact group $\mathcal{G}$ act on the affine variety $\mathcal{V}_B$ and let S be a $\mathcal{G}(\mathbb{R})$-invariant basic closed semi-algebraic set in $\mathcal{V}_B(\mathbb{R})$. Fix a filtration of finite dimensional subspaces of B, and assume that every finitely generated quadratic module in B describing S has only strongly stable generators with respect to that filtration. Then every finitely generated quadratic module in $A = \mathbb{R}[\mathcal{V}_B]^{\mathcal{G}}$ describing $\iota^*(S)$ has only strongly stable generators with respect to the induced filtration on A.*
In particular, if $\dim(\iota^*(S)) \geq 2$, then (SMP) is not finitely solvable for $\iota^*(S)$. So the invariant moment problem is not finitely solvable for S in that case.

Proof. If $\mathrm{QM}(a_1, \ldots, a_s) \subseteq A$ describes $\iota^*(S)$, then

$$\mathrm{QM}(\iota(a_1), \ldots, \iota(a_s)) \subseteq B$$

describes $S = (\iota^*)^{-1}(\iota^*(S))$. This uses that the fibres of ι^* are precisely the $\mathcal{G}(\mathbb{R})$-orbits and that S is $\mathcal{G}(\mathbb{R})$-invariant. Now apply Proposition 3.16. □

One checks that the geometric conditions from Theorem 6.24 in [CKS] imply, that the conditions from our Theorem 3.21 are fulfilled. Note also that the geometric conditions obtained above always imply the strong stability of any finite set of generators for S. So Theorem 3.21 yields a negative result concerning the invariant moment problem in all of these cases.

4 The Sequential Closure

In this chapter we deal with the ‡-property of quadratic modules. Remember that for a set C in a real vector space E, $C^{\ddagger}$ is defined to be the sequential closure of C with respect to the finest locally convex topology on E. It is contained in the closure $\overline{C}$ and can be characterized as the union of all finite dimensional closures of C (see the remark following Lemma 1.1). If C is a convex cone, by Proposition 1.3, $C^{\ddagger}$ consists of all elements f of E for which there is some $q \in E$, such that

$$f + \varepsilon q \in C \text{ for all } \varepsilon > 0.$$

We examine the sequence of iterated sequential closures of a convex cone C in E. Therefore define

$$C^{(0)} := C,\ C^{(\xi+1)} := \left(C^{(\xi)}\right)^{\ddagger}$$

for ordinals ξ, and

$$C^{(\mu)} := \bigcup_{\xi<\mu} C^{(\xi)}$$

for limit ordinals μ. Define

$$\ddagger(C) := \xi,$$

where ξ is the least ordinal such that $C^{(\xi)} = C^{(\xi+1)}$, and call it the ‡-*index of* C. If the vector space has countable dimension, then a set C is closed if and only if it is sequentially closed, by Proposition 1.2 and Lemma 1.1. So the transfinite sequence of iterated sequential closures of C terminates exactly at $\overline{C}$, and $\ddagger(C)$ is the least ordinal ξ such that $C^{(\xi)} = \overline{C}$.

A quadratic module M in an $\mathbb{R}$-algebra is said to have the ‡-property, if

$$M^{\ddagger} = M^{\text{sat}}$$

holds. This property implies (SMP) for M, and that was one of the reasons the authors of [KM, KMS] introduced and examined it. One of the most important results of these works concerning the ‡-property is Theorem 5.3 in [KMS]. It says that a quadratic module defining a cylinder with compact cross section has the ‡-property, under reasonable assumptions on the generators.

It was an open problem whether the ‡-property and (SMP) are equivalent for quadratic modules or preorderings. It was even unknown whether $M^{\ddagger} = \overline{M}$ could be true for quadratic modules or preorderings in general. We solve these questions to the negative, by providing a finitely generated preordering in the polynomial ring of two variables, that has (SMP) but not the ‡-property. This is done in Section 4.2. Section 4.3 contains some short remarks about the sequential closure on quotient algebras. In Section 4.4 we prove a fibre theorem that allows to check the ‡-property by looking at lower dimensional problems. The idea is to generalize the proof of Theorem 5.3 from [KMS]. However, we start by examining convex cones in countable dimensional vector spaces.

4.1 Examples of Sequential Closures of Convex Cones

During this section let

$$E = \bigoplus_{i=0}^{\infty} \mathbb{R} \cdot e_i = \{(f_i)_{i\in\mathbb{N}} \mid f_i \in \mathbb{R},\ \text{only finitely many } f_i \neq 0\}$$

be a countable dimensional $\mathbb{R}$-vector space. For $m \in \mathbb{N} \setminus \{0\}$ we write

$$W_m := \bigoplus_{i=0}^{m-1} \mathbb{R} \cdot e_i,$$

so the increasing sequence $(W_m)_{m\in\mathbb{N}}$ of finite dimensional subspaces exhausts the whole space E. In the following we construct examples of sets and convex cones with different ‡-indices.

For $n \in \{1, 2, \ldots\}$ and $l = (l_0, l_1, \ldots, l_n) \in (\mathbb{N} \setminus \{0\})^{n+1}$ define

$$V(l) := \underbrace{[\frac{1}{l_1}, 1] \times \cdots \times [\frac{1}{l_1}, 1]}_{l_0 \text{ times}} \times \underbrace{[\frac{1}{l_2}, 1] \times \cdots \times [\frac{1}{l_2}, 1]}_{l_1 \text{ times}} \times \cdots$$
$$\times \underbrace{[\frac{1}{l_n}, 1] \times \cdots \times [\frac{1}{l_n}, 1]}_{l_{n-1} \text{ times}}.$$

$V(l)$ is a compact subset of $W_{l_0+\cdots+l_{n-1}}$. Let

$$U(l) := V(l) \times \bigoplus_{i=l_0+\cdots+l_{n-1}}^{\infty} [0,1] \cdot e_i,$$

so $U(l) \subseteq E$ and $U(l) \cap W_m$ is compact for every $m \in \mathbb{N}$; indeed nonempty if and only if $m \geq l_0 + \cdots + l_{n-1}$. Now define

$$M_n := \bigcup_{l \in (\mathbb{N} \setminus \{0\})^{n+1}} U(l).$$

The intention behind this is, that M_n contains n "steps", and the application of the ‡-operator removes one at a time.

We have for $m \geq n \geq 2$

$$\overline{M_n \cap W_m} \subseteq M_{n-1}.$$

To see this take a converging sequence $(x_i)_i$ from $M_n \cap W_m$. So for each x_i there is some $l^{(i)} \in (\mathbb{N} \setminus \{0\})^{n+1}$ such that $x_i \in U(l^{(i)})$. As $U(l) \cap W_m$ is only nonempty if $l_0 + \cdots + l_{n-1} \leq m$, we can assume without loss of generality (by choosing a subsequence), that the $l^{(i)}$ coincide in all but the last component. This shows that the limit of the sequence $(x_i)_i$ belongs to M_{n-1} (indeed to $U(l_0^{(i)}, \ldots, l_{n-1}^{(i)}) \cap W_m$).

So $(M_n)^{\ddagger} \subseteq M_{n-1}$, and the other inclusion is obvious. We thus have for $n \geq 2$:

$$(M_n)^{\ddagger} = M_{n-1}.$$

In addition,

$$M_1 \subsetneq M_1^{\ddagger} = \bigoplus_{i=0}^{\infty} [0,1] \cdot e_i,$$

which is closed. This shows $\ddagger(M_n) = n$ and $\overline{M_n} = \bigoplus_{i=0}^{\infty} [0,1] \cdot e_i$ for all $n \geq 1$.

Let $\mathrm{cc}(M_n)$ denote the convex cone generated by M_n, i.e. $\mathrm{cc}(M_n)$ consists of all finite positive combinations of elements from M_n, including 0. We have for $n \geq 2$

$$\mathrm{cc}(M_n)^{\ddagger} = \mathrm{cc}(M_{n-1}).$$

To see "$\subseteq$" suppose $x \in \mathrm{cc}(M_n)^{\ddagger}$. Then we have a sequence $(x_i)_i$ in some $\mathrm{cc}(M_n) \cap W_m = \mathrm{cc}(M_n \cap W_m)$ that converges to x in W_m. Write

$$x_i = \lambda_1^{(i)} a_1^{(i)} + \cdots \lambda_N^{(i)} a_N^{(i)}$$

with all $a_j^{(i)} \in M_n \cap W_m$ and all $\lambda_j^{(i)} \geq 0$. We can choose the same sum length N for all x_i, by the conic version of Carathéodory's Theorem (see for example [Ba], Problem 6, p. 65). By choosing a subsequence of $(x_i)_i$ we can assume that for all $j \in \{1, \ldots, N\}$ the sequence $(a_j^{(i)})_i$ converges to some element a_j. This uses $M_n \cap W_m \subseteq [0,1]^m$. All elements a_j lie in $M_n^{\ddagger} = M_{n-1}$. As $n \geq 2$, the first component of each element $a_j^{(i)}$ is at least $\frac{1}{m}$. So all the sequences $(\lambda_j^{(i)})_i$ are bounded and therefore without loss of generality also convergent. This shows that x belongs to $\mathrm{cc}(M_{n-1})$.

To see "$\supseteq$" note that $M_n^{\ddagger} \subseteq \mathrm{cc}(M_n)^{\ddagger}$ and $\mathrm{cc}(M_n)^{\ddagger}$ is a convex cone. So

$$\mathrm{cc}(M_{n-1}) = \mathrm{cc}(M_n^{\ddagger}) \subseteq \mathrm{cc}(M_n)^{\ddagger}.$$

For $n = 1$ we have

$$\mathrm{cc}(M_1) = \left\{ f = (f_i)_i \in \bigoplus_{i=0}^{\infty} \mathbb{R}_{\geq 0} \cdot e_i \mid f_0 = 0 \Rightarrow f = 0 \right\},$$

so

$$\mathrm{cc}(M_1) \subsetneq \mathrm{cc}(M_1)^{\ddagger} = \bigoplus_{i=0}^{\infty} \mathbb{R}_{\geq 0} \cdot e_i,$$

which is closed. So we have:

Example 4.1. For $n \in \{1, 2, \ldots\}$, the convex cone $\mathrm{cc}(M_n)$ fulfills

$$\ddagger(\mathrm{cc}(M_n)) = n$$

and

$$\overline{\mathrm{cc}(M_n)} = \bigoplus_{i=0}^{\infty} \mathbb{R}_{\geq 0} \cdot e_i.$$

We proceed and now want to construct a set that does not have a finite ‡-index. The idea is to unite all the sets M_n with enough distance between them, so that the ‡-operator applies to each of them separately.

First note that for any set $A \subseteq E$ and any $x \in E$ we have

$$(x + A)^{\ddagger} = x + A^{\ddagger}.$$

This follows directly from continuity of addition. So with the above defined notions, for $x \in E$ and $k, n \in \{1, 2 \ldots\}$ we have

$$x \in (x + M_n)^{(k)} \Leftrightarrow k \geq n.$$

Indeed $x \in (x + M_n)^{(k)} = x + M_n^{(k)}$ if and only if $0 \in M_n^{(k)}$, which we have shown to hold precisely if $k \geq n$.

