

# Linear Elliptic Equations of Second Order

Lecture Notes

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# Preface

These lecture notes are intended as an introduction to second order elliptic partial differential equations. It can be considered as a continuation of a chapter on elliptic equations of the lecture notes [17] on partial differential equations. In [17] we focused our attention mainly on explicit solutions for standard problems for elliptic, parabolic and hyperbolic equations.

Unfortunately, in general there are no explicit solutions. The main topic of these lecture notes is to provide existence results for elliptic equations and qualitative results as regularity properties or asymptotic behaviour of solutions. Some of these considerations lead directly to numerical methods for solving the problems considered, but this is not the topic of these notes.

The first chapter concerns integral equation methods for boundary value problems of the Laplace equation. This method can be extended to a large class of linear elliptic equations and systems. In the following chapter we consider Perron's method for the Dirichlet problem for the Laplace equation. This method is based on the maximum principle and on estimates of derivatives of solutions of the Laplace equation. In order to extend maximum principle methods to more general linear and nonlinear elliptic equation we consider more general maximum principles.

For additional reading we recommend following books: W. I. Smirnov [21], I. G. Petrowski [20], D. Gilbarg and N. S. Trudinger [10], S. G. Michlin [14], P. R. Garabedian [9], W. A. Strauss [22], F. John [13], L. C. Evans [5] and R. Courant and D. Hilbert [4]. Some material of these lecture notes was taken from some of these books.



# Chapter 1

## Potential theory

The notation *potential* has its origin in Newton's attraction rule

$$K(x, y) = -G \frac{Mm}{|y-x|^2} \frac{y-x}{|y-x|},$$

where  $G = 6.67 \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$ , and  $K$  is the force acting between two mass points  $M$  and  $m$  located at  $x, y \in \mathbb{R}^3$ , respectively. Since  $\text{rot } K = 0$ , there is a scalar function  $Q(x, y)$ , called *potential*, such that  $\nabla_x Q(x, y) = K(x, y)$ . Thus,  $Q(x, y) = -GMm|y-x|^{-1}$  is a Newton potential. The function  $Q(x, y)$  defines the work which has to be done to move one of the mass points to infinity if the other one is fixed.

Let  $\Omega \subset \mathbb{R}^n$  be a bounded, connected and sufficiently regular domain. Consider for given  $f$  and  $h$  the boundary value problem

$$\begin{aligned} -\Delta v &= f \text{ in } \Omega \\ v &= h \text{ on } \partial\Omega. \end{aligned}$$

We can transform this problem into a boundary value problem for the Laplace equation by setting  $v = u - w$ , where

$$w(x) = \int_{\Omega} s(|x-y|)f(y) dy.$$

Here  $s(r)$  denotes the *singularity function*, see also [17],

$$s(r) := \begin{cases} -\frac{1}{2\pi} \ln r & : n = 2 \\ \frac{r^{2-n}}{(n-2)\omega_n} & : n \geq 3 \end{cases}$$

We recall that  $\omega_n = |\partial B_1(0)|$ . Since  $w \in C^1(\mathbb{R}^n)$  and  $-\Delta w = f$  in  $\Omega$  if  $f$  is sufficiently regular, see Section 5.1, we arrive at the problem  $\Delta u = 0$

in  $\Omega$  and  $v = h - w$  on  $\partial\Omega$ . Consequently, it is sufficient to consider the boundary value problem for the Laplace equation, which is a problem with a homogeneous differential equation.

The *Dirichlet problem* (first boundary value problem) is to find a solution  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  of

$$\Delta u = 0 \text{ in } \Omega \tag{1.1}$$

$$u = \Phi \text{ on } \partial\Omega, \tag{1.2}$$

where  $\Phi$  is given and continuous on  $\partial\Omega$ .

The *Neumann problem* (second boundary value problem) is to find a solution  $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$  of

$$\Delta u = 0 \text{ in } \Omega \tag{1.3}$$

$$\frac{\partial u}{\partial n} = \Psi \text{ on } \partial\Omega, \tag{1.4}$$

where  $\Psi$  is given and continuous on  $\partial\Omega$ .

In [17], Chapter 7, we derived an explicit formula for the solution of (1.1), (1.2) if  $\Omega$  is a ball. In general, one gets explicit solutions, provided the Green function is known for the domain  $\Omega$  considered.

We denote (1.1), (1.2) by  $(D_i)$  and (1.3), (1.4) by  $(N_i)$  to indicate that the problems considered concerns the interior of  $\Omega$ . Then  $(D_e)$  and  $(N_e)$  denote the associated exterior problems, that is we have to replace in (1.1) and (1.3) the domain  $\Omega$  by its complement  $\mathbb{R}^n \setminus \overline{\Omega}$ .

For the Dirichlet problems we make an ansatz with a dipole potential

$$W(z) = \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) dS_y \tag{1.5}$$

if  $n \geq 3$ . In the case that  $n = 2$  we have to replace  $|z-y|^{2-n}$  by  $-\ln(|z-y|)$ . In the formula above  $\nu(y)$  denotes the exterior unit normal at  $y \in \partial\Omega$  and  $\sigma(y)$  the *dipole density*.

For the Neumann problem we make an ansatz with a single layer potential

$$V(z) = \int_{\partial\Omega} \frac{\sigma(y)}{|z-y|^{n-2}} dS_y \tag{1.6}$$

if  $n \geq 3$ . In the case that  $n = 2$  we have to replace  $|z-y|^{2-n}$  by  $-\ln(|z-y|)$ .

Both potentials solve the Laplace equation in  $\mathbb{R}^n \setminus \partial\Omega$ .

In the rest of this chapter we assume that  $n \geq 3$ .

We will see that discontinuous properties of these surface potentials lead to integral equations which can be studied by using Fredholm's results on integral equations. Thus, the method of surface potentials provide a beautiful example for Fredholm's theory.

## 1.1 Preliminaries

Let  $\Omega \subset \mathbb{R}^n$  be a bounded and connected domain with a sufficiently regular boundary  $\partial\Omega$ .

**Definition.** We say that  $\partial\Omega \in C^{1,\lambda}$ ,  $0 < \lambda \leq 1$ , if:

(i) For each given  $x \in \partial\Omega$  there exists a  $\rho > 0$  and  $N = N(x, \rho)$  balls  $B_{2\rho}(x_i) \subset \mathbb{R}^n$ ,  $i = 1, \dots, N$ , with centers  $x_i \in \partial\Omega$ , where  $x_1 = x$ , such that

$$\partial\Omega \subset \bigcup_{i=1}^N B_{2\rho}(x_i).$$

(ii) Let  $T_{x_i}$  be a plane which contains  $x_i$  and denote by  $Z_{2\rho}(x_i)$  a circular cylinder parallel to the normal on  $T_{x_i}$  such that its intersection with the plane  $T_{x_i}$  is a ball in  $\mathbb{R}^{n-1}$  with radius  $2\rho$  and the center at  $x_i$ . We assume the intersection  $\partial\Omega \cap Z_{2\rho}(x_i)$  has a local representation  $\tau = f(\xi)$ ,  $f \equiv f_i$ , where  $\xi$  is in an  $(n-1)$ -dimensional ball  $D_{2\rho} = D_{2\rho}(0)$  with radius  $2\rho$  and the center at  $0 \in \mathbb{R}^{n-1}$ . Moreover, we assume

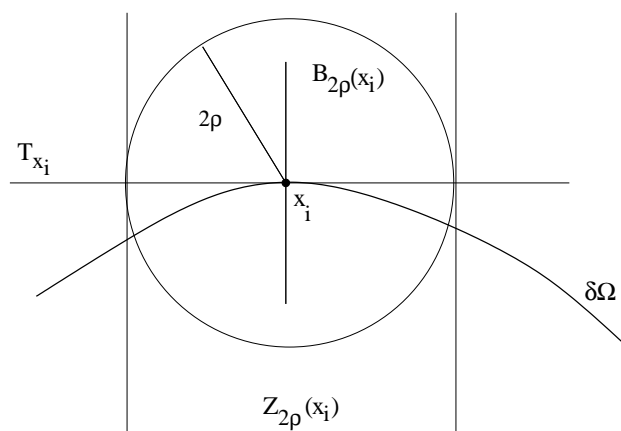
$$f \in C^{1,\lambda}(\overline{D_{2\rho}}), \quad f(0) = 0, \quad \nabla f(0) = 0.$$

**Lemma 1.1.1** (Partition of unity). *There exists  $\eta_i \in C_0^\infty(B_{2\rho}(x_i))$ ,  $0 \leq$*

*$\eta_i \leq 1$ , such that*

$$\sum_{i=1}^N \eta_i(x) = 1 \quad \text{if } x \in \bigcup_{i=1}^N B_{2\rho}(x_i).$$

*Proof.* For given  $B_{2\rho}(x_i)$  there exists  $\phi_i \in C_0^\infty(B_{2\rho}(x_i))$  with the properties

Figure 1.1: Definition of  $\partial\Omega \in C^{1,\lambda}$ 

that  $\phi_i = 1$  in  $B_\rho(x_i)$  and  $0 \leq \phi_i(x) \leq 1$ , see an exercise. Set

$$\begin{aligned} \eta_1 &= \phi_1 \\ &= 1 - (1 - \phi_1), \\ \eta_i &= \phi_i(1 - \phi_1) \cdot \dots \cdot (1 - \phi_{i-1}) \\ &= (1 - (1 - \phi_i))(1 - \phi_1) \cdot \dots \cdot (1 - \phi_{i-1}). \end{aligned}$$

Then

$$\sum_{i=1}^N \eta_i(x) = 1 - (1 - \phi_1) \cdot \dots \cdot (1 - \phi_N),$$

which implies that

$$\sum_{i=1}^N \eta_i(x) = 1, \quad \text{if } x \in \bigcup_{i=1}^N B_\rho(x_i)$$

since at least one of the factors is zero. □

Assume  $\partial\Omega \in C^{1,\lambda}$ , then we define the area integral by

$$\begin{aligned} \int_{\partial\Omega} g(y) dS_y &= \int_{\partial\Omega} \sum_{i=1}^N \eta_i(y) g(y) dS_y \\ &= \sum_{i=1}^N \int_{\partial\Omega} \eta_i(y) g(y) dS_y, \end{aligned}$$

where

$$\begin{aligned} & \int_{\partial\Omega} \eta_i(y) g(y) dS_y \\ &= \int_{D_{2\rho}} \eta_i(\xi, f_i(\xi)) g(\xi, f_i(\xi)) \sqrt{1 + |\nabla f_i(\xi)|^2} d\xi. \end{aligned}$$

Here we suppose that  $\partial\Omega \cap Z_{2\rho}(x_i)$  has the parametric representation  $y = (\xi, f_i(\xi))$ .

## 1.2 Dipole potential

**Dipole density.** The following consideration leads to the formula (1.5) for the dipole potential in the case of three dimensions.

Consider two parallel surfaces, one inside of  $\Omega$  and the other one outside, of distance  $\epsilon/2$ ,  $\epsilon > 0$  small, from  $\partial\Omega$ , see Figure 1.2. Assume there is a

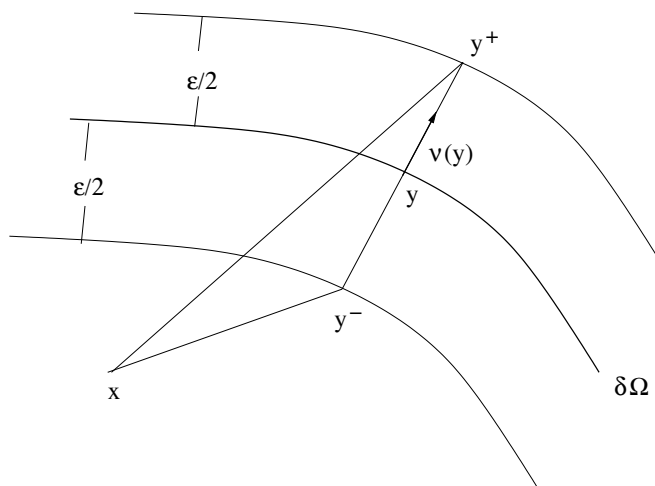


Figure 1.2: Double layer potential

loading of power  $\epsilon^{-1}$  at  $y^+ = y + (\epsilon/2)\nu(y)$  and a loading of power  $-\epsilon^{-1}$  at  $y^- = y - (\epsilon/2)\nu(y)$ . Set  $z^+ = y^+ - x$  and  $z^- = y^- - x$ , then the potential at  $x$  of the dipole is

$$\begin{aligned} u &= \frac{1}{\epsilon} \left( \frac{1}{|z^+|} - \frac{1}{|z^-|} \right) \\ &= \frac{1}{\epsilon} \frac{|z^-|^2 - |z^+|^2}{(|z^+| + |z^-|)|z^+||z^-|}. \end{aligned}$$

Since  $z^- = z^+ - \epsilon\nu(y)$ , we have

$$\begin{aligned} |z^-|^2 &= \langle z^+ - \epsilon\nu(y), z^+ - \epsilon\nu(y) \rangle \\ &= |z^+|^2 - 2\epsilon z^+ \cdot \nu(y) + \epsilon^2. \end{aligned}$$

Thus

$$u = -\frac{1}{\epsilon} \frac{2\epsilon z^+ \cdot \nu(y) + \epsilon^2}{(|z^+| + |z^-|)|z^+||z^-|}.$$

Set  $z = y - x$ , then

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} u &= -\frac{z \cdot \nu(y)}{|z|^3} \\ &= \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|y-x|} \right) \end{aligned}$$

is the potential of a single dipole with density  $\sigma(y) = 1$  at  $y$ . Multiplication with a density  $\sigma(y)$  and integration over  $\partial\Omega$  leads to (1.5).

The right hand side of (1.5) is called *dipole potential* or *potential of a double layer* with density  $\sigma$ . The dipole potential is in  $C^\infty(\mathbb{R}^n \setminus \partial\Omega)$  and a solution of the Laplace equation in  $\mathbb{R}^n \setminus \partial\Omega$ . In fact, see the following proposition, the right hand side of (1.5) is defined and continuous on  $\partial\Omega$  provided the boundary  $\partial\Omega$  is sufficiently smooth, but  $W(x)$  makes a jump across  $\partial\Omega$ .

Some of the following calculations are based on the formula for the directional derivative in direction  $\nu(y)$

$$\frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) = \frac{1}{|z-y|^n} \sum_{i=1}^n (z_i - y_i)(\nu(y))_i. \quad (1.7)$$

**Lemma 1.2.1.** *Assume  $\partial\Omega \in C^{1,\lambda}$  and  $\sigma \in C(\partial\Omega)$ . Then the right hand side of (1.5) is defined and continuous on  $\partial\Omega$ .*

*Proof.* Consider the case  $n \geq 3$ . Let  $x$  be the center of a local coordinate system and  $z \in \partial\Omega \cap Z_{2\rho}(x)$ , see Figure 1.3. We have to show that, see Section 1.1 for the definition of the surface integral and formula (1.7),

$$q(\zeta) := \int_{D_{2\rho}} \eta(\xi, f(\xi)) \sigma(\xi, f(\xi)) \frac{-(\zeta - \xi) \cdot \nabla f(\xi) + f(\zeta) - f(\xi)}{(|\zeta - \xi|^2 + |f(\zeta) - f(\xi)|^2)^{n/2}} d\xi$$

is continuous in a neighbourhood of  $0 \in \mathbb{R}^{n-1}$ . Here is  $D_{2\rho} = D_{2\rho}(0)$ ,  $z = (\zeta, f(\zeta))$  and  $y = (\xi, f(\xi))$  in local coordinates, and  $\eta \in C^\infty$  in its arguments.

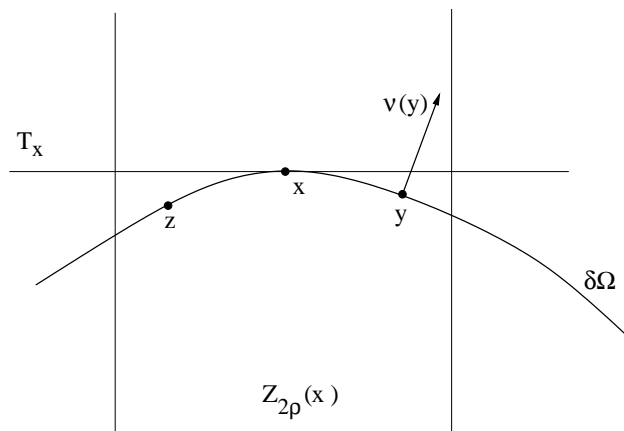


Figure 1.3: Local coordinates

Because of  $f \in C^{1,\lambda}(D_{2\rho})$ ,  $f(0) = 0$  and  $\nabla f(0) = 0$  we have

$$q(\zeta) = \int_{D_{2\rho}} \frac{A(\xi, \zeta)}{|\xi - \zeta|^{n-1-\lambda}} d\xi,$$

where  $A(\xi, \zeta)$  is bounded on  $\overline{D_{2\rho}} \times \overline{D_{2\rho}}$  and continuous if  $\xi \neq \zeta$ . Since the integrand is weakly singular, it follows that  $q(\zeta)$  is continuous on  $D_{2\rho}$ , see an exercise.  $\square$

Let  $x_0 \in \partial\Omega$  and  $x \in \mathbb{R}^n$ . Set

$$W(x) = W_1(x) + \sigma(x_0)W_0(x),$$

where

$$W_1(x) = \int_{\partial\Omega} (\sigma(y) - \sigma(x_0)) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y, \quad (1.8)$$

$$W_0(x) = \int_{\partial\Omega} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y. \quad (1.9)$$

The integral  $W_0(x)$  is called *Gauss integral*.

**Lemma 1.2.2.** *Suppose that  $\partial\Omega \in C^{1,\lambda}$ . Then*

$$W_0(x) = \begin{cases} -(n-2)\omega_n & : x \in \Omega \\ 0 & : x \notin \overline{\Omega} \\ -\frac{n-2}{2}\omega_n & : x \in \partial\Omega \end{cases}.$$

*Proof.* (i) If  $x \in \mathbb{R}^n \setminus \overline{\Omega}$  is fixed, then there is a domain  $\Omega_0 \supset \supset \Omega$  where  $|x - y|^{2-n} \in C^\infty(\Omega_0)$  and satisfies the Laplace equation. Then

$$\begin{aligned} 0 &= \int_{\Omega} \Delta_y \left( \frac{1}{|x - y|^{n-2}} \right) dy \\ &= \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y. \end{aligned}$$

(ii) Let  $x \in \Omega$  be fixed, then there is a ball  $B_\rho(x) \subset \Omega$ . Then

$$\begin{aligned} 0 &= \int_{\Omega \setminus B_\rho(x)} \Delta_y \left( \frac{1}{|x - y|^{n-2}} \right) dy \\ &= \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y - \int_{\partial B_\rho(x)} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y, \end{aligned}$$

where in the second integral  $\nu(y)$  denotes the exterior unit normal at the boundary of  $B_\rho(x)$ . Using polar coordinates with center at  $x$ , we find for the second integral

$$\begin{aligned} \int_{\partial B_\rho(x)} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y &= \int_{\partial B_1(x)} \frac{\partial}{\partial\rho} (\rho^{2-n}) \rho^{n-1} dS \\ &= (2 - n)\omega_n. \end{aligned}$$

(iii) Let  $x \in \partial\Omega$  and set for a sufficiently small  $\rho > 0$ ,

$$S_\rho = \Omega \cap \partial B_\rho(x), \quad C_\rho = \partial\Omega \setminus B_\rho(x),$$

see Figure 1.4. Then

$$\begin{aligned} 0 &= \int_{\Omega \setminus B_\rho(x)} \Delta_y \left( \frac{1}{|x - y|^{n-2}} \right) dy \\ &= \int_{C_\rho} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y - \int_{S_\rho} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y. \end{aligned}$$

Since, see an exercise,

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y = \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y,$$

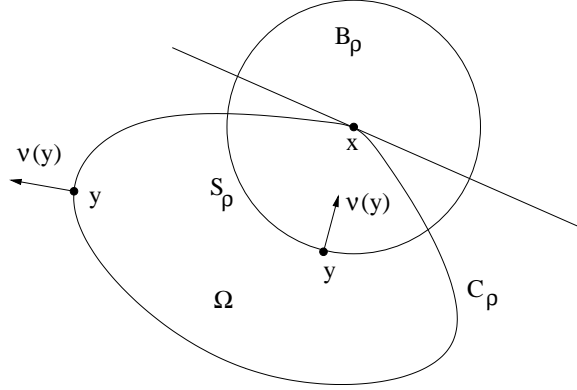


Figure 1.4: Figure to the proof of Lemma 1.2.2

it follows that

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y = \lim_{\rho \rightarrow 0} \int_{S_\rho} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y.$$

We have

$$\int_{S_\rho} \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y = -(n-2)\rho^{1-n} \int_{S_\rho} dS_y$$

and

$$\int_{S_\rho} dS_y = \frac{\omega_n}{2} \rho^{n-1} (1 + O(\rho^{2\lambda})).$$

The previous formula follows by introducing local coordinates at  $x$ . Let  $(\xi, h(\xi)) \in \partial B_\rho(x) \cap \partial\Omega$ , then

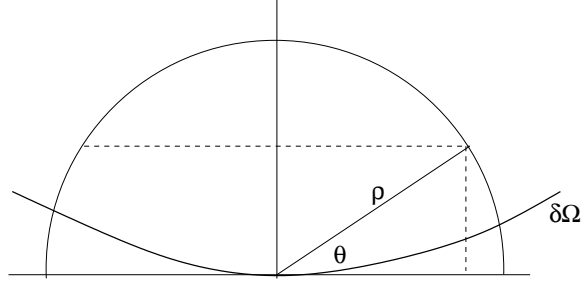
$$|h(\xi)| \leq c\rho^{1+\lambda}.$$

Let  $F$  be a layer of a sphere with radius  $\rho$  of height  $c\rho^{1+\lambda}$ , see Figure 1.5, then

$$\frac{\omega_n}{2} \rho^{n-1} - |F| \leq |S_\rho| \leq \frac{\omega_n}{2} \rho^{n-1} + |F|.$$

We have

$$\begin{aligned} |F| &= \frac{1}{2} \rho^{n-1} \omega_n (1 - \cos \theta) \\ &= \frac{1}{2} \rho^{n-1} \omega_n \left( 1 - (1 - c^2 \rho^{2\lambda})^{1/2} \right) \\ &= \frac{1}{2} \rho^{n-1} \omega_n O(\rho^{2\lambda}) \end{aligned}$$

Figure 1.5: Estimate of  $|S_\rho|$ 

as  $\rho \rightarrow 0$ . □

**Lemma 1.2.3.** *Let  $\partial\Omega \in C^{1,\lambda}$ . Then*

$$\int_{\partial\Omega} \left| \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y$$

is uniformly bounded for  $x \in \mathbb{R}^n$ .

*Proof.* (i) For fixed  $d > 0$  consider  $x$  such that  $\text{dist}(x, \partial\Omega) \geq d/2$ . Then, see formula (1.7),

$$\left| \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| \leq (n-2) \frac{2^{n-1}}{d^{n-1}},$$

which implies that

$$\int_{\partial\Omega} \left| \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y \leq \frac{(n-2)2^{n-1}}{d^{n-1}} |\partial\Omega|.$$

(ii) Consider  $x \in \mathbb{R}^n$  such that  $\text{dist}(x, \partial\Omega) < d/2$  for a  $d > 0$  and let

$$x_0 \in \partial\Omega : \quad |x - x_0| = \min_{y \in \partial\Omega} |x - y|.$$

Set  $S_d = \partial\Omega \cap B_d(x_0)$ . Then for  $y \in \partial\Omega \setminus S_d$  we have

$$|y - x| \geq |y - x_0| - |x - x_0| > d/2,$$

which implies that

$$\begin{aligned} \int_{\partial\Omega \setminus S_d} \left| \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y &\leq \frac{(n-2)2^{n-1}}{d^{n-1}} |\partial\Omega \setminus S_d| \\ &\leq \frac{(n-2)2^{n-1}}{d^{n-1}} |\partial\Omega|. \end{aligned}$$

(iii) Consider

$$I_d := \int_{S_d} \left| \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y.$$

In local coordinates, see Figure 1.6, we have, since  $x - x_0$  is perpendicular

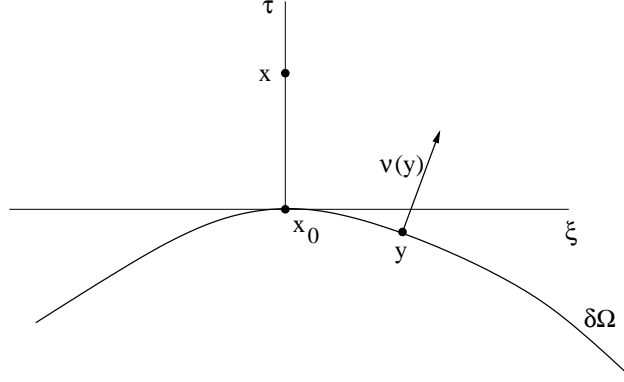


Figure 1.6: Local coordinates, case (iii)

on the tangent plane  $T_{x_0}$ , that  $x = (0, \dots, 0, \delta)$  and  $y = (\xi, f(\xi))$ , where  $f \in C^{1,\lambda}(\overline{D_\rho})$ ,  $f(0) = 0$ ,  $\nabla f(0) = 0$  and  $D_\rho$  is a ball in  $\mathbb{R}^{n-1}$  with the center at  $0 \in \mathbb{R}^{n-1}$  and the radius  $\rho$ . We choose  $d > 0$  such that  $\rho > d$ . Then, see formula (1.7),

$$\begin{aligned} \sqrt{1 + |\nabla f(\xi)|^2} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) &= \frac{\xi \cdot \nabla f(\xi) + (\delta - f(\xi))}{(|\xi|^2 + (\delta - f(\xi))^2)^{n/2}} \\ &= \frac{h_1(\xi) + \delta}{(|\xi|^2 + (\delta + h_2(\xi))^2)^{n/2}}, \end{aligned}$$

where  $h_i \in C(\overline{D_\rho(0)})$ ,  $|h_i(\xi)| \leq c|\xi|^{1+\lambda}$ .