Now define $x_n := (2n, 1, 0, 0, \ldots) \in E$ and

$$M := \bigcup_{n=1}^{\infty} (x_n + M_n) \ \subseteq E.$$

We claim that for all $k \in \mathbb{N}$ we have

$$M^{(k)} = \bigcup_{n=1}^{\infty} (x_n + M_n)^{(k)}.$$

Note that for all k, m,

$$(x_n + M_n)^{(k)} \cap W_m \subseteq [2n, 2n+1] \times [1,2] \times [0,1] \times \cdots \times [0,1].$$

So an easy induction argument shows that a converging sequence from $M^{(k)}$ lies without loss of generality in one fixed $(x_n + M_n)^{(k)}$, which proves the claim.

For all k, n we have

$$x_n \in M^{(k)} \Leftrightarrow x_n \in (x_n + M_n)^{(k)} \Leftrightarrow k \geq n.$$

This shows that the sequence of iterated sequential closures of M does *not* terminate after finitely many steps. Furthermore,

$$M^{(\omega)} = \bigcup_{k \in \mathbb{N}} M^{(k)} = \bigcup_{k \in \mathbb{N}} \bigcup_{n=1}^{\infty} x_n + M_n^{(k)} = \bigcup_{n=1}^{\infty} x_n + M_n^{(n)}$$

is closed. So:

Example 4.2. With M as above, we have

$$\ddagger(M) = \omega.$$

Now we want to find a convex cone with a similar property. We claim

$$\mathrm{cc}(M)^{(k)} = \mathrm{cc}(M^{(k)})$$

for all $k \in \mathbb{N}$. We prove this by induction; the case $k = 0$ is clear. Now we suppose it is true for some k and show it for $k+1$. For "$\subseteq$" take a converging sequence $(x_i)_i$ from $\mathrm{cc}(M)^{(k)} \cap W_m = \mathrm{cc}(M^{(k)}) \cap W_m = \mathrm{cc}(M^{(k)} \cap W_m)$ for some $m \geq 2$. As before write

$$x_i = \lambda_1^{(i)} a_1^{(i)} + \cdots \lambda_N^{(i)} a_N^{(i)}$$

with fixed N, all $\lambda_j^{(i)} \geq 0$ and all $a_j^{(i)} \in M^{(k)} \cap W_m$. So each $a_j^{(i)}$ belongs to some $\left(x_n + M_n^{(k)}\right) \cap W_m$, from which we conclude $n \leq m + k$ (otherwise the intersection would be empty). So, similar as before, we can assume that all the sequences $(a_j^{(i)})_i$ converge to some $a_j \in M^{(k+1)}$. As the first components of all $a_j^{(i)}$ are bigger or equal than 2, we can also bound the $\lambda_j^{(i)}$ and so assume that the sequences $(\lambda_j^{(i)})_i$ converge. This shows that the limit of the sequence $(x_i)_i$ lies in $\mathrm{cc}(M^{(k+1)})$. The inclusion "$\supseteq$" is again clear.

Now we have

$$\begin{aligned}\mathrm{cc}(M)^{(\omega)} &= \bigcup_{k\in\mathbb{N}} \mathrm{cc}(M^{(k)}) \\ &= \mathrm{cc}\left(\bigcup_{k\in\mathbb{N}} M^{(k)}\right) \\ &= \mathrm{cc}\left(\bigcup_{n=1}^{\infty} x_n + M_n^{(n)}\right).\end{aligned}$$

So one checks that $a = (1, 0, 0, \ldots)$ does not belong to $\mathrm{cc}(M)^{(\omega)}$. On the other hand, each $x_k = (2k, 1, 0, \ldots)$ and therefore $(1, \frac{1}{2k}, 0, \ldots)$ does. So

$$\mathrm{cc}(M)^{(\omega+1)} \supseteq \mathrm{cc}(M)^{(\omega)} + \mathbb{R}_{\geq 0} \cdot a \supsetneq \mathrm{cc}(M)^{(\omega)},$$

and the convex cone in the middle is closed. Indeed if a sequence $(x_i)_i$ converges in some W_m, with

$$x_i = \lambda_1^{(i)} a_1^{(i)} + \cdots \lambda_N^{(i)} a_N^{(i)} + \lambda^{(i)} a, \tag{9}$$

all $\lambda_j^{(i)}, \lambda^{(i)} \geq 0$, all $a_j^{(i)} \in \bigcup_{n=1}^{\infty} x_n + M_n^{(n)}$, then the sequences $(\lambda_j^{(i)})_i$ and $(\lambda^{(i)})_i$ converge without loss of generality (they are bounded, look at the first components of (9)). If such a sequence $(\lambda_j^{(i)})_i$ converges to zero, then the sequence $(\lambda_j^{(i)} a_j^{(i)})_i$ converges without loss of generality to an element from $\mathbb{R}_{\geq 0} \cdot a$. Otherwise it converges without loss of generality to an element from $\mathrm{cc}(M)^{(\omega)}$. So all in all we have proven the following surprising fact:

Example 4.3. For the convex cone $\mathrm{cc}(M)$ we have

$$\ddagger(\mathrm{cc}(M)) = \omega + 1.$$

Unfortunately, the $\ddagger$-index does not behave very predictable when going down to some quotient space E/W in general. For example, the image of M_n and $\mathrm{cc}(M_n)$ in E/W_m, under the canonical identification $E/W_m \cong E$, equals M_{n-m} and $\mathrm{cc}(M_{n-m})$, respectively. Here, we set $M_k = \overline{M_1}$ when $k \leq 0$. So the $\ddagger$-index can fall by an arbitrary natural number when going down to a quotient space.

On the other hand, it can also go up. For example look at

$$D_n := \left\{ (|l|, x) \mid l \in (\mathbb{N} \setminus \{0\})^{n+1}, x \in (x_n + U(l)) \right\}$$

as well as

$$D := \bigcup_{n=1}^{\infty} D_n$$

as sets in the space

$$E' := \mathbb{R} \oplus E.$$

Here, $|l|$ denotes $l_0 + \cdots + l_n$. D and all the D_n are checked to be closed. Factoring out the first component of E' makes the $\ddagger$-index of D_n rise from 0 to n, the index of D even rise from 0 to ω.

See Section 4.3 for some results about quadratic modules and quotient algebras.

4.2 A Preordering Counterexample

In this section we construct an example, which shows that (SMP) does not imply the ‡-property. This will answer Question 3 in [KM] and Question 2 in [KMS]. It will also give a negative answer to the question in [Sm3], whether the fibre theorem (Theorem 2.8 in our work) holds for the ‡-property instead of (SMP).

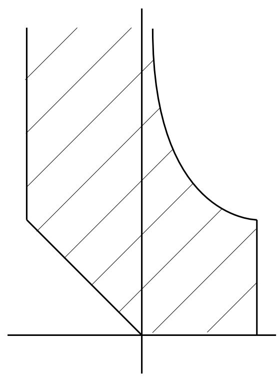
Consider $A = \mathbb{R}[X, Y]$, the real polynomial ring in two variables. We take the four polynomials

$$f_1 = Y^3, \; f_2 = Y + X, \; f_3 = 1 - XY \text{ and } f_4 = 1 - X^2$$

and write

$$P := \mathrm{PO}(f_1, f_2, f_3, f_4).$$

The corresponding basic closed semi-algebraic set $\mathcal{S}(P)$ in $\mathbb{R}^2$ looks like this:



Proposition 4.4. *The preordering P in $\mathbb{R}[X, Y]$ has (SMP).*

Proof. The basic closed semi-algebraic set $\mathcal{S}(P)$ is contained in the cylinder $[-1, 1] \times [0, \infty)$ in $\mathbb{R}^2$. The polynomial X is therefore bounded on S and we can apply Schmüdgen's Fibre Theorem (Theorem 2.8) to the preordering P.

For any $\lambda \in [-1, 1]$ write

$$P_\lambda = P + (X - \lambda).$$

For $0 < \lambda \leq 1$, P_λ describes a compact semi-algebraic set and therefore has (SMP) by Theorem 1.9 (even the ‡-property). For $\lambda \in [-1, 0]$, P_λ has (SMP) if and only if the preordering

$$\mathrm{PO}(Y^3, Y + \lambda, 1 - \lambda Y) \subseteq \mathbb{R}[Y]$$

has (SMP), by Proposition 2.1. This preordering describes the one dimensional semi-algebraic set $[-\lambda, \infty)$. As $Y + \lambda$ is the natural generator for this set, it is even saturated (see [KM], Theorem 2.2). So P_λ has (SMP) (it is indeed also saturated). So by Schmüdgen's Theorem, the whole preordering P has (SMP). □

The next result is a characterization of $P^\ddagger$. We write

$$\mathrm{PO}(a_1, \ldots, a_s)_d$$

for the set of elements having a representation in $\mathrm{PO}(a_1, \ldots, a_s)$ with sums of squares of elements of degree $\leq d$.

Proposition 4.5. *A polynomial $f \in \mathbb{R}[X, Y]$ belongs to $P^\ddagger$ if and only if there is some $d \in \mathbb{N}$, such that for all $\lambda \in [-1, 1]$,*

$$f(\lambda, Y) \in \mathrm{PO}(f_1(\lambda, Y), \ldots, f_4(\lambda, Y))_d \subseteq \mathbb{R}[Y].$$

Proof. The "if"-part is a consequence of Theorem 4.14 below (or can already be obtained by looking at the proof of Theorem 5.3. in [KMS]). We can use it with $s = 1, b_1 = X$, and find

$$f = p_\lambda + (X - \lambda) q_\lambda$$

where $p_\lambda \in P$ and $\deg(q_\lambda)$ is bounded for all λ. So Theorem 4.14 yields $f \in P^\ddagger$.

For the "only if"-part assume f belongs to $P^\ddagger$. So there is some $q \in \mathbb{R}[X, Y]$ and sums of squares $\sigma_e^{(\varepsilon)} \in \sum \mathbb{R}[X, Y]$ for all $\varepsilon > 0$ and $e \in \{0, 1\}^4$, such that

$$f + \varepsilon q = \sum_e \sigma_e^{(\varepsilon)} f_1^{e_1} \cdots f_4^{e_4}.$$

Note that the total degree of the $\sigma_e^{(\varepsilon)}$ may rise with ε getting smaller. However, the degree as polynomials in Y cannot rise; it is bounded by the Y-degree of $f+\varepsilon q$, which does not change with ε. This is because the set $\mathcal{S}(P)$ contains the cylinder $[-1,0] \times [1,\infty]$ and is therefore totally stable with respect to the $(0,1)$-grading (see Chapter 3).