Since

$$\begin{aligned} |\xi|^2 + |\delta + h_2(\xi)|^2 &= |\xi|^2 + \delta^2 + 2\delta h_2 + h_2^2 \\ &\geq |\xi|^2 + \frac{1}{2}\delta^2 - 7h_2^2 \\ &\geq \frac{1}{2}|\xi|^2 + \frac{1}{2}\delta^2, \end{aligned}$$

provided  $\rho$  satisfies  $7c\rho^{2\lambda} < 1/2$ . It follows that

$$I_d \leq 2^{n/2} \int_{D_\rho} \frac{c|\xi|^{1+\lambda} + |\delta|}{(|\xi|^2 + \delta^2)^{n/2}} d\xi. \quad (1.10)$$

The right hand side of (1.10) is uniformly bounded with respect to  $|\delta| < d$ . More precisely, we have

$$I_d \leq 2^{n/2} \omega_{n-1} \max\{c\lambda^{-1}\rho^\lambda, \pi/2\},$$

see an exercise.  $\square$

**Lemma 1.2.4.** *Assume  $\sigma \in C(\partial\Omega)$  and  $x_0 \in \partial\Omega$ . Then  $W_1(x)$ , see definition (1.8), is continuous at  $x_0$ .*

*Proof.* Set  $S_\rho = \partial\Omega \cap B_\rho(x_0)$  and  $W_1(x) = I_1 + I_2$ , where

$$\begin{aligned} I_1(x) &= \int_{S_\rho} (\sigma(y) - \sigma(x_0)) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y \\ I_2(x) &= \int_{\partial\Omega \setminus S_\rho} (\sigma(y) - \sigma(x_0)) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y. \end{aligned}$$

We have

$$|W_1(x) - W_1(x_0)| \leq |I_1(x)| + |I_1(x_0)| + |I_2(x) - I_2(x_0)|$$

and

$$|I_1(x)| \leq \int_{S_\rho} |\sigma(y) - \sigma(x_0)| \left| \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y.$$

Set, see Lemma 1.2.3,

$$C = \sup_{x \in \mathbb{R}^n} \int_{\partial\Omega} \left| \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y$$

and choose for given  $\epsilon > 0$  a  $\rho = \rho(\epsilon)$  such that

$$|\sigma(y) - \sigma(x_0)| < \frac{\epsilon}{3C}$$

if  $y \in S_\rho$ . Then  $|I_1(x)| < \epsilon/3$  and  $|I_1(x_0)| < \epsilon/3$ .

Consider  $x \in \mathbb{R}^n$  such that  $|x - x_0| < \rho/2$ , then

$$|y - x| \geq |y - x_0| - |x - x_0| \geq \rho/2,$$

provided that  $y \in \partial\Omega \setminus S_\rho$ . Since  $I_2$  is continuous in  $B_{\rho/2}(x_0)$ , there is a  $\delta = \delta(\epsilon)$  such that

$$|I_2(x) - I_2(x_0)| < \epsilon/3$$

if  $|x - x_0| < \delta(\epsilon)$ . Summarizing, we have

$$|W_1(x) - W_1(x_0)| < \epsilon$$

if  $|x - x_0| < \min\{\rho(\epsilon)/2, \delta(\epsilon)\}$ .  $\square$

Let  $x_0 \in \partial\Omega$  and denote by  $W_i(x_0)$  the limit of  $W(x)$  from interior to  $x_0$  and by  $W_e(x_0)$  the limit of  $W(x)$  from exterior to  $x_0$ .

**Proposition 1.2.1.** *Suppose that  $\sigma \in C(\partial\Omega)$  and  $x_0 \in \partial\Omega$ . The limits  $W_i(x_0)$  and  $W_e(x_0)$  exist and satisfy the jump relations*

$$\begin{aligned} W_i(x_0) &= -\frac{(n-2)\omega_n}{2}\sigma(x_0) + W(x_0), \\ W_e(x_0) &= \frac{(n-2)\omega_n}{2}\sigma(x_0) + W(x_0). \end{aligned}$$

*Proof.* We will prove the first of the jump relations. For  $x \in \Omega$  we set  $W(x) = W_1(x) + \sigma(x_0)W_0(x)$ , where  $W_1(x)$  is continuous at  $x_0$ , see Lemma 1.2.4, and  $W_0(x)$  is the Gauss integral, see Lemma 1.2.2. Thus

$$\begin{aligned} W_i(x_0) &= \lim_{x \rightarrow x_0, x \in \Omega} (W_1(x) + \sigma(x_0)W_0(x)) \\ &= W_1(x_0) - (n-2)\sigma(x_0) \\ &= \int_{\partial\Omega} (\sigma(y) - \sigma(x_0)) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x_0 - y|^{n-2}} \right) dS_y - (n-2)\sigma(x_0) \\ &= W(x_0) - \frac{(n-2)\omega_n}{2}\sigma(x_0). \end{aligned}$$

$\square$

**Corollary.** *The double layer potential*

$$W(x) = \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y,$$

where  $\sigma \in C(\partial\Omega)$ , defines a solution of the interior Dirichlet problem ( $D_i$ ) if and only if  $\sigma \in C(\partial\Omega)$  is a solution of the integral equation

$$\Phi(x) = -\frac{(n-2)\omega_n}{2}\sigma(x) + \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y,$$

where  $x \in \partial\Omega$ , and  $W(x)$  is a solution of the exterior Dirichlet problem  $(D_e)$  if and only if  $\sigma \in C(\partial\Omega)$  satisfies the integral equation

$$\Phi(x) = \frac{(n-2)\omega_n}{2}\sigma(x) + \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y$$

We recall that  $W$  is a solution of  $(D_i)$  if and only if

$$\Phi(x) = \lim_{z \rightarrow x, z \in \Omega} W(z),$$

and of  $(D_e)$  if and only if

$$\Phi(x) = \lim_{z \rightarrow x, z \in \mathbb{R}^n \setminus \bar{\Omega}} W(z).$$

### 1.3 Single layer potential

Consider the *single layer potential*

$$V(x) = \int_{\partial\Omega} \frac{\sigma(y)}{|x-y|^{n-2}} dS_y,$$

where  $\sigma \in C(\partial\Omega)$ .

**Lemma 1.3.1.**  $V \in C(\mathbb{R}^n)$ .

*Proof.* It remains to show that  $V(x)$  is continuous if  $x \in \partial\Omega$ . Let  $x \in \partial\Omega$ , set  $S_\rho = \partial\Omega \cap B_\rho(x)$ ,  $\rho > 0$  sufficiently small, and

$$V(x) = V_1(x) + V_2(x),$$

where

$$\begin{aligned} V_1(x) &= \int_{S_\rho} \frac{\sigma(y)}{|x-y|^{n-2}} dS_y, \\ V_2(x) &= \int_{\partial\Omega \setminus S_\rho} \frac{\sigma(y)}{|x-y|^{n-2}} dS_y. \end{aligned}$$

Consider  $z \in \mathbb{R}^n$ ,  $z$  in a neighbourhood of  $x$ . We have

$$|V(z) - V(x)| \leq |V_1(z)| + |V_1(x)| + |V_2(z) - V_2(x)|.$$

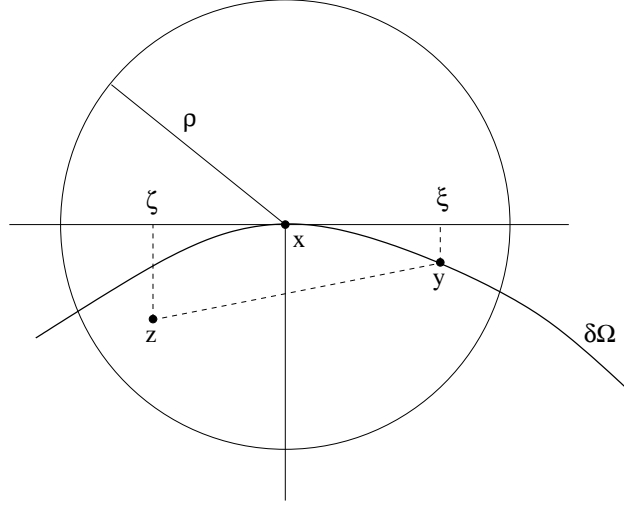


Figure 1.7: Proof of Lemma 1.3.1

In local coordinates it is  $y = (\xi, f(\xi))$ ,  $z = (\zeta, \delta)$ , where  $\xi, \zeta \in D_\rho = D_\rho(0)$ , see Figure 1.7, and

$$\begin{aligned}
 |V_1(z)| &\leq \int_{D_\rho} \frac{|\sigma(\xi, f(\xi))| \sqrt{1 + |\nabla f(\xi)|^2}}{(|\zeta - \xi|^2 + (\delta - f(\xi))^2)^{(n-2)/2}} d\xi \\
 &\leq c \int_{D_\rho} \frac{d\xi}{|\xi - \zeta|^{n-2}} \\
 &\leq c \int_{D_{2\rho}} \frac{d\xi}{|\xi|^{n-2}} \\
 &= 2c\omega_{n-1}\rho.
 \end{aligned}$$

Let  $\epsilon > 0$  be given and set  $\rho = \rho(\epsilon) = \epsilon/(6c\omega_{n-1})$ , then  $|V_1(z)| < \epsilon/3$  if  $|z - x| < \rho(\epsilon)$ . Consequently, for those  $z$  we have

$$|V(z) - V(x)| \leq \frac{2}{3}\epsilon + |V_2(z) - V_2(x)|.$$

For fixed  $\rho > 0$  there is a  $\delta = \delta(\epsilon) > 0$  such that

$$|V_2(z) - V_2(x)| < \frac{\epsilon}{3}$$

if  $|z - x| < \delta(\epsilon)$ . Summarizing, we have

$$|V(z) - V(x)| < \epsilon,$$

provided that  $|z - x| < \min\{\rho(\epsilon), \delta(\epsilon)\}$ .  $\square$

**Definition.** Assume  $u \in C^1(\Omega)$  and  $\partial\Omega \in C^{1,\lambda}$ . We say that there exists a *regular interior normal derivative* of  $u$  on  $\partial\Omega$  if the limit

$$\left(\frac{\partial u(x)}{\partial \nu(x)}\right)_i := \lim_{z \rightarrow x} \frac{\partial u(z)}{\partial \nu(x)}$$

exists for each  $x \in \partial\Omega$ . Here is  $z \in \Omega$  on the line defined by the exterior normal  $\nu_x$  at  $x$ , see Figure 1.8, and this limit is uniform with respect  $x \in \partial\Omega$  and it is a continuous function on  $\partial\Omega$ . Analogously, we define the *regular*

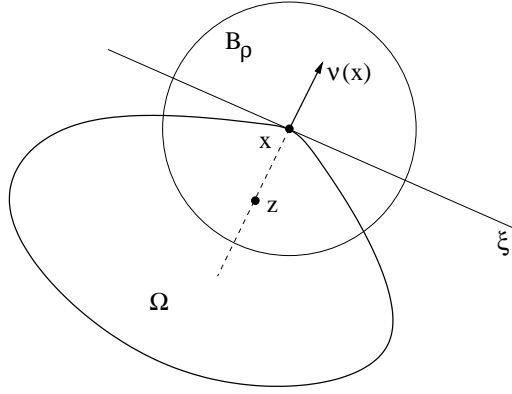


Figure 1.8: Normal derivative

*exterior normal derivative* of  $u \in C^1(\mathbb{R}^n \setminus \bar{\Omega})$  on  $\partial\Omega$  by

$$\left(\frac{\partial u(x)}{\partial \nu(x)}\right)_e := \lim_{z \rightarrow x} \frac{\partial u(z)}{\partial \nu(x)},$$

where  $z \in \mathbb{R}^n \setminus \bar{\Omega}$  is on the line defined by  $\nu(x)$  and  $x$ .