By evaluating in $X = \lambda$, this means that $f(\lambda, Y) + \varepsilon q(\lambda, Y)$ belongs to

$$\mathrm{PO}(f_1(\lambda, Y), \ldots, f_4(\lambda, Y))_d$$

for some fixed d and all $\lambda \in [-1,1], \varepsilon > 0$. As mentioned in the previous chapter, $\mathrm{PO}(f_1(\lambda, Y), \ldots, f_4(\lambda, Y))_d$ is a closed set in a finite dimensional subspace of $\mathbb{R}[Y]$ (Proposition 2.6 in [PSc]). So we get $f(\lambda, Y) \in \mathrm{PO}(f_1(\lambda, Y), \ldots, f_4(\lambda, Y))_d$ for all $\lambda \in [-1,1]$, the desired result. □

Corollary 4.6. *P does not have the ‡-property.*

Proof. The polynomial Y is obviously nonnegative on the semi-algebraic set $\mathcal{S}(P)$. However, it does not belong to

$$\mathrm{PO}(f_1(1, Y), \ldots, f_4(1, Y)) = \mathrm{PO}(Y^3, Y+1, 1-Y) \subseteq \mathbb{R}[Y].$$

Indeed, writing down a representation and evaluating in $Y = 0$, this shows that Y^2 divides Y, a contradiction. So in view of Proposition 4.5, Y cannot belong to $P^\ddagger$. □

Note that Y is not in $P^\ddagger$, as it fails to be in the preordering corresponding to the fibre $X = 1$. However, Proposition 4.5 even demands all the polynomials $f(\lambda, Y)$ to have representations in the fibre-preorderings

$$\mathrm{PO}(f_1(\lambda, Y), \ldots, f_4(\lambda, Y))$$

with *simultaneous* degree bounds, for f to be in $P^\ddagger$. Indeed, there are examples of polynomials belonging to all the fibre-preorderings, but failing the degree-bound condition (and so also not belonging to $P^{\ddagger}$). We will give one here, as it gives a justification for one of the assumptions in Theorem 4.14 below.

Example 4.7. Take $f = 2Y + X$, which belongs to P^{sat}. For any $\lambda \in [-1, 1]$, $f(\lambda, Y) = 2Y + \lambda$ belongs to

$$\text{PO}(f_1(\lambda, Y), \ldots, f_4(\lambda, Y)) \subseteq \mathbb{R}[Y];$$

for $\lambda > 0$ as $f(\lambda, Y)$ is strictly positive on the corresponding compact semi-algebraic set (so use Theorem 1.9), for $\lambda \in [-1, 0]$, as the fibre preordering is saturated, see the proof of Proposition 4.4.

However, for $\lambda \searrow 0$, there can be no bound on the degree of the sums of squares in the representation. Indeed, for $\lambda > 0$, write down a representation

$$2Y + \lambda = \sum_{e \in \{0,1\}^3} \sigma_e^{(\lambda)} Y^{3e_1} (Y + \lambda)^{e_2} (1 - \lambda Y)^{e_3}, \tag{10}$$

where the $\sigma_e^{(\lambda)}$ are sums of squares in Y. Evaluating in $Y = 0$, this shows

$$\sigma_{(0,1,0)}^{(\lambda)}(0) + \sigma_{(0,1,1)}^{(\lambda)}(0) \leq 1. \tag{11}$$

Now if the degrees of the $\sigma_e^{(\lambda)}$ could be bounded for all $\lambda > 0$, we could write down a first order logic formula, saying that we have representations as in (10) for all $\lambda > 0$. We add the statement (11) to the formula. By Tarski's Transfer Principle, it holds in any real closed extension field of $\mathbb{R}$. So take such a representation in some non-archimedean real closed extension field R, for some $\lambda > 0$ which is infinitesimal with respect to $\mathbb{R}$. The same argument as for example in [KMS], Example 4.4(a) shows, that we can apply the residue map $\mathcal{O} \to \mathcal{O}/\mathfrak{m} = \mathbb{R}$ to the coefficients of all the polynomials occurring in this representation. Here, $\mathcal{O}$ denotes the convex hull of $\mathbb{R}$ in R. This is a valuation ring with maximal ideal $\mathfrak{m}$. So we would get a representation

$$2Y = \sigma_{(0,0,0)} + \sigma_{(1,0,0)}Y^3 + \sigma_{(0,1,0)}Y + \sigma_{(0,0,1)} + \sigma_{(1,1,0)}Y^4 + \sigma_{(1,0,1)}Y^3 + \sigma_{(0,1,1)}Y + \sigma_{(1,1,1)}Y^4,$$

with sums of squares σ_e in $\mathbb{R}[Y]$ fulfilling

$$\sigma_{(0,1,0)}(0) + \sigma_{(0,1,1)}(0) \leq 1. \qquad (12)$$

As no cancellation of highest degree terms can occur, we get

$$0 = \sigma_{(0,0,0)} = \sigma_{(1,0,0)} = \sigma_{(0,0,1)} = \sigma_{(1,1,0)} = \sigma_{(1,0,1)} = \sigma_{(1,1,1)}$$

as well as

$$\sigma_{(0,1,0)} + \sigma_{(0,1,1)} = 2.$$

This last fact obviously contradicts (12).

So for $f = 2Y + X$, the degree bound condition on the fibres fails, although the polynomial belongs to all of the fibre preorderings. In view of Proposition 4.5, f does not belong to $P^\ddagger$. This shows that the "degree bound"-assumption in Theorem 4.14 below is really necessary.

The above example answers open question 3 in [KM], whether (SMP) implies the ‡-property. It also answers open question 2 in [KMS], whether the strong assumptions in their Theorem 5.3 are really necessary; they indeed are. Finally, it answers the question in [Sm3], whether the fibre theorem holds for the ‡-property instead of (SMP). We have shown in the proof of Proposition 4.4, that all the fibre preorderings P_λ do not only have (SMP), but even the ‡-property. As P itself does not have the ‡-property, this gives a negative answer to the question. In Section 4.4 we will prove a result, that sometimes allows to use a dimension reduction when examining the ‡-property.

We conclude this section with the following result:

Proposition 4.8. *For the preordering P we have*

$$\ddagger(P) = 2.$$

Proof. We first claim that $Y + \varepsilon \in P^{\ddagger}$ for all $\varepsilon > 0$. This follows from Proposition 4.5, once we have shown

$$Y + \varepsilon \in \sum(W; Y^3, Y + \lambda, Y^3(1 - \lambda Y))$$

for a fixed finite dimensional subspace W of $\mathbb{R}[Y]$ and all $\lambda \in [-1, 1]$ (see Chapter 3 for the notation). But this is indeed the case. For $\lambda \leq \varepsilon$ it is obvious with $W = \mathbb{R}$. Now write down a representation

$$Y + \varepsilon = \sigma_0 + \sigma_1 Y^3 + \sigma_2 Y^3(1 - \varepsilon Y),$$

which is possible using the fact that in dimension one, each quadratic module describing a compact set is archimedean (see for example Theorem 6.3.8 in [PD]). For $\lambda \geq \varepsilon$ we have

$$\begin{aligned} Y + \varepsilon &= \sigma_0 + \sigma_1 Y^3 + \sigma_2 Y^3(1 - \lambda Y) + \sigma_2 Y^3(\lambda - \varepsilon)Y \\ &= \tau_0 + \sigma_1 Y^3 + \sigma_2 Y^3(1 - \lambda Y), \end{aligned}$$

where $\tau_0 = \sigma_0 + (\lambda - \varepsilon)Y^4\sigma_2$. This proves our first claim.

Now let $f \in P^{\mathrm{sat}}$. For all $\varepsilon > 0$ there is some $\delta > 0$ such that

$$f + \varepsilon \in \mathrm{PO}(Y + \delta, X + Y, 1 - XY, 1 - X^2)^{\mathrm{sat}}.$$

Now we find

$$f + \varepsilon + \varepsilon' q \in \mathrm{PO}(Y + \delta, X + Y, 1 - XY, 1 - X^2)$$

for a suitable q and all $\varepsilon' > 0$. This is Theorem 5.3 from [KMS] or Theorem 4.14 below. By looking at the proofs we can indeed choose the same element q for all ε (and corresponding δ) . See also Remark 4.15. So in particular

$$f + \varepsilon(q + 1) \in \mathrm{PO}(Y + \delta, X + Y, 1 - XY, 1 - X^2) \subseteq P^{\ddagger},$$

so $f \in P^{(2)}$.

□

4.3 Sequential Closures and Quotients

We include a short section on sequential closures on quotients, similar to Section 2.1 above. We show that we can always factor out ideals contained in $\sqrt[rr]{M \cap -M}$ without changing the situation too much. Therefore let A be an $\mathbb{R}$-algebra and M a quadratic module in A. Let I be an ideal of A, contained in the real radical of the support of M, and

$$\pi \colon A \to A/I$$

the canonical projection. We already noted

$$\pi(M)^{\text{sat}} = \pi(M^{\text{sat}}) \text{ and } \overline{\pi(M)} = \pi(\overline{M}).$$

Lemma 2.1 and Corollary 3.12 from [Sc4] tell us $I + \varepsilon \subseteq M$ for all $\varepsilon > 0$, so in particular $I \subseteq M^{\ddagger}$.

Proposition 4.9. *Let A be an $\mathbb{R}$-algebra and M a quadratic module in A. Let I be an ideal of A, contained in $\sqrt[rr]{M \cap -M}$, and $\pi \colon A \to A/I$ the canonical projection. Then for all ordinals ξ we have*

$$\pi(M)^{(\xi)} = \pi(M^{(\xi)})$$

and

$$\ddagger(\pi(M)) = \ddagger(M + I).$$

If $\ddagger(\pi(M)) \geq 1$, then $\ddagger(M) = \ddagger(\pi(M))$, whereas $\ddagger(\pi(M)) = 0$ implies $\ddagger(M) \leq 1$. M has the $\ddagger$-property in A if and only if $\pi(M)$ has the $\ddagger$-property in A/I, which is again the case if and only if $M + I$ has the $\ddagger$-property in A.

Proof. First suppose $\pi(f) \in \pi(M)^{\ddagger}$ for some $f \in A$. So there is some $q \in A$, such that for all $\varepsilon > 0$ there is some $m_\varepsilon \in M$ and $i_\varepsilon \in I$, such that

$$f + \varepsilon q = m_\varepsilon + i_\varepsilon.$$

So

$$f + \varepsilon(q + 1) = m_\varepsilon + i_\varepsilon + \varepsilon \in M,$$

using $I + \varepsilon \subseteq M$ as explained above. So $f \in M^{\ddagger}$ and therefore $\pi(f) \in \pi(M^{\ddagger})$. The inclusion $\pi(M^{\ddagger}) \subseteq \pi(M)^{\ddagger}$ is obvious. A straightforward transfinite induction now shows

$$\pi(M)^{(\xi)} = \pi(M^{(\xi)})$$

for all ordinals ξ.

Now whenever $\ddagger(M) \leq \xi$, then $M^{(\xi)} = M^{(\xi+1)}$, so

$$\pi(M)^{(\xi)} = \pi(M^{(\xi)}) = \pi(M^{(\xi+1)}) = \pi(M)^{(\xi+1)},$$

which shows $\ddagger(\pi(M)) \leq \xi$, so

$$\ddagger(\pi(M)) \leq \ddagger(M).$$

The same result applied to $M + I$ instead of M yields

$$\ddagger(\pi(M)) \leq \ddagger(M + I).$$

For arbitrary ordinals ξ we have

$$\pi^{-1}(\pi(M)^{(\xi)}) = \pi^{-1}(\pi(M^{(\xi)})) = M^{(\xi)} + I.$$

So whenever $\ddagger(\pi(M)) = \xi$, then

$$M^{(\xi)} + I = \pi^{-1}(\pi(M)^{(\xi)}) = \pi^{-1}(\pi(M)^{(\xi+1)}) = M^{(\xi+1)} + I = M^{(\xi+1)}$$

using $I \subseteq M^{\ddagger} \subseteq M^{(\xi+1)}$. If $\xi \geq 1$ we get $M^{(\xi)} = M^{(\xi+1)}$, so $\ddagger(M) \leq \xi$, so

$$\ddagger(M) = \ddagger(\pi(M)) = \ddagger(M + I).$$

If $\xi = 0$, then $M + I = M^{\ddagger} = \pi^{-1}(\pi(M))$ is sequentially closed, so $\ddagger(M + I) = 0, \ddagger(M) \leq 1$. The remark concerning the $\ddagger$-property is clear from these considerations. $\square$

For arbitrary ideals we have the following result:

Proposition 4.10. *Let A be a finitely generated $\mathbb{R}$-algebra and M a finitely generated quadratic module in A. If M has the ‡-property and I is an ideal in A, then*

$$(M+I)^{(2)} = (M+I)^{\text{sat}}.$$

Proof. Let $f \in (M+I)^{\text{sat}}$. By Corollary 2.6 in [Sc6], for all $\varepsilon > 0$ there is some $i_\varepsilon \in I$, such that $f + \varepsilon + i_\varepsilon \in M^{\text{sat}}$. So

$$f + \varepsilon + i_\varepsilon + \delta q_\varepsilon \in M$$

for some suitable q_ε and all $\delta > 0$. Therefore

$$f + \varepsilon \in (M+I)^{\ddagger}$$

and so $f \in (M+I)^{(2)}$. □

Under an additional assumption we can improve on this:

Proposition 4.11. *Let A be a finitely generated $\mathbb{R}$-algebra and M a finitely generated quadratic module in A. Assume M has the following property: For every finite dimensional subspace W of A there is some $q_W \in A$, such that whenever $f \in M^{\text{sat}} \cap W$, then $f + \varepsilon q_W \in M$ for all $\varepsilon > 0$.*

Then for any ideal I of A, the quadratic module $M + I$ has the same property. In particular, it has the ‡-property.

Proof. Let $f \in (M+I)^{\text{sat}} \cap W$ for some finite dimensional subspace W of A. Then for every $\varepsilon > 0$ there is some $i_\varepsilon \in I$, such that $f + \varepsilon + i_\varepsilon \in M^{\text{sat}}$. This follows again from Corollary 2.6 in [Sc6]. We even can choose all the elements i_ε from a finite dimensional subspace of A depending on M, I and W, but not on ε. This follows, using a standard ultrapower argument, from the fact that Corollary 2.6 from [Sc6] holds for finitely generated algebras over arbitrary real closed fields. So

$$f + \varepsilon + i_\varepsilon \in M^{\text{sat}} \cap W'$$

for some finite dimensional subspace W' of A and all $\varepsilon > 0$. So

$$f + \varepsilon(1 + q_{W'}) = f + \varepsilon + \varepsilon q_{W'} \in M + I$$

for all $\varepsilon > 0$, what was to be shown. □

In almost all of our coming examples of quadratic modules having the ‡-property, we will indeed have the slightly stronger property demanded in Proposition 4.11.

To what extent we can check the ‡-property by looking at a suitable family of fibre modules is the content of the next section.

4.4 A Fibre Theorem for Sequential Closures

The setup in this section is similar to the one in Chapter 2. We consider arbitrary $\mathbb{R}$-algebras A, B and a Hausdorff space X, which is now assumed to be compact. We suppose to have algebra homomorphisms $\varphi\colon A \to B$ and $\hat{}\colon A \to C(X, \mathbb{R})$:

$$\begin{array}{ccc} B & & \\ \uparrow{\scriptstyle\varphi} & & \\ A & \xrightarrow{\ \hat{}\ } & C(X,\mathbb{R}) \end{array}$$

For some quadratic module M in B we want to examine $M^\ddagger$ in terms of the fibre modules $M_x = M + I_x$ as in Chapter 2. As we have seen, we cannot expect a result like

$$M^\ddagger = \bigcap_{x \in X} M_x^\ddagger$$

to hold under reasonable assumptions. Indeed, even

$$M^\ddagger \supseteq \bigcap_{x \in X} M_x$$

is not true in the example from the section 4.2. We will need some kind of additional degree bound condition, as in Proposition 4.5. This is done in Theorem 4.13, which is the main result in this section. We derive from it conditions for a quadratic module to have the ‡-property. First we prove a helpful (but technical) proposition:

Proposition 4.12. *Let A, B be $\mathbb{R}$-algebras and $\varphi\colon A \to B$ an algebra homomorphism. Let X be a non-empty compact Hausdorff space and $\hat{}\colon A \to C(X,\mathbb{R})$ a homomorphism whose image separates points of X. Assume $w, w_1, \ldots, w_s \in B$ and $\varepsilon > 0$ are such that for all $x \in X$ there is a representation*

$$w = \sum_{j=1}^{s} \varphi(a_j^{(x)}) \cdot w_j,$$

with $a_j^{(x)} \in A$ and $|\widehat{a_j^{(x)}}(x)| < \varepsilon$ for all j. Then there are $a_1, \ldots, a_s \in A$ with $|\hat{a}_j| < \varepsilon$ on X for all j and

$$w = \sum_{j=1}^{s} \varphi(a_j) \cdot w_j.$$

Proof. Every $x \in X$ has an open neighborhood U_x, such that $|\widehat{a_j^{(x)}}| < \varepsilon$ on U_x for all $j = 1, \ldots, s$. By compactness of X there are $x_1, \ldots, x_t \in X$, such that

$$X = U_{x_1} \cup \cdots \cup U_{x_t}.$$

If $t = 1$, then the result follows, so assume $t \geq 2$. Choose a partition of unity $e_1, \ldots, e_t$ subordinate to that cover, i.e. all e_k are continuous functions from X to $[0,1]$, $\operatorname{supp}(e_k) \subseteq U_{x_k}$ for all k, and $e_1(x) + \cdots + e_t(x) = 1$ for all $x \in X$. Then for

$$f_j := e_1 \cdot \widehat{a_j^{(x_1)}} + \cdots + e_t \cdot \widehat{a_j^{(x_t)}}$$

we have

$$\| f_j \| < \varepsilon,$$

where $\| \ \|$ denotes the sup-norm on $C(X, \mathbb{R})$. Let

$$\delta := \min\{\ \varepsilon - \| f_j \| \mid j = 1, \ldots, s\}$$

and choose a positive real number N, big enough to bound the sup-norm of all $\widehat{a_j^{(x_k)}}$.

The image of A in $C(X, \mathbb{R})$ is dense, by the Stone-Weierstrass Theorem. So we find $q_1, \ldots, q_{t-1} \in A$ such that

$$\| e_k - \hat{q}_k \| < \frac{\delta}{N(t-1)t}$$

for $k = 1, \ldots, t-1$, and we define

$$q_t := 1 - \sum_{k=1}^{t-1} q_k.$$

So we have for $k = 1, \ldots, t$

$$\| e_k - \hat{q}_k \| < \frac{\delta}{Nt}.$$

We define

$$a_j := q_1 \cdot a_j^{(x_1)} + \cdots + q_t \cdot a_j^{(x_t)}$$

for $j = 1, \ldots, s$. So

$$\begin{aligned} \| \hat{a}_j \| &\leq \| f_j \| + \| \hat{a}_j - f_j \| \\ &\leq \| f_j \| + \sum_{k=1}^{t} \| e_k - \hat{q}_k \| \cdot \| \widehat{a_j^{(x_k)}} \| \\ &< \| f_j \| + \delta \\ &\leq \ \varepsilon. \end{aligned}$$

Now as $\sum_{k=1}^{t} q_k = 1$ we have

$$\begin{aligned} w &= \varphi(\sum_{k=1}^{t} q_k) \cdot w \\ &= \sum_{k=1}^{t} \left(\varphi(q_k) \cdot \sum_{j=1}^{s} \varphi(a_j^{(x_k)}) w_j \right) \\ &= \sum_{j=1}^{s} \varphi \left(\sum_{k=1}^{t} q_k a_j^{(x_k)} \right) \cdot w_j \\ &= \sum_{j=1}^{s} \varphi(a_j) \cdot w_j, \end{aligned}$$

which proves the proposition. □

As in Section 2, we denote by I_x the ideal in B generated by the set

$$Z_x = \{\varphi(a) \mid \hat{a}(x) = 0\},$$

for $x \in X$. If W is a subset of B, then we write

$$I_x(W) = \left\{ \sum_{j=1}^{s} z_j v_j \mid s \in \mathbb{N}, z_j \in Z_x, v_j \in W \right\}.$$

For $W = B$, $I_x(W)$ obviously equals I_x.

The following is the main theorem in this Chapter. Its proof contains and generalizes the idea from [KMS], Theorem 5.3.