Assume  $z \notin \partial\Omega$ , then

$$\frac{\partial V(z)}{\partial l} = \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial l} \left( \frac{1}{|z - y|^{n-2}} \right) dS_y,$$

where  $l$  is any direction. If  $x \in \partial\Omega$  we define

$$\frac{\partial V(x)}{\partial \nu(x)} := \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y. \quad (1.11)$$

In some of the following considerations we need the formula

$$\frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x-y|^{n-2}} \right) = -\frac{n-2}{|x-y|^n} \sum_{i=1}^n (x_i - y_i) (\nu(x))_i. \quad (1.12)$$

**Lemma 1.3.2.** *The right hand side of (1.11) exists if  $x \in \partial\Omega$ .*

*Proof.* We introduce a local coordinate system with the center at  $x$  as in previous considerations and show that

$$I_\rho(x) := \int_{S_\rho} \sigma(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y$$

exists, where  $S_\rho = \partial\Omega \cap B_\rho(x)$ ,  $\rho > 0$  sufficiently small. In local coordinates it is  $y = (\xi, f(\xi))$ . Using formula (1.12), we obtain

$$\begin{aligned} |I_\rho(x)| &\leq c_1 \int_{D_\rho} \frac{|f(\xi)|}{|\xi|^n} d\xi \\ &\leq c_2 \int_{D_\rho} |\xi|^{-n+1+\lambda} d\xi \\ &= c_2 \omega_{n-1} \lambda^{-1} \rho^\lambda. \end{aligned}$$

□

Let  $x \in \partial\Omega$  and consider the sum

$$\begin{aligned} s(z) &= \frac{\partial V(z)}{\partial \nu(x)} + W(z) \\ &= \int_{\partial\Omega} \sigma(y) \left( \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z-y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) \right) dS_y, \end{aligned}$$

where  $W$  is the dipole potential and  $z$  is on the line defined by  $\nu(x)$ , see Figure 1.8.

**Lemma 1.3.3.** *The sum  $s(z)$  is continuous at  $x$ .*

*Proof.* Set  $S_\rho = \partial\Omega \cap B_\rho(x)$ ,  $\rho > 0$  sufficiently small, and

$$s(z) = s_1(z) + s_2(z),$$

where

$$\begin{aligned} s_1(z) &= \int_{S_\rho} \sigma(y) \left( \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z-y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) \right) dS_y, \\ s_2(z) &= \int_{\partial\Omega \setminus S_\rho} \sigma(y) \left( \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z-y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) \right) dS_y. \end{aligned}$$

We have

$$|s(z) - s(x)| \leq |s_1(z)| + |s_1(x)| + |s_2(z) - s_2(x)|.$$

Thus the lemma is shown if for given  $\epsilon > 0$  there exists a  $\rho = \rho(\epsilon) > 0$  such that  $|s_1(z)| < \epsilon/3$  if  $|x - z| < \rho(\epsilon)$ , see the proof of Lemma 1.3.1. It is, see (1.7), (1.12),

$$\begin{aligned} & \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z-y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) \\ &= (n-2) \frac{1}{|z-y|^n} \left( \sum_{i=1}^n (z_i - y_i)(\nu(y))_i - \sum_{i=1}^n (z_i - y_i)(\nu(x))_i \right), \end{aligned}$$

where, in local coordinates,  $x = (0, \dots, 0, 0)$ ,  $z = (0, \dots, 0, \delta)$ ,  $\nu(x) = (0, \dots, 0, 1)$  and

$$\nu(y) = \frac{1}{\sqrt{1 + |\nabla f(\xi)|^2}} (-f_{\xi_1}, \dots, -f_{\xi_{n-1}}, 1).$$

It follows that

$$\begin{aligned} |s_1(z)| &\leq c_1 \int_{S_\rho} \left| \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z-y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) \right| dS_y \\ &\leq c_2 \int_{D_\rho(0)} \frac{|\xi \cdot \nabla f(\xi) + (\delta - f(\xi)) - \delta \sqrt{1 + |\nabla f(\xi)|^2}|}{(|\xi|^2 + |\delta - f(\xi)|^2)^{n/2}} d\xi \\ &\leq c_3 \int_{D_\rho(0)} \frac{|\xi|^{1+\lambda} + |\delta| |\xi|^{2\lambda}}{(|\xi|^2 + \delta^2)^{n/2}} d\xi \\ &\leq c_3 \omega_{n-1} \max\{\lambda^{-1} \rho^\lambda, \pi \rho^{2\lambda}/2\}, \end{aligned}$$

where the constants  $c_i$  are independent of  $\rho$ . □

**Proposition 1.3.1.** *Suppose that  $\partial\Omega \in C^{1,\lambda}$ . Then there exists a regular interior and a regular exterior normal derivative of  $V$ , and these derivatives*

satisfy the jump relations

$$\begin{aligned}\left(\frac{\partial V(x)}{\partial \nu(x)}\right)_i &= \frac{(n-2)\omega_n}{2}\sigma(x) + \frac{\partial V(x)}{\partial \nu(x)} \\ \left(\frac{\partial V(x)}{\partial \nu(x)}\right)_e &= -\frac{(n-2)\omega_n}{2}\sigma(x) + \frac{\partial V(x)}{\partial \nu(x)},\end{aligned}$$

where  $x \in \partial\Omega$ .

*Proof.* The existence of regular normal derivatives follow from Lemm 1.3.3 and Proposition 1.2.1 since

$$\frac{\partial V(z)}{\partial \nu(x)} = \left(\frac{\partial V(z)}{\partial \nu_x} + W(z)\right) - W(z),$$

where  $z$  is on the line defined by  $\nu(x)$ . From Lemma 1.3.3 it follows also

$$\begin{aligned}\left(\frac{\partial V(x)}{\partial \nu(x)}\right)_i + W_i(x) &= \left(\frac{\partial V(x)}{\partial \nu(x)}\right)_e + W_e(x) \\ &= \frac{\partial V(x)}{\partial \nu(x)} + W(x),\end{aligned}$$

where

$$\begin{aligned}\frac{\partial V(x)}{\partial \nu(x)} &:= \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial \nu(x)} \left(\frac{1}{|x-y|^{n-2}}\right) dS_y, \\ W(x) &:= \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left(\frac{1}{|x-y|^{n-2}}\right) dS_y.\end{aligned}$$

Using Lemma 1.3.1, we obtain

$$\begin{aligned}\left(\frac{\partial V(x)}{\partial \nu(x)}\right)_i &= W(x) - W_i(x) + \frac{\partial V(x)}{\partial \nu(x)} \\ &= \frac{(n-2)\omega_n}{2}\sigma(x) + \frac{\partial V(x)}{\partial \nu(x)}\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\partial V(x)}{\partial \nu(x)}\right)_e &= W(x) - W_e(x) + \frac{\partial V(x)}{\partial \nu(x)} \\ &= -\frac{(n-2)\omega_n}{2}\sigma(x) + \frac{\partial V(x)}{\partial \nu(x)}.\end{aligned}$$

□

**Remark.** Let  $x \in \partial\Omega$ , then it follows immediately that

$$\left(\frac{\partial V(x)}{\partial\nu(x)}\right)_i - \left(\frac{\partial V(x)}{\partial\nu(x)}\right)_e = (n-2)\omega_n\sigma(x).$$

**Corollary.** *The single layer potential*

$$V(x) = \int_{\partial\Omega} \frac{\sigma(y)}{|x-y|^{n-2}} dS_y,$$

where  $\sigma \in C(\partial\Omega)$  defines a solution of the interior Neumann problem  $(N_i)$  if and only if  $\sigma \in C(\partial\Omega)$  is a solution of the integral equation

$$\Psi(x) = \frac{(n-2)\omega_n}{2}\sigma(x) + \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(x)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y,$$

where  $x \in \partial\Omega$ , and  $V(x)$  is a solution of the exterior Neumann problem  $(N_e)$  if and only if  $\sigma \in C(\partial\Omega)$  satisfies the integral equation

$$\Psi(x) = -\frac{(n-2)\omega_n}{2}\sigma(x) + \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(x)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y.$$

We recall that  $V$  is a solution of  $(N_i)$  if and only if

$$\Psi(x) = \left(\frac{\partial V(x)}{\partial\nu(x)}\right)_i,$$

and of  $(N_e)$  if and only if

$$\Psi(x) = \left(\frac{\partial V(x)}{\partial\nu(x)}\right)_e.$$

## 1.4 Integral equations

Denote by  $H = L^2(\partial\Omega)$  the Hilbert space with the inner product

$$\langle \sigma, \mu \rangle = \int_{\partial\Omega} \sigma(x) \overline{\mu(x)} dS_x,$$

and, if  $n \geq 3$ , we define the linear operator  $T$  from  $H$  into  $H$  by

$$(T\sigma)(x) = \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y.$$

Set

$$(T^*\sigma)(x) = \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(x)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y,$$

then

$$\langle T\sigma, \mu \rangle = \langle \sigma, T^*\mu \rangle$$

for all  $\sigma, \mu \in H$ . Below we will show that  $T$  is bounded. Then it follows that  $T^*$  is the adjoint operator to  $T$ .

According to the above corollaries to Proposition 1.2.1 and Proposition 1.3.1, the potentials  $W$  and  $V$  are solutions of the boundary value problems  $(D_i)$ ,  $(D_e)$ ,  $(N_i)$  and  $(N_e)$  if the density  $\sigma$  is continuous on  $\partial\Omega$  and satisfies the integral equations, respectively,

$$\sigma - \frac{2}{(n-2)\omega_n} T\sigma = -\frac{2}{(n-2)\omega_n} \Phi \quad (D_i)_I$$

$$\sigma + \frac{2}{(n-2)\omega_n} T\sigma = \frac{2}{(n-2)\omega_n} \Phi \quad (D_e)_I$$

$$\sigma + \frac{2}{(n-2)\omega_n} T^*\sigma = \frac{2}{(n-2)\omega_n} \Psi \quad (N_i)_I$$

$$\sigma - \frac{2}{(n-2)\omega_n} T^*\sigma = -\frac{2}{(n-2)\omega_n} \Psi \quad (N_e)_I.$$

**Remark.** Since we make the ansatz with above potentials for the exterior problems, we *prescribe* in fact the behaviour  $|u(z)| \leq c|z|^{1-n}$ ,  $|u(z)| \leq c|z|^{2-n}$ , respectively, as  $z \rightarrow \infty$ .

The above integral equations are defined for  $\sigma \in L^2(\partial\Omega)$ . In the following we will discuss whether or not there exist solutions in  $L^2(\partial\Omega)$ . From a regularity result which says that an  $L^2$ -solution is in fact in  $C(\partial\Omega)$ , we recover that the potentials define solutions of the boundary value problem, see the corollaries to Proposition 1.2.1 and Proposition 1.3.1.