Theorem 4.13. *Let A, B be $\mathbb{R}$-algebras and $\varphi\colon A \to B$ an algebra homomorphism. Let X be a compact Hausdorff space and $\hat{}\colon A \to C(X, \mathbb{R})$ a homomorphism whose image separates points of X. Let $M \subseteq B$ be a quadratic module and assume*

$$\hat{a} > 0 \text{ on } X \Rightarrow \varphi(a) \in M$$

holds for all $a \in A$. Then for all finitely generated A-submodules W of B, we have

$$\bigcap_{x \in X} M + I_x(W) \subseteq M^{\ddagger}.$$

Proof. Fix such a W. Assume $f \in B$ has a representation

$$f = m_x + i_x$$

with $m_x \in M$ and $i_x \in I_x(W)$, for all $x \in X$. As $I_x(W) \subseteq W$, we can assume without loss of generality $m_x \in M \cap W$ for all x. Let $w_1, \ldots, w_s$ be generators of W as an A-module. Due to the identity $w = (\frac{w+1}{2})^2 - (\frac{w-1}{2})^2$ we can assume that all w_j are squares in B (by possibly enlarging W). We will now show

$$f + \varepsilon \sum_{j=1}^{s} w_j \in M$$

for all $\varepsilon > 0$. Therefore fix one such $\varepsilon > 0$. We take representations

$$i_x = \sum_{j=1}^{s} \varphi\left(c_j^{(x)}\right) \cdot w_j, \quad m_x = \sum_{j=1}^{s} \varphi\left(d_j^{(x)}\right) \cdot w_j$$

where all $c_j^{(x)}, d_j^{(x)} \in A$ and $\widehat{c_j^{(x)}}(x) = 0$. Now each $x \in X$ has an open neighborhood U_x, such that

$$|\widehat{c_j^{(x)}}| < \frac{\varepsilon}{2} \text{ on } U_x$$

for $j = 1, \ldots, s$. By compactness of X we have

$$X = U_{x_1} \cup \cdots \cup U_{x_t}$$

for some $x_1, \ldots, x_t \in X$. Let $e_1, \ldots, e_t$ be a continuous partition of unity subordinate to that cover. Using the Stone-Weierstrass Theorem, we approximate the square root of each e_k (which is again a continuous function) by elements g_k from A, such that

$$\sum_{k=1}^{t} \parallel e_k - \hat{g}_k^2 \parallel \cdot \parallel \hat{d}_j^{(x_k)} \parallel < \frac{\varepsilon}{2}$$

holds for all $j = 1, \ldots, s$. Here, $\| \ \|$ denotes the sup-norm on $C(X, \mathbb{R})$ again. Define

$$w = f - \underbrace{\sum_{k=1}^{t} \varphi(g_k)^2 \cdot m_{x_k}}_{\in M} .$$

The proof is complete if we show $w + \varepsilon \sum_{j=1}^{s} w_j \in M$. Fix $x \in X$. Then

$$\begin{aligned} w &= \sum_{k=1}^{t} e_k(x) \cdot f - \sum_{k=1}^{t} \varphi(g_k^2) \cdot m_{x_k} \\ &= \sum_{k=1}^{t} e_k(x) \cdot (\underbrace{f - m_{x_k}}_{= i_{x_k}}) + \sum_{k=1}^{t} \big(e_k(x) - \varphi(g_k^2)\big)\, m_{x_k} \\ &= \sum_{k=1}^{t} e_k(x) \sum_{j=1}^{s} \varphi(c_j^{(x_k)}) w_j + \sum_{k=1}^{t} \big(e_k(x) - \varphi(g_k^2)\big) \sum_{j=1}^{s} \varphi(d_j^{(x_k)}) w_j \\ &= \sum_{j=1}^{s} \left(\sum_{k=1}^{t} e_k(x) \varphi(c_j^{(x_k)}) \right) \cdot w_j \\ &\qquad + \sum_{j=1}^{s} \left(\sum_{k=1}^{t} \big(e_k(x) - \varphi(g_k^2)\big)\, \varphi(d_j^{(x_k)}) \right) \cdot w_j \\ &= \sum_{j=1}^{s} \varphi(a_j^{(x)}) \cdot w_j , \end{aligned}$$

where we define

$$a_j^{(x)} = \sum_{k=1}^{t} e_k(x) \cdot c_j^{(x_k)} + \big(e_k(x) - g_k^2\big) \cdot d_j^{(x_k)} .$$

By the above considerations we have

$$|\widehat{a_j^{(x)}}(x)| < \varepsilon$$

for all j. So we can apply Proposition 4.12 to $w, w_1, \ldots, w_s$ and find

$$w = \sum_{j=1}^{s} \varphi(a_j) \cdot w_j$$

for some $a_j \in A$ with $|\hat{a}_j| < \varepsilon$ on X. Thus

$$w + \varepsilon \sum_{j=1}^{s} w_j = \sum_{j=1}^{s} \varphi(a_j + \varepsilon) \cdot w_j \in M,$$

as all $\widehat{a_j + \varepsilon}$ are strictly positive on X and all w_j are squares. □

We can apply this in the same way as we applied Theorem 2.5 in Chapter 2. Therefore let B be an arbitrary $\mathbb{R}$-algebra and $M \subseteq B$ a quadratic module. Assume we have $b_1, \ldots, b_s \in B$ such that $C_i - b_i, b_i - c_i \in M$ for some real numbers $C_i \geq c_i$. Consider the subalgebra $A = \mathbb{R}[b_1, \ldots, b_s]$ of B generated by these elements, and the quadratic module $\widetilde{M}$ in A generated by $C_i - b_i, b_i - c_i$ $(i = 1 \ldots, s)$. It is archimedean. The role of φ is again played by the canonical inclusion

$$\iota \colon A \hookrightarrow B$$

and we have $\iota(\widetilde{M}) \subseteq M$. The space $X := \mathcal{S}(\widetilde{M}) \subseteq \mathcal{V}_A(\mathbb{R})$ is compact and we have the usual morphism

$$\hat{\ } \colon A \to C(X, \mathbb{R}),$$

whose image separates points. Whenever $\hat{a} > 0$ on X, then $a \in \widetilde{M}$, by Theorem 1.10, so $\iota(a) \in M$. Note that for $\alpha \in X$ and W an A-submodule of B, we have

$$I_\alpha(W) = \left\{ \sum_{i=1}^{s} (b_i - \alpha(b_i)) w_i \mid w_i \in W \right\}$$

and $(\alpha(b_1), \ldots, \alpha(b_s)) \in \prod_{i=1}^{s} [c_i, C_i]$. For $\lambda \in \mathbb{R}^n$ write

$$I_\lambda(W) = \left\{ \sum_{i=1}^{s} (b_i - \lambda_i) w_i \mid w_i \in W \right\}$$

and $I_\lambda := I_\lambda(B)$. We get the following ‡-counterpart to Theorem 2.6:

Theorem 4.14. *Let B be an $\mathbb{R}$-algebra and $M \subseteq B$ a quadratic module. Suppose $b_1, \dots, b_s \in B$ are such that*

$$C_1 - b_1, b_1 - c_1, \dots, C_s - b_s, b_s - c_s \in M$$

for some real numbers $C_i \geq c_i$ $(i = 1, \dots, s)$. Then for every finitely generated $\mathbb{R}[b_1, \dots, b_s]$-submodule W of B we have

$$\bigcap_{\lambda \in \Lambda} M + I_\lambda(W) \subseteq M^\ddagger,$$

where $\Lambda = \prod_{i=1}^{s}[c_i, C_i]$. In particular, if M is finitely generated and all the (finitely generated) quadratic modules $M + I_\lambda$ are closed and stable with the same stability map, then $M^\ddagger = \overline{M}$. If all $M + I_\lambda$ are saturated and stable with the same stability map, then M has the ‡-property. (Here, the stability map with respect to the canonical generators of each $M + I_\lambda$ is meant.)

Proof. The first part of the theorem is clear from the above considerations and Theorem 4.13. For the second part, assume M is finitely generated, say by $f_1, \dots, f_t$. Then $M + I_\lambda$ is finitely generated as a quadratic module, by the canonical generators

$$f_1, \dots, f_t, \pm(b_1 - \lambda_1), \dots, \pm(b_s - \lambda_s).$$

Assume all $M + I_\lambda$ are closed (or saturated, respectively) and stable with the same stability map. Suppose some f belongs to $\overline{M}$ (or M^{sat}, respectively). Then f belongs to all $\overline{M + I_\lambda}$ (or $(M + I_\lambda)^{\text{sat}}$, respectively), so to all $M + I_\lambda$ by our assumption. Now by the assumed stability there is a *fixed* finite dimensional $\mathbb{R}$-subspace W of B, such that f belongs to all $M + I_\lambda(W)$. So the first part of the theorem yields $f \in M^\ddagger$. □

Remark 4.15. If all the quadratic modules $M + I_\lambda$ in Theorem 4.14 are saturated and stable with the same stability map, we even get a little bit more than the ‡-property for M. Indeed, we get the property that was assumed in Proposition 4.11. That is, for any $f \in M^{\text{sat}}$ we find $f + \varepsilon q \in M$ for all $\varepsilon > 0$, and we can choose the element q to only depend on the finite dimensional subspace f is taken from, not on the explicit choice of f. Indeed, the proof of Theorem 4.14 shows that q only depends on the A-module W, which only depends on the space f is taken from.

Remark 4.16. As in Theorem 2.8, if B is a finitely generated $\mathbb{R}$-algebra and $M \subseteq B$ a finitely generated preordering, we can let the intersection in Theorem 4.14 run over $M + I_\alpha(W)$ for $\alpha \in \mathcal{S}(M)$, instead of $M + I_\lambda(W)$ for $\lambda \in \Lambda$.

Proof. Suppose f belongs to $\bigcap_{\alpha \in \mathcal{S}(M)} M + I_\alpha(W)$ for some finitely generated $\mathbb{R}[b_1, \ldots, b_s]$-module W. We show that f also belongs to $\bigcap_{\lambda \in \Lambda} M + I_\lambda(W')$ for some finitely generated $\mathbb{R}[b_1, \ldots, b_s]$-module W'.

If for $\lambda \in \Lambda$ the semi-algebraic set corresponding to $M + I_\lambda$ is nonempty, then $\lambda = (\beta(b_1), \ldots, \beta(b_s))$ for some $\beta \in \mathcal{S}(M)$, exactly as in the proof of Theorem 2.8. So f belongs to $M + I_\lambda(W)$.

If the semi-algebraic set is empty, then $-1 \in M + I_\lambda$. Indeed even $-1 \in M + I_\lambda(U)$, where U is a finitely generated $\mathbb{R}[b_1, \ldots, b_s]$-submodule of B that does *not* depend on λ (only on the generators of M and on $b_1, \ldots, b_s$). This follows for example from [PD], Remark 4.2.13 and a standard ultrapower argument. Using the identity $f = (\frac{f+1}{2})^2 - (\frac{f-1}{2})^2$ we get

$$f \in M + I_\lambda\left(\left(\frac{f-1}{2}\right)^2 \cdot U\right).$$

So we can take

$$W' := \left(\frac{f-1}{2}\right)^2 \cdot U + W.$$

□

Note that we really need the finitely generated submodule W in Theorem 4.14 (and therefore also in Theorem 4.13) in general. In Example 4.7, the polynomial $f = 2Y + X$ belongs to all fibre preorderings $P + (X - \lambda)$, but fails to be in $P^\ddagger$. As we have shown, this is because it is not possible to find one fixed finitely generated $\mathbb{R}[X]$-module W, such that $f \in P + I_\lambda(W)$ for all $\lambda > 0$.

The condition $C_i - b_i, b_i - c_i \in M$ from Theorem 4.14 (and therefore also the condition $\hat{a} > 0$ on $X \Rightarrow \varphi(a) \in M$ from Theorem 4.13) is also necessary in general. This is shown by Example 5.1 below.

4.5 Applications

In this section we give some applications of the fibre theorem from the last section. The first one is the Cylinder Theorem (Theorem 5.3 combined with Corollary 5.5) from [KMS]. See [KM, KMS] for the definition of natural generators for semi-algebraic subsets of $\mathbb{R}$.

Corollary 4.17. *Let $P = \mathrm{PO}(f_1, \ldots, f_t)$ be a finitely generated preordering in the polynomial ring $\mathbb{R}[X_1, \ldots, X_n, Y]$. Assume $N - \sum_{i=1}^n X_i^2 \in P$ for some $N > 0$. Now for all $\lambda \in \mathbb{R}^n$, the preordering*

$$\mathrm{PO}(f_1(\lambda, Y), \ldots, f_t(\lambda, Y)) \subseteq \mathbb{R}[Y]$$

describes a basic closed semi-algebraic set S_λ in $\mathbb{R}$. Suppose the natural generators for S_λ are among the $f_1(\lambda, Y), \ldots, f_t(\lambda, Y)$, whenever S_λ is not empty. Then P has the $\ddagger$-property, even the slightly stronger property described in Remark 4.15.

If all the fibre sets S_λ are of the form $\emptyset, (-\infty, \infty), (-\infty, p]$, $[q, \infty), (-\infty, p] \cup [q, \infty)$ or $[p, q]$, then the result holds with P replaced by $M = \mathrm{QM}(f_1, \ldots, f_t)$.

Proof. The assumptions imply that all the preorderings

$$P + (X_1 - \lambda_1, \ldots, X_n - \lambda_n)$$

(or the corresponding quadratic modules, respectively) are saturated and stable with the same stability map for all λ. See [KMS], Section 4. An easy calculation, as for example in [KM], Note 2.3 (4), shows

$$\sqrt{N} - X_i, X_i + \sqrt{N} \in P$$

for all i. So we can apply Theorem 4.14. □

We can also use Theorem 4.14 in the case that the natural generators are not among the $f_i(\lambda, Y)$. This can be seen as a slight generalization of Corollary 5.4 from [KMS]:

Corollary 4.18. *Let $M = \mathrm{QM}(f_1, \ldots, f_t)$ be a finitely generated quadratic module in $\mathbb{R}[X_1, \ldots, X_n, Y]$ and assume*

$$N - \sum_{i=1}^{n} X_i^2 \in M$$

for some $N > 0$. Suppose for all $\lambda \in \mathbb{R}^n$ the set

$$S_\lambda := \mathcal{S}(M) \cap \{(x_1, \ldots, x_n, y) \mid x_1 = \lambda_1, \ldots, x_n = \lambda_n\}$$

is either empty or unbounded. Then

$$M^\ddagger = \overline{M}$$

holds.

Proof. Again $\sqrt{N} - X_i, X_i + \sqrt{N} \in P$ for all i. Furthermore. the assumptions imply that all the quadratic modules

$$M + (X_1 - \lambda_1, \ldots, X_n - \lambda_n)$$

are closed and stable with the same stability map for all λ (for the empty fibers use Theorem 4.5 from [KMS]). Now apply Theorem 4.14. □

We want to get results for more complicated fibres. [Sc2] gives a criterion for quadratic modules on curves to be stable and closed. However, we need some result to obtain the *uniform* stability asked for in Theorem 4.14. So we consider the following setup. Let $b \in \mathbb{R}[X, Y]$ be a polynomial of degree $d > 0$. We assume that the highest degree homogeneous part of b factors as

$$\prod_{i=1}^{d} (r_i X + s_i Y),$$

where all the $(r_i : s_i)$ are pairwise disjoint points of $\mathbb{P}^1(\mathbb{R})$. In particular, b is square free. Let C denote the affine curve defined by b and $\widetilde{C}$ its projective closure. So $\widetilde{C}$ is defined by $\tilde{b}$, the homogenization of b with respect to the new variable Z. The assumption on the highest degree part of b implies, that all the points at infinity of b, namely

$$P_1 = (-s_1 : r_1 : 0), \ldots, P_d = (-s_d : r_d : 0) \in \mathbb{P}^2,$$

are real regular points (of the projective curve $\widetilde{C}$). So the local rings of $\widetilde{C}$ at all these points are discrete valuation rings (a well known fact, see for example [F], Chapter 3). Indeed, the projective curve $\widetilde{C}$ is the so called "good completion" (see for example [P2]) of the affine curve C. We denote the valuation corresponding to the local ring at P_i by ord_i. For a polynomial $h \in \mathbb{R}[X, Y]$, we write $\mathrm{ord}_{P_i}(h)$ and mean the value with respect to the valuation ord_{P_i} of $h(\frac{X}{Z}, \frac{Y}{Z})$ as a rational function on $\widetilde{C}$.
We start with the following result:

Proposition 4.19. *Let b, C and $\widetilde{C}$ be as above. Suppose*

$$\mathrm{ord}_{P_i}(h) \geq -n$$

for some $h \in \mathbb{R}[X, Y], n \in \mathbb{N}$ and all i. Then there is some $h' \in \mathbb{R}[X, Y]$ with $\deg(h') \leq n$ and $h \equiv h' \mod (b)$.

Proof. Let m be the degree of h and $\tilde{b} = Z^d b(\frac{X}{Z}, \frac{Y}{Z})$ as well as $\tilde{h} = Z^m h(\frac{X}{Z}, \frac{Y}{Z})$ be the homogenization of b and h, respectively. Assume without loss of generality

$$P_1 = (1 : y : 0)$$

for some $y \in \mathbb{R}$.

For any homogeneous polynomial g in the variables X, Y, Z we have

$$0 \leq \mathrm{ord}_{P_1}\left(\frac{g}{X^{\deg(g)}}\right) = I(P_1; \tilde{b} \cap g),$$

where I denotes the intersection number. This is [F], Chapter 3.3.

As

$$\mathrm{ord}_{P_1}(h) = \mathrm{ord}_{P_1}\left(\frac{\tilde{h}}{X^m}\right) - m \cdot \mathrm{ord}_{P_1}\left(\frac{Z}{X}\right),$$

we have

$$\begin{aligned} -n &\leq \mathrm{ord}_{P_1}(h) \\ &= I(P_1; \tilde{b} \cap \tilde{h}) - m \cdot I(P_1; \tilde{b} \cap Z) \\ &\leq I(P_1; \tilde{b} \cap \tilde{h}) - m. \end{aligned}$$

Now whenever $m \geq n + 1$, then

$$1 \leq I(P_1; \tilde{b} \cap \tilde{h}),$$

so $\tilde{h}$ must vanish at P_1.

The same argument applies to all points at infinity of b. So if $m \geq n+1$, then the highest degree part of b divides the highest degree part of h in $\mathbb{R}[X, Y]$. Thus h can be reduced modulo b to a polynomial h' of strictly smaller degree. □

In the following proposition, the pure closedness and stability result follows from [Sc2], Proposition 6.5.

Proposition 4.20. *Let* $M = \mathrm{QM}(f_1, \dots, f_t) \subseteq \mathbb{R}[X,Y]$ *be a finitely generated quadratic module. Let* $b \in \mathbb{R}[X,Y]$ *be a polynomial whose highest degree part factors as above. For some* $\lambda \in \mathbb{R}$ *assume that all the (real regular) points at infinity of the curve* C_λ *defined by* $b = \lambda$ *lie in the closure of* $\mathcal{S}(M) \cap C_\lambda(\mathbb{R})$. *Then the finitely generated quadratic module*

$$M + (b - \lambda)$$

is closed and stable, with a stability map that depends only on b *and* M, *but not on* λ.

Proof. Without loss of generality, let $P_1 = (1 : y : 0)$ be a point at infinity of C_λ. Denote by ord_{P_1} the valuation with respect to the local ring of $\widetilde{C_\lambda}$ at P_1. Let $h \in \mathbb{R}[X,Y]$ have degree m, and let $\tilde{h}$ as well as $\widetilde{b - \lambda}$ be the homogenizations, as in the previous proof. Then

$$\begin{aligned} \mathrm{ord}_{P_1}(h) &= \mathrm{ord}_{P_1}\left(\frac{\tilde{h}}{X^m}\right) - m \cdot \mathrm{ord}_{P_1}\left(\frac{Z}{X}\right) \\ &\geq -m \cdot I\left(P_1; \widetilde{b - \lambda} \cap Z\right) \\ &= -m \cdot I\left(P_1; \tilde{b} \cap Z\right), \end{aligned}$$

where the last equality uses property (7) in [F], p. 75, for intersection numbers. So there is some N, not depending on λ, such that

$$\mathrm{ord}_P(h) \geq -m \cdot N$$

for all the points at infinity of C_λ.

Now the proof of Proposition 6.5 from [Sc2] shows that whenever $h \in \overline{M + (b - \lambda)}$, then we can find a representation

$$h = \sum_{k=0}^{t} \sigma_k f_k + g \cdot (b - \lambda) \tag{13}$$

with sums of squares σ_k built of polynomials that have order greater than $-m \cdot N$ in all points at infinity of C_λ. Applying Proposition 4.19 we can reduce these elements modulo $b - \lambda$ and obtain a representation as in (13) with sums of squares of elements of degree $\leq m \cdot N$. So of course also the degree of g is bounded suitably, independent of λ. This shows that the stability map does not depend on λ. □

So the following Theorem is an immediate consequence of Theorem 4.14 and Proposition 4.20.

Theorem 4.21. *Let $M \subseteq \mathbb{R}[X,Y]$ be a finitely generated quadratic module. Let $b \in \mathbb{R}[X,Y]$ with $R - b, b - r \in M$ for some $r \leq R$, and assume the highest degree part of b factors as above. Suppose for all $\lambda \in [r, R]$, all the (real regular) points at infinity of the curve C_λ defined by $b = \lambda$ lie in the closure of $\mathcal{S}(M) \cap C_\lambda(\mathbb{R})$. Then*

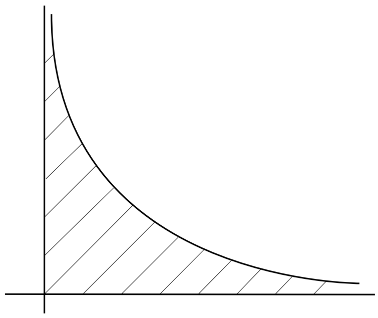
$$M^\ddagger = \overline{M}$$

holds. If all the fibre modules $M + (b - \lambda)$ have (SMP) in addition, then M has the $\ddagger$-property.

5 Examples in the plane

We conclude this work with a collection of examples in the plane. They illustrate our main results. Our $\mathbb{R}$-algebra will always be $A = \mathbb{R}[X, Y]$, the real polynomial ring in two variables.

Example 5.1. We look at the semi-algebraic set in $\mathbb{R}^2$ defined by the inequalities $0 \leq x, 0 \leq y$ and $xy \leq 1$:



A lot of interesting phenomena can be observed for this set. There are different quadratic modules describing it, we consider the following ones:

$$
\begin{aligned}
M_1 &:= \mathrm{QM}\,(X, Y, 1 - XY) \\
M_2 &:= \mathrm{QM}\,(X, Y, XY, 1 - XY) \\
M_3 &:= \mathrm{QM}\left(X, Y^3, XY, 1 - XY\right) \\
P_1 &:= \mathrm{PO}\,(X, Y, 1 - XY) \\
P_2 &:= \mathrm{PO}\left(X, Y, (1 - XY)^3\right)
\end{aligned}
$$

The quadratic module M_1 is stable. Indeed, take the monomial ordering that first sorts by degree and then lexicographically with $X > Y$. No two generators of M_1 have the same degree modulo $2 \cdot (\mathbb{Z} \oplus \mathbb{Z})$. So Proposition 3.11 combined with Theorem 3.13 yields total stability with respect to the corresponding grading. This grading induces a filtration of finite dimensional subspaces, so in particular, M_1 is stable. By Theorem 3.2 and Theorem 3.3, M_1 is closed and does not have (SMP).