**Proposition 1.4.1.** *Suppose that  $\partial\Omega \in C^{1,\lambda}$ . Then  $T$  is a completely continuous operator from  $H$  into  $H$ .*

*Proof.* (i)  $T$  is bounded. It is sufficient, see Section 1.1, to show that

$$(P\mu)(\zeta) := \int_{D_\rho} a(\xi) \mu(\xi) K(\xi, \zeta) d\xi$$

is bounded from  $L^2(D_\rho)$  into  $L^2(D_\rho)$ . Here is  $D_\rho = D_\rho(0) \subset \mathbb{R}^{n-1}$ ,  $a \in C_0^\infty(D_\rho)$ ,  $\mu(\xi) = \sigma(\xi, f(\xi))$  and

$$K(\xi, \zeta) = \frac{(n-2)(-(\xi - \zeta) \cdot \nabla f(\xi) + f(\zeta) - f(\xi))}{(|\xi - \zeta|^2 + (f(\xi) - f(\zeta))^2)^{n/2}}.$$

Set  $q(\zeta) := (P\mu)(\zeta)$ , then

$$q(\zeta) = \int_{D_\rho} \mu(\xi) \frac{A(\xi, \zeta)}{|\xi - \zeta|^{n-1-\lambda}} d\xi,$$

where  $A$  is bounded on  $\overline{D_\rho} \times \overline{D_\rho}$  and continuous if  $\xi \neq \zeta$ . Let  $\kappa = n-1-\lambda$ , then we have, with constants  $c_i$  independent of  $\mu$  and  $\rho$ , that

$$\begin{aligned} |q(\zeta)| &\leq c_1 \int_{D_\rho} \frac{|\mu(\xi)|}{|\xi - \zeta|^{\kappa/2}} \frac{1}{|\xi - \zeta|^{\kappa/2}} d\xi, \\ |q(\zeta)|^2 &\leq c_2 \int_{D_\rho} \frac{|\mu(\xi)|^2}{|\xi - \zeta|^\kappa} d\xi \int_{D_\rho} \frac{d\xi}{|\xi - \zeta|^\kappa} \\ &\leq c_3 \rho^\lambda \int_{D_\rho} \frac{|\mu(\xi)|^2}{|\xi - \zeta|^\kappa} d\xi, \\ \int_{D_\rho} |q(\zeta)|^2 d\zeta &\leq c_3 \rho^\lambda \int_{D_\rho} |\mu(\xi)|^2 \left( \int_{D_\rho} \frac{d\zeta}{|\xi - \zeta|^\kappa} \right) d\xi \\ &\leq c_4 \rho^{2\lambda} \int_{D_\rho} |\mu(\xi)|^2 d\xi. \end{aligned}$$

(ii)  $T$  is completely continuous. According to a lemma due to Kolmogoroff, see for example [21], pp. 246, or [1], pp. 31,  $P$  is completely continuous if for given  $\epsilon_1 > 0$  there exists an  $h_0(\epsilon_1) > 0$  such that

$$\int_{D_\rho} |q(\zeta + h) - q(\zeta)|^2 d\zeta \leq \epsilon_1^2$$

for all  $h \in \mathbb{R}^{n-1}$  such that  $|h| \leq h_0(\epsilon_1)$ , and uniformly for  $\|\mu\|_{L^2(D_\rho)} \leq M$ , where  $M < \infty$ . That is, the set  $\|\mu\|_{L^2(D_\rho)} \leq M$  is uniformly continuous in the mean. Above we set  $q(\zeta) = 0$  if  $\zeta \notin D_\rho$ . Let  $\eta \in C(\mathbb{R}_+)$ ,  $0 \leq \eta \leq 1$ , such that for given  $\epsilon > 0$

$$\eta(t) = \begin{cases} 1 & : 0 \leq t \leq \epsilon/2 \\ 0 & : t \geq \epsilon \end{cases}.$$

Set

$$q(\zeta) = q_1(\zeta) + q_2(\zeta),$$

where

$$\begin{aligned} q_1(\zeta) &= \int_{D_\rho(0)} \mu(\xi) \frac{A(\xi, \zeta) \eta(|\xi - \zeta|)}{|\xi - \zeta|^\kappa} d\xi \\ q_2(\zeta) &= \int_{D_\rho(0)} \mu(\xi) \frac{A(\xi, \zeta) (1 - \eta(|\xi - \zeta|))}{|\xi - \zeta|^\kappa} d\xi. \end{aligned}$$

We have

$$|q(\zeta + h) - q(\zeta)| \leq |q_1(\zeta + h) - q_1(\zeta)| + |q_2(\zeta + h) - q_2(\zeta)|. \quad (1.13)$$

Let  $\epsilon > 0$  be fixed, then for given  $\tau > 0$  there is a  $h_0 = h_0(\tau) > 0$  such that

$$|q_2(\zeta) - q_2(\zeta + h)| < \tau \quad (1.14)$$

for all  $|h| \leq h_0$ . Concerning  $q_1(\zeta)$  we have

$$\begin{aligned} |q_1(\zeta)| &\leq c \int_{D_\rho(0) \cap B_\epsilon(\zeta)} |\mu(\xi)| \frac{d\xi}{|\xi - \zeta|^\kappa} \\ |q_1(\zeta)|^2 &\leq c^2 \int_{D_\rho(0) \cap B_\epsilon(\zeta)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - \zeta|^\kappa} \\ &\quad \int_{D_\rho(0) \cap B_\epsilon(\zeta)} \frac{d\xi}{|\xi - \zeta|^\kappa} \\ &\leq c^2 \omega_{n-1} \epsilon^\lambda \lambda^{-1} \int_{D_\rho(0) \cap B_\epsilon(\zeta)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - \zeta|^\kappa} \\ \int_{D_\rho(0)} |q_1(\zeta)|^2 d\zeta &\leq c^2 \omega_{n-1} \epsilon^\lambda \lambda^{-1} \int_{D_\rho(0)} \int_{D_\rho(0)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - \zeta|^\kappa} d\zeta \\ &= c^2 \omega_{n-1} \epsilon^\lambda \lambda^{-1} \int_{D_\rho(0)} |\mu(\xi)|^2 d\xi \int_{D_\rho(0)} \frac{d\zeta}{|\xi - \zeta|^\kappa} \\ &\leq c^2 \omega_{n-1} \epsilon^\lambda \lambda^{-1} \int_{D_\rho(0)} |\mu(\xi)|^2 d\xi \int_{D_{2\rho}(\xi)} \frac{d\zeta}{|\xi - \zeta|^\kappa} \\ &= c^2 \omega_{n-1}^2 \epsilon^\lambda \lambda^{-2} (2\rho)^\lambda \|\mu\|_{L^2(D_\rho)}^2. \end{aligned}$$

Analogously we have

$$\begin{aligned}
|q_1(\zeta + h)|^2 &\leq c^2 \int_{D_\rho(0) \cap B_\epsilon(\zeta+h)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - (\zeta + h)|^\kappa} \\
&\quad \int_{D_\rho(0) \cap B_\epsilon(\zeta+h)} \frac{d\xi}{|\xi - (\zeta + h)|^\kappa} \\
&\leq c^2 \omega_{n-1} \epsilon^\lambda \lambda^{-1} \int_{D_\rho(0) \cap B_\epsilon(\zeta+h)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - (\zeta + h)|^\kappa} \\
\int_{D_\rho(0)} |q_1(\zeta + h)|^2 d\zeta &\leq c^2 \omega_{n-1} \epsilon^\lambda \lambda^{-1} \int_{D_\rho(0)} \int_{D_\rho(0)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - (\zeta + h)|^\kappa} d\zeta \\
&= c^2 \omega_{n-1} \epsilon^\lambda \lambda^{-1} \int_{D_\rho(0)} |\mu(\xi)|^2 d\xi \int_{D_\rho(0)} \frac{d\zeta}{|\xi - (\zeta + h)|^\kappa} \\
&\leq c^2 \omega_{n-1} \epsilon^\lambda \lambda^{-1} \int_{D_\rho(0)} |\mu(\xi)|^2 d\xi \\
&\quad \int_{D_{3\rho}(\xi-h)} \frac{d\zeta}{|\zeta - (\xi - h)|^\kappa} \\
&= c^2 \omega_{n-1}^2 \epsilon^\lambda \lambda^{-2} (3\rho)^\lambda \|\mu\|_{L^2(D_\rho)}^2.
\end{aligned}$$

Combining these  $L^2$ -estimates with (1.13) and (1.14) we obtain that the mapping  $T$  is completely continuous.  $\square$

From a result from the functional analysis it follows

**Corollary.**  $T^*$  is bounded with the same norm as  $T$  and  $T^*$  is completely continuous.

In the following we study the question of the existence of solutions  $\sigma \in L^2(\partial\Omega)$  of the above integral equations. To recover that the associated surface potentials define solutions of the original boundary value problems, we need more regularity, namely  $\sigma \in C(\partial\Omega)$ . We obtain this property by using the integral equations.

**Proposition 1.4.2** (Regularity). *Let  $w \in L^2(D_\rho)$  be a solution of the integral equation*

$$w(\zeta) - \int_{D_\rho} w(\xi) \frac{A(\xi, \zeta)}{|\xi - \zeta|^\kappa} d\xi = b(\zeta),$$

where  $\kappa = n - 1 - \lambda$ ,  $D_\rho = D_\rho(0) \subset \mathbb{R}^{n-1}$ ,  $A$  is bounded in  $\overline{D_\rho} \times \overline{D_\rho}$  and

continuous if  $\xi \neq \zeta$ . The function  $b \in C(\overline{D_\rho})$  is given. Then it follows that  $w \in C(\overline{D_\rho})$ .

*Proof.* Let  $\eta \in C(\mathbb{R}_+)$ ,  $0 \leq \eta \leq 1$ , such that for given  $\epsilon > 0$

$$\eta(t) = \begin{cases} 1 & : 0 \leq t \leq \epsilon/2 \\ 0 & : t \geq \epsilon \end{cases}.$$

Set

$$\frac{A(\xi, \zeta)}{|\xi - \zeta|^\kappa} = K_1(\xi, \zeta) + K_2(\xi, \zeta),$$

where

$$\begin{aligned} K_1(\xi, \zeta) &= \frac{A(\xi, \zeta)\eta(|\xi - \zeta|)}{|\xi - \zeta|^\kappa} \\ K_2(\xi, \zeta) &= \frac{A(\xi, \zeta)(1 - \eta(|\xi - \zeta|))}{|\xi - \zeta|^\kappa}. \end{aligned}$$

Then

$$w(\zeta) - \int_{D_\rho} w(\xi)K_1(\xi, \zeta) d\xi = g(\zeta),$$

where

$$g(\zeta) = \int_{D_\rho} w(\xi)K_2(\xi, \zeta) d\xi + b(\zeta)$$

is a continuous function on  $\overline{D_\rho}$ . Define the integral operator  $T_1$  from  $L^2(D_\rho)$  into  $L^2(D_\rho)$  by

$$(T_1 w)(\zeta) = \int_{D_\rho} w(\xi)K_1(\xi, \zeta) d\xi,$$

then we can write the above integral equation as  $(I - T_1)w = g$ , where  $I$  denotes the identity operator. The  $L^2$ -norm of  $T_1$  satisfies the inequality  $\|T_1\| < 1$ , provided  $\epsilon > 0$  is sufficiently small, see an exercise. It follows that  $w$  is given by the Neumann series

$$w = (I - T_1)^{-1}g = \sum_{n=0}^{\infty} T_1^n g,$$

which is a uniformly convergent series of continuous functions, provided  $\epsilon > 0$  was chosen sufficiently small, see an exercise.  $\square$

**FREDHOLM THEOREMS.** Here we recall some results from functional analysis, see for example [23]. Let  $H$  be a Hilbert space over  $\mathbb{C}$  and  $T : H \mapsto$

$H$  a completely continuous linear operator. Consider for given  $f, g \in H$  and  $\lambda \in \mathbb{C}$  the equations

$$u + \lambda Tu = f \quad (I)$$

$$v + \bar{\lambda} T^*v = g \quad (I^*)$$

and the associated homogeneous equations

$$u + \lambda Tu = 0 \quad (I_h)$$

$$v + \bar{\lambda} T^*v = 0 \quad (I_h^*).$$

Equations  $(I^*), (I_h^*)$  are called adjoint to  $(I), (I_h)$ , respectively.

(i) *Let  $\lambda$  be an eigenvalue of  $(I_h)$ , then the linear space of solutions has finite dimension.*

(ii) *The eigenvalue problem  $(I_h)$  has at most a countable set of eigenvalues with at most one limit element at infinity.*

(iii)  *$\lambda$  is an eigenvalue of  $(I_h)$  if and only if  $\bar{\lambda}$  is an eigenvalue of  $(I_h^*)$  and  $\dim N(I + \lambda T) = \dim N(I + \bar{\lambda} T)$ .*

(iv)  *$(I)$  has a solution if and only if  $f \perp N(I + \bar{\lambda} T^*)$  and  $(I^*)$  has a solution if and only if  $g \perp N(I + \lambda T)$ .*

We recall that  $N(A)$  denotes the null space  $N(A) = \{w \in H : Aw = 0\}$  of a linear operator  $A$ .

**Proposition 1.4.3.** *Suppose that  $\partial\Omega \in C^2$ , then  $\lambda_0 = -2/((n-2)\omega_n)$  is no eigenvalue of the homogeneous integral equation to  $(D_i)$ .*

*Proof.* Suppose that  $\lambda_0$  is an eigenvalue and  $\mu_0 \in L^2(\partial\Omega)$  an associated eigenvector of the adjoint problem  $(N_e)_I$ . From Proposition 1.4.2 we have that  $\mu_0 \in C(\partial\Omega)$ . Consider the single layer potential

$$V(x) := \int_{\partial\Omega} \frac{\mu_0(y)}{|x-y|^{n-2}} dS_y.$$

From a jump relation of Proposition 1.3.1 it follows that

$$\left( \frac{\partial V(x)}{\partial \nu(x)} \right)_e = 0 \quad (1.15)$$

since  $\mu_0$  is an eigenvector of  $(N_e)_I$ .

Set for a (small)  $h > 0$

$$\Omega_h = \Omega \cup \{y \in \mathbb{R}^n : y = x + s\nu(x), x \in \partial\Omega, 0 \leq s < h\},$$

see Figure 1.9. The surface  $\partial\Omega_h$  is called *parallel surface* to  $\partial\Omega$ .

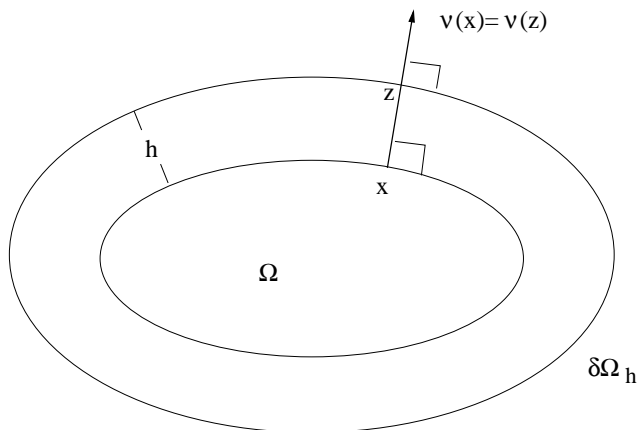


Figure 1.9: Parallel surface

Consider a ball  $B_R = B_R(0)$  such that  $\overline{\Omega_h} \subset B_R$ , then

$$\int_{B_R \setminus \Omega_h} |\nabla V|^2 dx = \int_{\partial B_R} V(x) \frac{\partial V(x)}{\partial \nu(x)} dS_x - \int_{\partial\Omega_h} V(z) \frac{\partial V(z)}{\partial \nu(z)} dS_z. \quad (1.16)$$

We have on  $\partial\Omega_h$

$$\nu(z) = \nu(x). \quad (1.17)$$

To show this equation, we consider the surface  $\partial\Omega$  which is given (locally) by  $x = x(u)$ , where  $u \in U$  and  $U$  is an  $(n-1)$ -dimensional parameter domain. Then the parallel surface  $\partial\Omega_h$  is defined by  $z(u) = x(u) + h\nu(x(u))$ . Then we consider a  $C^1$ -curve  $X(t)$  on  $\partial\Omega$  with  $X(0) = x$ , and let  $Z(t)$  be the associated curve on  $\partial\Omega_h$ . Then

$$|X(t) - Z(t)|^2 = h^2.$$

It follows

$$(X(t) - Z(t)) \cdot X'(t) - (X(t) - Z(t)) \cdot Z'(t) = 0,$$

which proves (1.17) since the first term is zero.

Combining (1.16), (1.17), (1.15) and

$$\lim_{h \rightarrow 0} \frac{\partial V(z)}{\partial \nu(x)} = \left( \frac{\partial V(x)}{\partial \nu(x)} \right)_e,$$

we obtain

$$\lim_{h \rightarrow 0} \int_{B_R \setminus \Omega_h} |\nabla V|^2 dx = \int_{\partial B_R} V(x) \frac{\partial V(x)}{\partial \nu(x)} dS_x$$

since the surface element  $dS_z$  converges uniformly to  $dS_x$  on  $U$  as  $h \rightarrow 0$ .<sup>1</sup> We have  $V = O(R^{2-n})$  and  $\partial V / \partial \nu(x) = O(R^{1-n})$ , consequently

$$\lim_{R \rightarrow \infty} \left( \lim_{h \rightarrow 0} \int_{B_R \setminus \Omega_h} |\nabla V|^2 dx \right) = 0.$$

Thus  $V = \text{const.}$  on  $\mathbb{R}^n \setminus \overline{\Omega}$ . From the behaviour of  $V$  at infinity it follows that  $V \equiv 0$  on  $\mathbb{R}^n \setminus \overline{\Omega}$ . Because of  $V \in C(\mathbb{R}^n)$ , see Lemma 1.3.1.

From the maximum principle we find that  $V \equiv 0$  in  $\Omega$  since  $\Delta V = 0$  in  $\Omega$  and  $V = 0$  on  $\partial\Omega$ . Thus, also the interior regular normal derivative on  $\partial\Omega$  is zero. Finally the jump relations, see Proposition 1.3.1, imply that  $\mu_0(x) \equiv 0$  on  $\partial\Omega$ .  $\square$

Proposition 1.4.3 and Fredholm's theorems imply

**Theorem 1.4.1.** *Let  $\Omega$  be bounded and  $\partial\Omega \in C^2$ . Then there exists for given  $\Phi, \Psi \in C(\partial\Omega)$  a unique solution of the interior Dirichlet problem  $(D_i)$  and the exterior Neumann problem  $(N_e)$ , respectively.<sup>2</sup>*

*Proof.*  $N(I + \lambda_0 T) = N(I + \overline{\lambda_0} T) = \{0\}$ .  $\square$

**Proposition 1.4.4.** *Let  $\Omega$  be bounded and  $\partial\Omega \in C^2$ . The number  $\lambda_0 = 2 / ((n-2)\omega_n)$  is a simple eigenvalue of  $(D_e)_I$  to the eigenvector  $\sigma \equiv 1$ .*

*Proof.* From, see Lemma 1.2.2,

$$\int_{\partial\Omega} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y = -\frac{(n-2)\omega_n}{2}$$

<sup>1</sup>In the case  $\mathbb{R}^3$  we have  $dS_z = \sqrt{EG - F^2} du_1 du_2$ , where  $E = z_{u_1} \cdot z_{u_1}$ ,  $G = z_{u_2} \cdot z_{u_2}$ ,  $F = z_{u_1} \cdot z_{u_2}$  and  $z(u) = x(u) + h\nu(x(u))$ .

<sup>2</sup>In this and in the following two theorems it is sufficient to assume that  $\partial\Omega \in C^{1,1}$  by *Rademacher's Theorem*: A Lipschitz continuous function is differentiable almost everywhere, see for example [5], pp. 280 or [6].

we see that  $\lambda_0$  is an eigenvalue and  $\sigma \equiv 1$  is an associated eigenvector. From Fredholm's theorems it follows that there exists at least one eigenfunction  $\mu_0(x)$  to  $\lambda_0$  of  $(N_i)_h$ . We will show that  $\dim N(I + \lambda_0 T^*) = 1$ . Set

$$V(x) = \int_{\partial\Omega} \frac{\mu_0(y)}{|x-y|^{n-2}} dS_y.$$

From the jump relations, see Proposition 1.3.1 and from the fact that  $\mu_0$  is an eigenvector it follows that  $(\partial V(x)/\partial\nu(x))_i = 0$ . We obtain as in the proof of Proposition 1.4.3 that  $V(x) = \text{const.} =: c_0$  in  $\Omega$ . This constant is different from zero. If not, then  $V = 0$  on  $\partial\Omega$ . Then the maximum principle implies that  $V \equiv 0$  in  $\mathbb{R}^n \setminus \bar{\Omega}$ . We recall that  $V = O(|x|^{2-n})$  as  $|x| \rightarrow \infty$ . Consequently, we have also  $(\partial V(x)/\partial\nu(x))_e = 0$ , which implies that  $\mu_0 = 0$ , see the jump relations of Proposition 1.3.1.

Let  $\mu_1$  be another eigenvector to  $\lambda_0$ , then we can assume that  $\mu_1 \in C(\partial\Omega)$  according to the regularity result Proposition 1.4.2. Set

$$V_1(x) = \int_{\partial\Omega} \frac{\mu_1(y)}{|x-y|^{n-2}} dS_y.$$

As above, we conclude that  $V_1(x) \equiv \text{const.} =: c_1$  in  $\Omega$ , where  $c_1 \neq 0$ . The linear combination  $\mu_2 := c_1\mu_0 - c_0\mu_1$  is contained in the null space  $N(I + \lambda_0 T^*)$ . Set

$$\begin{aligned} V_2(x) &= \int_{\partial\Omega} \frac{\mu_2(y)}{|x-y|^{n-2}} dS_y \\ &= c_1V_0(x) - c_0V_1(x). \end{aligned}$$

In  $\Omega$  we have  $V_2(x) = c_1c_0 - c_0c_1$ , and from the jump relations we find as above that  $\mu_2(x) \equiv 0$ . Thus we have shown

$$\mu_1(x) = \frac{c_1}{c_0}\mu_0(x).$$

□

Then it follows from Fredholm's theorems

**Theorem 1.4.2.** *Let  $\Omega$  be bounded and  $\partial\Omega \in C^2$ . Then there exists a solution of  $(N_i)$  if and only if*

$$\int_{\partial\Omega} \Psi(y) dS_y = 0.$$

In fact, we obtain also the existence of a solution of  $(D_e)_I$  under the assumption that

$$\int_{\partial\Omega} \mu_0(y) \Phi(y) dS_y = 0,$$

where  $\mu_0$  is the eigenvector from above. It turns out that there is a solution of the exterior Dirichlet problem without this restriction if we look for solutions with a weaker decay at infinity. We make the ansatz of a sum of a double layer and a single layer potential

$$\begin{aligned} u(x) &= \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y \\ &\quad + d \int_{\partial\Omega} \frac{\mu_0(y)}{|x-y|^{n-2}} dS_y, \end{aligned}$$

where  $d$  is a constant which we will determine later. The ansatz defines a solution of the exterior Dirichlet problem if and only if

$$\lim_{y \rightarrow x, y \in \mathbb{R}^n \setminus \bar{\Omega}} u(y) = \phi(x),$$

where  $x \in \partial\Omega$ . From a jump relation of Proposition 1.2.1 we see that the unknown density  $\sigma$  must satisfy the integral equation

$$\begin{aligned} \Phi(x) &= \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y \\ &\quad + \frac{(n-2)\omega_n}{2} \sigma(x) + d \int_{\partial\Omega} \frac{\mu_0(y)}{|x-y|^{n-2}} dS_y. \end{aligned}$$

Above we have shown that, if  $x \in \partial\Omega$ ,

$$\int_{\partial\Omega} \frac{\mu_0(y)}{|x-y|^{n-2}} dS_y = \text{const.} = c_0$$

with a constant  $c_0 \neq 0$ . Thus we have to consider the integral equation

$$\frac{(n-2)\omega_n}{2} \sigma(x) + \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial\nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y = \Phi(x) - dc_0.$$

This equation has a solution if

$$\int_{\partial\Omega} (\Phi(x) - dc_0) \mu_0(x) dS_x = 0$$

is satisfied. We can find an appropriate constant  $d$  such that this equation is satisfied since

$$\int_{\partial\Omega} \mu_0(x) dS_x \neq 0.$$

This inequality is a consequence of a jump relation and of the fact that  $\mu_0$  is an eigenvector:

$$\begin{aligned} \left( \frac{\partial V_0(x)}{\partial \nu(x)} \right)_e &= -\frac{(n-2)\omega_n}{2} \mu_0(x) + \int_{\partial\Omega} \mu_0(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y \\ 0 &= \frac{(n-2)\omega_n}{2} \mu_0(x) + \int_{\partial\Omega} \mu_0(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x-y|^{n-2}} \right) dS_y, \end{aligned}$$

which implies that

$$\left( \frac{\partial V_0(x)}{\partial \nu(x)} \right)_e = -(n-2)\omega_n \mu_0(x).$$

Suppose that

$$\int_{\partial\Omega} \mu_0(x) dS_x = 0,$$

then

$$\int_{\partial\Omega} \left( \frac{\partial V_0(x)}{\partial \nu(x)} \right)_e dS_x = 0,$$

which implies, see the proof of Proposition 1.4.3 for notations,

$$\int_{B_R \setminus \Omega_h} |\nabla V_0|^2 dx = \int_{\partial B_R} V_0(x) \frac{\partial V_0(x)}{\partial \nu(x)} dS_x - \int_{\partial\Omega_h} V_0(z) \frac{\partial V_0(z)}{\partial \nu(z)} dS_z.$$

Letting  $h \rightarrow 0$  and  $R \rightarrow \infty$ , it follows  $V_0 = \text{const.}$  in  $\mathbb{R}^n \setminus \bar{\Omega}$  and  $V_0 = c_0$  since  $V_0(x) = \text{const.} = c_0$  on  $\partial\Omega$ .