To the quadratic module M_2 we can apply Theorem 4.21 with the bounded polynomial $b = XY$: we have $b, 1-b \in M_2$. For $\lambda \in [0,1]$, the finitely generated quadratic module

$$\mathrm{QM}(X, Y, XY, 1-XY) + (XY - \lambda) = \mathrm{QM}(X, Y) + (XY - \lambda)$$

has (SMP). It is indeed even saturated. This is an easy calculation for $\lambda > 0$; for $\lambda = 0$ it is Example 3.26 from [P2]. So M_2 has the ‡-property, and in particular (SMP).

Note that the fibre modules of M_1 and M_2 are the same:

$$M_1 + (XY - \lambda) = M_2 + (XY - \lambda)$$

for all $\lambda \in [0,1]$. As M_1 does neither have the ‡-property nor (SMP), this shows that the condition $N - b, b - n \in M$ in Theorem 4.21, as well as the corresponding conditions in Theorem 4.14 and Theorem 2.6 cannot be omitted.

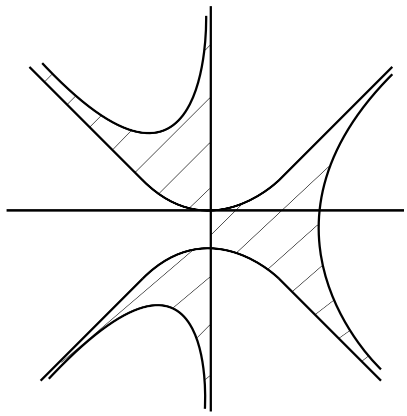
Now consider M_3. The quadratic module $\mathrm{QM}(Y^3) \subseteq \mathbb{R}[Y]$, obtained by evaluating in $X = 0$, does not have (SMP) (see for example [KM]). So in view of Proposition 2.1 and Proposition 2.2, M_3 does not have (SMP). On the other hand, we can still apply Theorem 4.21 with $b = XY$, and obtain

$$M_3^{\ddagger} = \overline{M_3}.$$

The preordering P_1 obviously contains M_2 and therefore also has the ‡-property. This solves the question posed in [KMS], Example 8.4.

P_2 finally illustrates that we can always replace *bounded* generators of a preordering by odd powers, without losing (SMP). Indeed, by Proposition 2.7, the polynomial $1 - XY$ belongs to $\overline{P_2}$, so $\overline{P_2} = \overline{P_1}$ and P_2 has (SMP).

Example 5.2. We consider the semi-algebraic set defined by the inequalities $0 \leq x(x+y)(x-y) - xy \leq 1$:



We can apply Theorem 4.21 to the quadratic module

$$M = \mathrm{QM}(b, 1-b),$$

where $b = X(X+Y)(X-Y) - XY$. We use b as the bounded polynomial and obtain

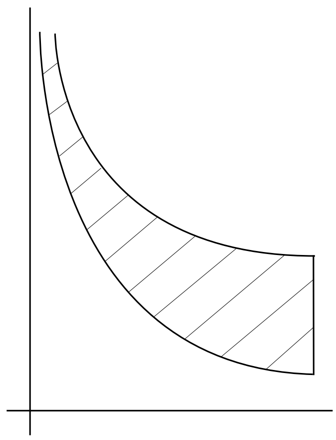
$$M^{\ddagger} = M^{\vee\vee}.$$

However, M does not have (SMP). Indeed, the quadratic module

$$M + (b)$$

does not have (SMP). This follows (together with Proposition 2.1) from [P2], Theorem 3.17 and Proposition 6.5 from [Sc2]. So in view of Proposition 2.2, M does not have (SMP).

Example 5.3. This example is Example 3 from [Sm3] and illustrates the use of Corollary 2.9. The semi-algebraic set $\mathcal{S}$ we are looking at is defined by the inequalities $0 \leq x, x \leq 2, 1 \leq xy, xy \leq 2$:



The two polynomials X and XY are bounded on $\mathcal{S}$ and separate its points. So by Corollary 2.9, *every* finitely generated preordering describing this set has (SMP).

We can use Corollary 4.17 in that example to see that the quadratic module

$$M = \mathrm{QM}(X, 2 - X, XY - 1, 2 - XY)$$

has the ‡-property. But we can also apply Theorem 4.14 with the two polynomials X and XY simultaneously to get the same result. We demonstrate how to do this.

Therefore take $\lambda = (\lambda_1, \lambda_2) \in [0, 2] \times [1, 2]$ and consider the fibre module

$$M_\lambda := M + (X - \lambda_1, XY - \lambda_2) = \sum \mathbb{R}[X, Y]^2 + (X - \lambda_1, XY - \lambda_2).$$

If $\lambda_1 > 0$, the corresponding semi-algebraic set is the singleton $\{(\lambda_1, \lambda_2/\lambda_1)\}$. Whenever some $f \in \mathbb{R}[X, Y]$ is nonnegative on this point, then

$$f = f(\lambda_1, \frac{\lambda_2}{\lambda_1}) + f_1 \cdot (X - \lambda_1) + f_2 \cdot (Y - \frac{\lambda_2}{\lambda_1})$$

for some polynomials f_1, f_2 with $\deg(f_1), \deg(f_2) \leq \deg(f)$. From the identity

$$Y - \frac{\lambda_2}{\lambda_1} = -\frac{1}{\lambda_1} Y \cdot (X - \lambda_1) + \frac{1}{\lambda_1} \cdot (XY - \lambda_2)$$

we see that f belongs to M_λ with the required degree bounds independent of λ. If $\lambda_1 = 0$, then the semi-algebraic set defined by M_λ is empty. In fact we have

$$1 = \frac{1}{\lambda_2} Y \cdot X - \frac{1}{\lambda_2} \cdot (XY - \lambda_2),$$

which shows that every $f \in \mathbb{R}[X, Y]$ belongs to M_λ with the required degree bounds independent of λ. So Theorem 4.14 applied to M shows

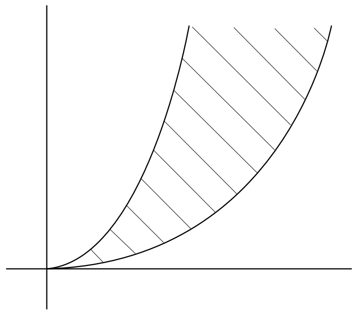
$$M^{\ddagger} = M^{\mathrm{sat}}.$$

Note that we cannot use Theorem 4.21 with the bounded polynomial $b = X$ or $b = XY$ to obtain this result, as not all the points at infinity of these polynomials lie in the closure of $\mathcal{S}(M) \cap C_\lambda(\mathbb{R})$.

Example 5.4. Now we give some examples for the geometric stability results from Chapter 3.

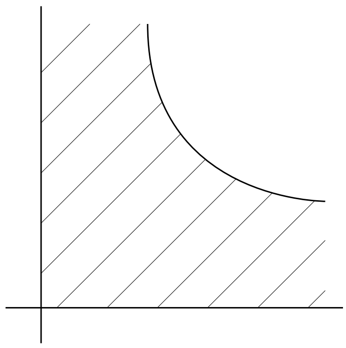
The first set we look at is defined by the three inequalities

$$0 \leq x, x^2 \leq y, y \leq 2x^2 :$$



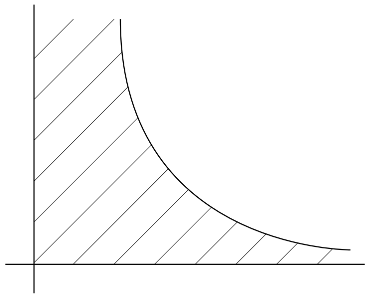
It contains a set $T_{K,(1,2)}$ (see Section 3.3 for the notation). Therefore every finitely generated quadratic module describing this set is stable, thus also closed and does not have (SMP).

The second set is described by $0 \leq x, 0 \leq y, (x-1)(y-1) \leq 1$:



It contains a full dimensional cylinder in each direction of coordinates (that is, sets $T_{K_1,(1,0)}$ and $T_{K_2,(0,1)}$), and so every finitely generated quadratic module describing it is stable, closed and cannot have (SMP). This is one way to answer Open Question 4 from [KMS]. Another way to solve this open question is due to Claus Scheiderer. One applies Theorem 3.10 from [PSc].

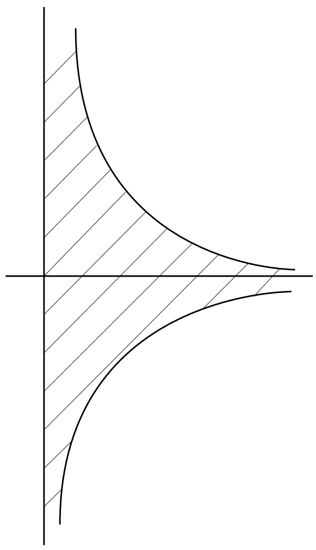
We can weaken the geometric situation and still obtain stability. Look at the inequalities $0 \leq x, 0 \leq y, (x-1)y \leq 1$:



This set contains a full dimensional cylinder in direction of y (a set $T_{K_1,(0,1)}$) and a set $T_{K_2,(1,-1)}$. The $(0,1)$- and the $(1,-1)$-gradings cover the usual grading, by Proposition 3.19 (or the fact that there are no nontrivial bounded polynomials; see Theorem 3.20). So every finitely generated quadratic module describing this set is stable, therefore also closed and cannot have (SMP).

We can still go one step further in narrowing the tentacles going to infinity. Look at the semi-algebraic set defined by

$$0 \leq x, x^2 y \leq 1, -1 \leq xy :$$



It contains a set $T_{K_1,(-1,2)}$ (corresponding to the tentacle going to infinity in positive direction of y), and a set $T_{K_2,(1,-1)}$ (corresponding to the part of the tentacle going to infinity in direction of x that lies below the x-axis). As

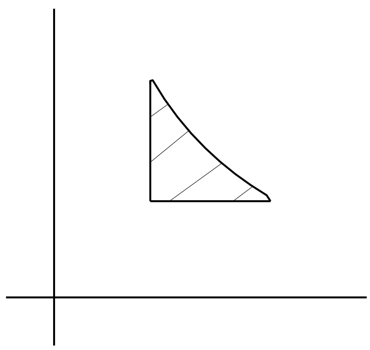
$$2 \cdot (-1,2) + 3 \cdot (1,-1) = (1,1)$$

is positive in each coordinate, the results from Chapter 3 show that every finitely generated quadratic module describing this set is stable, and therefore also closed and does not have (SMP). The considerations also show that there are no nontrivial bounded polynomials on this set, which is not completely obvious in this case.