From the decay behaviour of  $V_0$  at infinity and since  $V \in C(\mathbb{R}^n)$  we find that  $V_0 = 0$  in  $\mathbb{R}^n \setminus \Omega$ . Thus we have  $c_0 = 0$ , a contradiction to a consideration above.

Thus, we have shown

**Theorem 1.4.3.** *Let  $\Omega$  be bounded and  $\partial\Omega \in C^2$ . Then for given  $\Phi \in C(\partial\Omega)$  there exists a unique solution  $u$  of  $(D_e)$  which the property  $u = O(|x|^{2-n})$  as  $|x| \rightarrow \infty$ .*

*Proof.* It remains to show that the solution is unique. This follows from the maximum principle.  $\square$

**Remark.** There is no uniqueness without the decay assumption. Counterexample: Let  $\Omega = B_R(0)$  be a ball in  $\mathbb{R}^3$ . Then  $u = 1/|x|$  and  $u = 1/R$  are two solutions of the Laplace equation with the same boundary values on  $\partial\Omega$ .

## 1.5 Volume potential

Set

$$\Gamma(x, y) = s(|x - y|),$$

where  $s(r)$  is the singularity function

$$s(r) := \begin{cases} -\frac{1}{2\pi} \ln r & : n = 2 \\ \frac{r^{2-n}}{(n-2)\omega_n} & : n \geq 3 \end{cases}$$

We recall that  $\omega_n = |\partial B_1(0)|$ . Let  $\Omega \in \mathbb{R}^n$  a bounded and sufficiently regular domain, then we define for given  $f$  the volume potential (or Newton potential)

$$V(x) = \int_{\Omega} \Gamma(x, y) f(y) dy.$$

If  $f$  is bounded in  $\Omega$  and  $f \in C^1(\Omega)$ , then  $V \in C^2(\Omega)$  and  $-\Delta V = f$  in  $\Omega$ , see for example [17]. This result holds under the weaker assumption that  $f$  is bounded and locally Hölder continuous in  $\Omega$ , see Proposition 1.5.1. In Chapter 4 on Schauder estimates we will see that also the second derivatives are Hölder continuous (with the same Hölder exponent).

**Definition.** Let  $f$  be a real function, defined in a fixed bounded neighbourhood  $D$  of  $x_0 \in \mathbb{R}^n$ . Then  $f$  is called *Hölder continuous* at  $x_0$  if there exists a real number  $\alpha$ ,  $0 < \alpha \leq 1$ , such that

$$[f]_{\alpha, x_0} := \sup_{x \in D \setminus \{x_0\}} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} < \infty.$$

The constant  $[f]_{\alpha, x_0}$  is called *Hölder constant* and  $\alpha$  *Hölder exponent*.

The function  $f$  is said to be *uniformly Hölder continuous* with respect to  $\alpha$  in  $D$  if

$$[f]_{\alpha, D} := \sup_{x, y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty,$$

and  $f$  is called *locally Hölder continuous* in a domain  $\Omega$  if  $f$  is uniformly Hölder continuous on compact subsets of  $\Omega$ .

In the following we will use the abbreviations  $D_i = \partial/\partial x_i$  and  $D_{ij} = \partial^2/\partial x_i \partial x_j$ .

**Proposition 1.5.1.** (i) *Let  $f$  be bounded and integrable over  $\Omega$ . Then  $V \in C^1(\mathbb{R}^n)$  and for any  $x \in \Omega$*

$$D_i V(x) = \int_{\Omega} D_i \Gamma(x, y) f(y) dy.$$

(ii) *Let  $f$  be bounded and locally Hölder continuous in  $\Omega$  with exponent  $0 < \alpha \leq 1$ . Then  $V \in C^2(\Omega)$ ,  $-\Delta V = f$  in  $\Omega$ , and for any  $x \in \Omega$*

$$\begin{aligned} D_{ij} V(x) &= \int_{\Omega_0} D_{ij} \Gamma(x, y) (f(y) - f(x)) dy \\ &\quad - f(x) \int_{\partial \Omega_0} D_i \Gamma(x, y) (\nu(y))_j dS_y, \end{aligned} \quad (1.18)$$

where  $\Omega_0 \supset \Omega$  is any domain for which the divergence theorem holds and  $f$  is extended to vanish outside  $\Omega$ .

*Proof.* See [10], Chapter 4. (i) Set for  $x \in \mathbb{R}^n$

$$v(x) = \int_{\Omega} D_i \Gamma(x, y) f(y) dy.$$

This function is well defined since  $|D_i \Gamma| \leq |x - y|^{1-n}/\omega_n$  holds. We will show that  $v = D_i V$  and  $v \in C(\mathbb{R}^n)$ . Let  $\eta \in C^1(\mathbb{R})$  be a fixed function satisfying  $0 \leq \eta \leq 1$ ,  $0 \leq \eta' \leq 2$ ,  $\eta(t) = 0$  if  $t \leq 1$  and  $\eta(t) = 1$  if  $t \geq 2$ . For a (small)  $\epsilon > 0$  set  $\eta_\epsilon = \eta(|x - y|/\epsilon)$  and consider the regularized potential

$$V_\epsilon(x) = \int_{\Omega} \Gamma(x, y) \eta_\epsilon f(y) dy.$$

Then  $V_\epsilon \in C^1(\mathbb{R}^n)$  and

$$v(x) - D_i V_\epsilon(x) = \int_{B_{2\epsilon}(x)} D_i ((1 - \eta_\epsilon) \Gamma) f(y) dy.$$

We obtain

$$\begin{aligned} |v(x) - D_i V_\epsilon(x)| &\leq \sup_{\Omega} |f| \int_{B_{2\epsilon}(x)} (|D_i \Gamma| + 2|\Gamma|/\epsilon) dy \\ &\leq \sup_{\Omega} |f| \begin{cases} 4(1 + |\ln(2\epsilon)|)\epsilon & : n = 2 \\ 2n\epsilon/(n-2) & : n \geq 3 \end{cases}. \end{aligned}$$

It follows that  $V_\epsilon$  and  $D_i V_\epsilon$  converge uniformly in compact subsets of  $\mathbb{R}^n$  to  $V$  and  $v$ , respectively, as  $\epsilon \rightarrow 0$ . Consequently  $V \in C^1(\mathbb{R}^n)$  and  $D_i V = v$ .

(ii) Set for  $x \in \Omega$

$$\begin{aligned} u(x) &= \int_{\Omega_0} D_{ij} \Gamma(x, y) (f(y) - f(x)) dy \\ &\quad - f(x) \int_{\partial \Omega_0} D_i \Gamma(x, y) (\nu(y))_j dS_y. \end{aligned}$$

The right hand side is well-defined since  $f$  is locally Hölder continuous and since  $|D_{ij} \Gamma| \leq (1+n)|x-y|^n/\omega_n$  holds. Set  $v = D_i V$  and define for  $\epsilon > 0$  the regularized function

$$v_\epsilon(x) = \int_{\Omega_0} D_i \Gamma \eta_\epsilon f(y) dy.$$

Then

$$\begin{aligned} D_j v_\epsilon &= \int_{\Omega_0} D_j (D_i \Gamma \eta_\epsilon) f(y) dy \\ &= \int_{\Omega_0} D_j (D_i \Gamma \eta_\epsilon) (f(y) - f(x)) dy \\ &\quad + f(x) \int_{\Omega_0} D_j (D_i \Gamma \eta_\epsilon) dy \\ &= \int_{\Omega_0} D_j (D_i \Gamma \eta_\epsilon) (f(y) - f(x)) dy \\ &\quad - f(x) \int_{\partial \Omega_0} D_i \Gamma (\nu(y))_j dS_y, \end{aligned}$$

provided  $\epsilon > 0$  is small enough such that  $\eta_\epsilon = 1$  on  $\partial \Omega_0$ , see Figure 1.10. Then

$$u(x) - D_j v_\epsilon(x) = \int_{B_{2\epsilon}(x)} D_j ((1 - \eta_\epsilon) D_i \Gamma) (f(y) - f(x)) dy.$$

We suppose that  $2\epsilon < \text{dist}(x, \partial \Omega)$  if  $x \in \Omega_c$ ,  $\Omega_c \subset \subset \Omega$ . Then

$$\begin{aligned} |u(x) - D_j v_\epsilon(x)| &\leq [f]_{\alpha, \Omega_c} \int_{B_{2\epsilon}(x)} (|D_{ij} \Gamma| + 2|D_i \Gamma|/\epsilon) |x-y|^\alpha dy \\ &\leq [f]_{\alpha, \Omega_c} (4 + n/\alpha) (2\epsilon)^\alpha. \end{aligned}$$

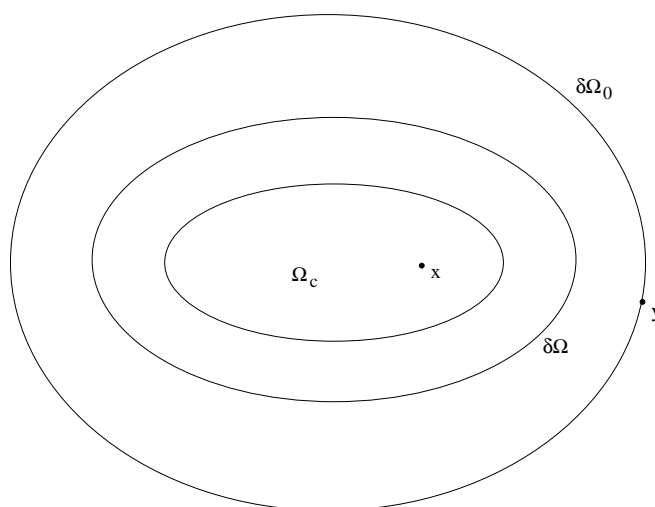


Figure 1.10: Proof of Proposition 1.5.1

It follows that  $V \in C^2(\Omega)$  and  $u = D_{ij}V$  since  $D_i v_\epsilon$  converges to  $u$  uniformly on compact subsets of  $\Omega$ .

Set in formula (1.18)  $\Omega_0 = B_R(x)$ ,  $R$  sufficiently large, then

$$\begin{aligned} \Delta V &= -f(x) \int_{\partial B_R(x)} \sum_{i=1}^n D_i \Gamma(x, y) (\nu(y))_i dS_y \\ &= -f(x) \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(x)} \nu(y) \cdot \nu(y) dS_y \\ &= -f(x). \end{aligned}$$

□

From Proposition 1.5.1 and Theorem 1.4.1 we obtain

**Theorem 1.5.1.** *Let  $\Omega$  be a bounded domain with  $\partial\Omega \in C^2$ ,  $f$  be bounded and locally Hölder continuous in  $\Omega$  and  $\Phi \in C(\partial\Omega)$ . Then there exists a unique solution  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  of the Dirichlet problem  $-\Delta u = f$  in  $\Omega$ ,  $u = \Phi$  on  $\partial\Omega$ .*

*Proof.* Set  $u = V + w$ , where

$$V(x) = \int_{\Omega} \Gamma(x, y) f(y) dy.$$

Then  $u$  is a solution of the Dirichlet problem if and only if  $\Delta w = 0$  in  $\Omega$  and  $w = \Phi - V$  on  $\partial\Omega$ . The existence of a  $w$  follows from Theorem 1.4.1. The uniqueness of  $u$  is a consequence of the maximum principle. Moreover,  $w$  is given by a dipole potential, see Theorem 1.4.1.  $\square$

## 1.6 Exercises

1. Let  $S$  be a surface in  $\mathbb{R}^3$  defined by  $z = f(x)$ ,  $x = (x_1, x_2)$ ,  $x \in U$ , where  $U$  is a neighbourhood of  $x = 0$ . Assume  $f \in C^{1,\lambda}(U)$ ,  $f(0) = 0$ ,  $\nabla f(0) = 0$ . Consider the intersection  $I = S \cap \partial B_{\rho_0}(0)$ ,  $\rho_0 > 0$  sufficiently small, that is,

$$I = \{(x, z) : z = f(x) \text{ and } x_1^2 + x_2^2 + z^2 = \rho_0^2\}.$$

Show that there is a function  $\epsilon(\rho, \phi)$ ,  $0 < \rho \leq \rho_0$ ,  $\phi \in [0, 2\pi)$ ,  $2\pi$ -periodic in  $\phi$  and in  $C^1$  with respect to  $\phi$ , such that  $\epsilon(\rho, \phi) = O(\rho^\lambda)$  as  $\rho \rightarrow 0$ , uniformly in  $\phi \in [0, 2\pi)$ , and

$$\begin{aligned} x_1 &= \rho(1 + \epsilon(\rho, \phi)) \cos \phi \\ x_2 &= \rho(1 + \epsilon(\rho, \phi)) \sin \phi. \end{aligned}$$

*Hint:* Implicit function theorem.