Example 5.5. This last example illustrates a non-geometric stability result one more time. Exactly the same argument as applied to M_1 in the first example shows, that the quadratic module

$$M = \mathrm{QM}(X - \frac{1}{2}, Y - \frac{1}{2}, 1 - XY)$$

is stable and therefore closed. In contrast to M_1, it describes a compact set:



This quadratic module is Example 6.3.1 from [PD], for a non-archimedean quadratic module describing a compact set. We can see here that M is not only non-archimedean, but indeed does not have (SMP), which is stronger.

Zusammenfassung auf Deutsch

Endlich viele reelle Polynome $f_1, \ldots, f_t \in \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \ldots, X_n]$ definieren eine abgeschlossene semi-algebraische Menge

$$\mathcal{S}(f_1, \ldots, f_t) := \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_t(x) \geq 0\}\,.$$

Man möchte nun die Menge

$$\operatorname{Pos}(\mathcal{S}(f_1, \ldots, f_t)) = \{f \in \mathbb{R}[\underline{X}] \mid f \geq 0 \text{ auf } \mathcal{S}(f_1, \ldots, f_t)\}\,,$$

also die Menge der auf $\mathcal{S}(f_1, \ldots, f_t)$ nichtnegativen Polynome genauer untersuchen. Dazu betrachtet man zunächst die Präordnung, die von den Polynomen $f_1, \ldots, f_t$ definiert wird. Sie entsteht aus den f_i und Quadratsummen von Polynomen durch Addieren und Multiplizieren, also

$$\operatorname{PO}(f_1, \ldots, f_t) = \left\{ \sum_{e \in \{0,1\}^t} \sigma_e f_1^{e_1} \cdots f_t^{e_t} \mid \sigma_e \in \sum \mathbb{R}[\underline{X}]^2 \right\}.$$

Offensichtlich ist $\operatorname{PO}(f_1, \ldots, f_t)$ in $\operatorname{Pos}(\mathcal{S}(f_1, \ldots, f_t))$ enthalten, im Allgemeinen gilt jedoch keine Gleichheit. Man erweitert nun $\operatorname{PO}(f_1, \ldots, f_t)$ durch

$$\begin{aligned}\operatorname{PO}(f_1, \ldots, f_t)^{\ddagger} := \{f \in \mathbb{R}[\underline{X}] \mid & \exists q \in \mathbb{R}[\underline{X}]\ \forall \varepsilon > 0 \\ & f + \varepsilon q \in \operatorname{PO}(f_1, \ldots, f_t)\},\end{aligned}$$

sowie

$$\begin{aligned}\operatorname{PO}(f_1, \ldots, f_t)^{\vee\vee} := \{f \in \mathbb{R}[\underline{X}] \mid & L(f) \geq 0 \text{ für alle} \\ & L\colon \mathbb{R}[\underline{X}] \to \mathbb{R} \text{ linear mit} \\ & L\left(\operatorname{PO}(f_1, \ldots, f_t)\right) \subseteq [0, \infty)\},\end{aligned}$$

und erhält so folgende Kette:

$$\begin{aligned}\operatorname{PO}(f_1, \ldots, f_t) &\subseteq \operatorname{PO}(f_1, \ldots, f_t)^{\ddagger} \\ &\subseteq \operatorname{PO}(f_1, \ldots, f_t)^{\vee\vee} \\ &\subseteq \operatorname{Pos}(\mathcal{S}(f_1, \ldots, f_t)).\end{aligned}$$

Dabei kann $\mathrm{PO}(f_1,\dots,f_t)^{\vee\vee}$ auch als der Abschluss und $\mathrm{PO}(f_1,\dots,f_t)^{\ddagger}$ als der Folgenabschluss von $\mathrm{PO}(f_1,\dots,f_s)$ in der feinsten lokalkonvexen Topologie auf $\mathbb{R}[\underline{X}]$ charakterisiert werden.

Die Untersuchung von $\mathrm{PO}(f_1,\dots,f_t)^{\vee\vee}$ ist stark durch das sogenannte Momentenproblem motiviert, also der Frage, wann ein lineares Funktional ein darstellendes Maß besitzt. Der folgende Satz, eine unmittelbare Folgerung aus dem Satz von Haviland [H], zeigt diesen Zusammenhang:

Satz. Gilt $\mathrm{PO}(f_1,\dots,f_t)^{\vee\vee} = \mathrm{Pos}(\mathcal{S}(f_1,\dots,f_t))$, so hat jedes lineare Funktional des Polynomrings $\mathbb{R}[\underline{X}]$, welches auf $\mathrm{PO}(f_1,\dots,f_t)$ nichtnegativ ist, ein darstellendes Maß auf $\mathcal{S}(f_1,\dots,f_t)$. Das heißt, es gibt für jedes solche L ein Maß μ mit

$$L(f) = \int_{\mathcal{S}(f_1,\dots,f_t)} f d\mu \quad \forall f \in \mathbb{R}[\underline{X}].$$

Ein erstes wichtiges Resultat hierzu liefert Schmüdgens berühmter Satz aus dem Jahr 1991 (siehe [Sm2]). Er besagt (unter anderem) die Gleichheit

$$\mathrm{PO}(f_1,\dots,f_t)^{\ddagger} = \mathrm{PO}(f_1,\dots,f_t)^{\vee\vee} = \mathrm{Pos}(\mathcal{S}(f_1,\dots,f_t))$$

für den Fall, dass $\mathcal{S}(f_1,\dots,f_t)$ kompakt ist.

Für den nichtkompakten Fall gibt es Schmüdgens Fasersatz aus dem Jahr 2003:

Satz (Schmüdgen, [Sm3])**.** Sei $b \in \mathbb{R}[\underline{X}]$ mit $c \le b \le C$ auf $\mathcal{S}(f_1,\dots,f_t)$ für gewisse $c \le C$. Dann gilt

$$\mathrm{PO}(f_1,\dots,f_t)^{\vee\vee} = \bigcap_{r\in[c,C]} \mathrm{PO}(f_1,\dots,f_t, b-r, r-b)^{\vee\vee}.$$

Falls also

$$\mathrm{PO}(f_1,\dots,f_t,b-r,r-b)^{\vee\vee} = \mathrm{Pos}(\mathcal{S}(f_1,\dots,f_t,b-r,r-b))$$

für alle r gilt, so gilt auch

$$\mathrm{PO}(f_1, \ldots, f_t)^{\vee\vee} = \mathrm{Pos}(\mathcal{S}(f_1, \ldots, f_t)).$$

Die Faserpräordnungen $\mathrm{PO}(f_1, \ldots, f_t, b - r, r - b)$ definieren im Allgemeinen niedrigerdimensionale semi-algebraische Mengen. Oft ist deshalb mehr über sie bekannt, und der Fasersatz erlaubt die Übertragung des Wissens auf die komplizierteren Präordnungen $\mathrm{PO}(f_1, \ldots, f_t)$ und $\mathrm{PO}(f_1, \ldots, f_t)^{\vee\vee}$.

Das erste Hauptkapitel der vorliegenden Arbeit (Kapitel 2) beschäftigt sich mit diesem Fasersatz. Der ursprüngliche Beweis in [Sm3] benützt tiefliegende funktionalanalytische Resultate. Eine direkte Integralzerlegung von GNS Repräsentationen wird verwendet, um ein lineares Funktional des Polynomrings als Integral über andere Funktionale darzustellen. In Kapitel 2 wird Schmüdgens Satz nun elementarer bewiesen. Zusätzlich kann das ursprüngliche Resultat verallgemeinert werden. Unter gewissen Voraussetzungen gilt die Aussage auch für sogenannte quadratische Moduln, und in Algebren von abzählbarer Vektorraumdimension, anstatt nur in $\mathbb{R}[\underline{X}]$. Diese Ergebnisse folgen aus einem allgemeinen Hauptsatz, in dessen Beweis der Satz von Radon-Nikodym ein wesentlicher Bestandteil ist. Kapitel 2 endet mit einigen Anwendungen von Schmüdgens Fasersatz.

In Kapitel 3 wird Stabilität von Präordnungen und quadratischen Moduln untersucht. Dieser Begriff geht auf die Arbeit [PSc] zurück. Dabei heißt die Präordnung $\mathrm{PO}(f_1, \ldots, f_t)$ *stabil*, wenn man gewisse Gradschranken für die Quadratsummen in der Darstellung von Polynomen finden kann. Stabilität ist eine interessante Eigenschaft. In [PSc, Sc4] wird bewiesen, dass sie häufig

$$\mathrm{PO}(f_1, \ldots, f_t) = \mathrm{PO}(f_1, \ldots, f_t)^{\vee\vee} \subsetneq \mathrm{Pos}(\mathcal{S}(f_1, \ldots, f_t))$$

impliziert. In Kapitel 3 definieren wir nun Stabilität bezüglich Graduierungen, und setzen den Begriff mit dem urspünglichen in Verbindung. Eine Charakterisierung der Stabilität bezüglich Graduierungen liefert dann sowohl geometrische als auch kombinatorische Kriterien, um die Stabilität eines quadratischen Moduls oder einer Präordnung zu entscheiden. Alle diese Kriterien sind sehr einfach anzuwenden. Als Hauptergebnis (Theorem 3.20) erhalten wir, dass für eine gewisse Klasse von semialgebraischen Mengen die Nichtexistenz von nichttrivialen beschränkten Polynomen die Stabilität jedes korrespondierenden quadratischen Moduls impliziert.

Das letzte Hauptkapitel der Arbeit (Kapitel 4) ist der Untersuchung von $\mathrm{PO}(f_1,\dots,f_t)^\ddagger$ gewidmet. Wir beantworten die in [KM, KMS] gestellte Frage, ob

$$\mathrm{PO}(f_1,\dots,f_t)^\ddagger = \mathrm{PO}(f_1,\dots,f_t)^{\vee\vee}$$

immer stimmt, mit einem Gegenbeispiel (Abschnitt 4.2). Das Beispiel zeigt ebenfalls, dass die in [Sm3] gestellte Frage, ob der oben genannte Fasersatz auch für $\ddagger$ anstelle von $\vee\vee$ gilt, negativ beantwortet werden muss.

Unter stärkeren Voraussetzungen ist es aber möglich, ein Faserkriterium für die Zugehörigkeit eines Polynoms zu $\mathrm{PO}(f_1,\dots,f_t)^\ddagger$ zu geben. Dies ist der Inhalt von Theorem 4.14, dem Hauptsatz des Kapitels 4. Er erlaubt Anwendungen, die über die bisher bekannten Fasersätze für Zylinder aus [KM, KMS] hinausgehen (siehe Abschnitt 4.5).

Die Arbeit endet mit Kapitel 5, in dem eine Sammlung von zweidimensionalen Beispielen die vorangegangenen Resultate verdeutlicht.

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