2. Let  $B_{2R}(x_0) \subset \mathbb{R}^n$  be a ball with radius  $2R$  and the center at  $x_0$ . Show that there is a function  $\eta \in C_0^\infty(B_{2R}(x_0))$  which satisfies  $0 \leq \eta \leq 1$  in  $B_{2R}(x_0)$  and  $\eta \equiv 1$  in  $B_R(x_0)$ .

*Hint:* Set  $r = |x|$  and define

$$\phi(r) = e^{-1/((3/(2R)-r)^2 - r^2/4)}$$

if  $R < r < 2R$  and set  $\phi(r) = 0$  if  $0 < r < R$  or  $r \geq 2R$ . Let

$$\psi(r) = \frac{\int_0^r \phi(t) dt}{\int_0^\infty \phi(t) dt}$$

and  $\chi(r) = 1 - \psi(r)$ . Show that  $\eta(x) = \chi(|x - x_0|)$  is a function which satisfies the above properties.

3. Suppose that  $\partial\Omega \in C^{1,\lambda}$  and  $x \in \partial\Omega$ . Show that

$$\lim_{\rho \rightarrow 0} \int_{\partial\Omega \setminus B_\rho(x)} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z - y|^{n-2}} \right) dS_y = 0$$

*Hint:* Local coordinates and formula (1.7).

4. Let  $\Omega \subset \mathbb{R}^n$  be a bounded and sufficiently regular domain and set

$$q(\zeta) = \int_{\Omega} \frac{A(\xi, \zeta)}{|\xi - \zeta|^{n-\lambda}} d\xi,$$

where  $A$  is bounded in  $\overline{\Omega} \times \overline{\Omega}$  and continuous if  $\xi \neq \zeta$  and  $0 < \lambda \leq 1$ . Show that  $q \in C(\overline{\Omega})$ .

*Hint:* Let  $\eta \in C(\mathbb{R})$  be a fixed function satisfying  $0 \leq \eta \leq 1$ ,  $\eta(t) = 0$  if  $t \leq 1$  and  $\eta(t) = 1$  if  $t \geq 2$ . For (small)  $\epsilon > 0$  set  $\eta_{\epsilon} = \eta(|\zeta - \xi|/\epsilon)$ . Then consider the regularized function

$$q_{\epsilon}(\zeta) = \int_{\Omega} \eta_{\epsilon} \frac{A(\xi, \zeta)}{|\xi - \zeta|^{n-\lambda}} d\xi$$

and prove that  $q_{\epsilon}$  converges uniformly to  $q$  in  $\Omega$ .

5. Show that

$$\|K_1 w\|_{L^2(D_{\rho})} \leq c\epsilon^{\lambda} \|w\|_{L^2(D_{\rho})}.$$

For the definition of  $K_1$  see the proof of Proposition 1.4.2.

6. Assume  $g \in C(\overline{D_{\rho}})$ . Prove that

$$(i) |K_1^l g| \leq (c\epsilon^{\lambda})^l \max_{\overline{D_{\rho}}} |g(\zeta)|.$$

$$(ii) K_1^l g \text{ are continuous on } \overline{D_{\rho}}.$$

(iii)  $\sum_{l=1}^{\infty} K_1^l g$  is uniformly convergent on  $\overline{D_{\rho}}$ , provided that  $\epsilon > 0$  is small enough.

7. The solution  $u$  of the interior Dirichlet problem  $\Delta u = 0$  in  $\Omega$  and  $u = \Phi$  on  $\partial\Omega$ , where  $\Omega \subset \mathbb{R}^2$  and  $\Phi \in C(\partial\Omega)$  is given by

$$u(x) = - \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \ln(|x - y|) dS_y.$$

Here is  $\sigma(x)$ ,  $x \in \partial\Omega$ , the solution of the integral equation

$$\pi\sigma(x) + \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \ln(|x - y|) dS_y = -\Phi.$$

Find the density  $\sigma$  if  $\Omega$  is a disk  $B_R(0)$ .

*Hint:* Show that

$$\frac{\partial}{\partial \nu(y)} \ln(|x - y|) = \frac{1}{2R}$$

if  $x, y \in \partial B_R(0)$ . This formula is a consequence of

$$\begin{aligned} \frac{\partial}{\partial \nu(y)} \ln(|x - y|) &= \frac{1}{|y - x|^2} (y - x) \cdot \nu(y) \\ &= \frac{1}{|y - x|} \cos \beta, \end{aligned}$$

see Figure 1.11 for notations.

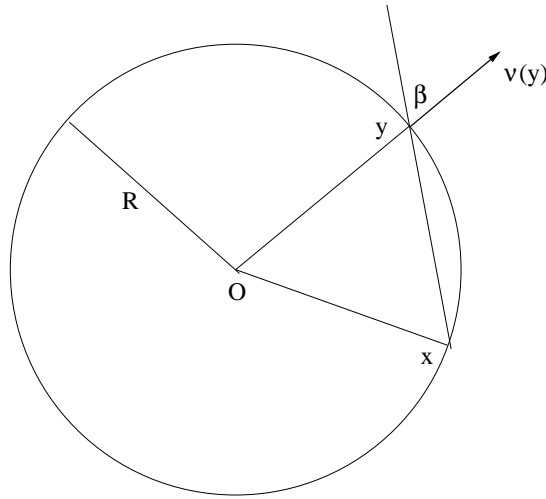


Figure 1.11: Notations to the exercise

8. Show that

$$C^\alpha[a, b] := \{u \in C[a, b] : \|u\|_\alpha < \infty\},$$

where  $-\infty < a < b < \infty$  and  $0 < \alpha \leq 1$ , defines a Banach space, where

$$\|u\|_\alpha := \max_{x \in [a, b]} |u(x)| + \sup_{x, y \in [a, b], x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

9. Show that  $C^\infty[a, b]$  is not dense in  $C^\alpha[a, b]$ . That is, there is a  $u \in C^\alpha[a, b]$  such that there does not exist a sequence  $u_n \in C^\infty[a, b]$  such that  $\|u_n - u\|_\alpha$  tends to zero.

*Hint:* Consider  $u = \sqrt{x}$  and  $[a, b] = [0, 1]$ . We have  $\sqrt{x} \in C^{1/2}[0, 1]$ . Assume

$$\sup_{x, y \in [a, b], x \neq y} \frac{|u_n(x) - u(x) - (u_n(y) - u(y))|}{|x - y|^\alpha} \rightarrow 0$$

if  $n \rightarrow \infty$ , where  $\alpha = 1/2$ . Then

$$\left| \frac{u_{n_0}(x) - u_{n_0}(0)}{\sqrt{x}} - 1 \right| \leq \epsilon$$

for a given  $\epsilon > 0$  and an integer  $n_0 = n_0(\epsilon)$ .

10. Show that  $C_0^\infty(a, b)$  is not dense in  $C^\alpha[a, b]$ .

*Hint:* Consider  $(a, b) = (-1, 1)$  and

$$u(x) = \begin{cases} \sqrt{x} & : 0 \leq x \leq 1 \\ 0 & : -1 \leq x \leq 0 \end{cases} .$$

## Chapter 2

# Perron's method

Perron's method is a maximum principle based existence theory for second order linear or quasilinear elliptic equations. In this chapter we consider the Dirichlet problem for the Laplace equation  $\Delta u = 0$  in  $\Omega$  and  $u = \Phi$  on  $\partial\Omega$ , where  $\Omega \subset \mathbb{R}^n$  is bounded and connected and  $\Phi$  is a given function defined on  $\partial\Omega$ .

In contrast to many other existence theories the Perron method provides results under rather weak assumptions on the boundary  $\partial\Omega$  since the problem of existence is separated from the question of the boundary behaviour.

### 2.1 A maximum principle

We know that a harmonic function  $u$  must be a constant if  $u$  achieves its supremum or infimum in a connected domain. This result is a consequence of the mean value formula for harmonic functions, see [17], Chapter 7, for example. Fortunately, there is a related principle for functions which satisfy  $\Delta u \geq 0$  or  $\Delta u \leq 0$  throughout in  $\Omega$ .

**Lemma 2.1.1** (Mean value theorems). *Suppose that  $u \in C^2(\Omega)$  satisfies  $\Delta u = 0$ ,  $\Delta u \geq 0$ ,  $\Delta u \leq 0$  in  $\Omega$ , respectively. Then for any ball  $B = B_R(x) \subset\subset \Omega$*

$$u(x) = (\leq, \geq) \frac{1}{|\partial B|} \int_{\partial B} u(y) dS_y \quad (2.1)$$

$$u(x) = (\leq, \geq) \frac{1}{|B|} \int_B u(y) dy. \quad (2.2)$$

*Proof.* See [10], Chapter 2, for example. Let  $\rho \in (0, R)$  and  $B_\rho = B_\rho(x)$ , then

$$\begin{aligned} \int_{B_\rho} \Delta u \, dy &= \int_{\partial B_\rho} \frac{\partial u}{\partial \nu(y)} \, dS_y \\ &= (\geq, \leq) 0, \end{aligned}$$

respectively. Here is  $\nu(y)$  the exterior unit normal at  $y$  on  $\partial B_\rho$ .

Set  $r = |x - y|$ ,  $\omega = (y - x)/r$ , then  $u(y) = u(x + r\omega)$ . Thus

$$\begin{aligned} \int_{\partial B_\rho} \frac{\partial u}{\partial \nu(y)} \, dS_y &= \int_{\partial B_\rho} u_{y_i}(x + \rho\omega)\omega_i \, dS_y \\ &= \int_{\partial B_\rho} \left. \frac{\partial u(x + r\omega)}{\partial r} \right|_{r=\rho} \, dS_y \\ &= \rho^{n-1} \int_{\partial B_1(0)} \left. \frac{\partial u(x + r\omega)}{\partial r} \right|_{r=\rho} \, d\omega \\ &= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{\partial B_1(0)} u(x + \rho\omega) \, d\omega \\ &= \rho^{n-1} \frac{\partial}{\partial \rho} \left( \rho^{1-n} \int_{\partial B_\rho} u(y) \, dS_y \right) \\ &= (\geq, \leq) 0. \end{aligned}$$

Consequently for any  $\rho \in (0, R)$

$$\frac{\partial}{\partial \rho} \left( \rho^{1-n} \int_{\partial B_\rho} u(y) \, dS_y \right) = (\geq, \leq) 0.$$

It follows

$$\rho^{1-n} \int_{\partial B_\rho} u(y) \, dS_y = (\leq, \geq) R^{1-n} \int_{\partial B_R} u(y) \, dS_y$$

or

$$\frac{1}{|\partial B_\rho|} \int_{\partial B_\rho} u(y) \, dS_y = (\leq, \geq) \frac{1}{|\partial B_R|} \int_{\partial B_R} u(y) \, dS_y.$$

Letting  $\rho \rightarrow 0$ , we obtain

$$u(x) = (\leq, \geq) \frac{1}{|\partial B_R|} \int_{\partial B_R} u(y) \, dS_y.$$

Formula (2.2) follows from

$$\rho^{n-1}\omega_n u(x) = (\leq, \geq) \int_{\partial B_\rho} u(y) dS_y,$$

where  $0 < \rho \leq R$ . We recall that  $\omega_n = |\partial B_1(0)|$ . Integrating over  $(0, R)$ , we obtain

$$\frac{\omega_n R^n}{n} u(x) = (\leq, \geq) \int_{B_R} u(y) dy,$$

which is formula (2.2) since  $|B_R| = R^n/(n\omega_n)$ .  $\square$

A consequence of Lemma 2.1.1 we have the following generalization of the maximum principle for harmonic functions.

**Theorem 2.1.1** (Strong maximum principle). *Assume  $\Omega \subset \mathbb{R}^n$  is a connected domain and  $u \in C^2(\Omega)$ . Let  $\Delta u \geq 0$  ( $\Delta u \leq 0$ ) in  $\Omega$  and suppose there exists a point  $y \in \Omega$  for which  $u(y) = \sup_\Omega u$  ( $u(y) = \inf_\Omega u$ ). Then  $u$  is a constant.*

*Proof.* Consider the case  $\Delta u \geq 0$  in  $\Omega$ . Let  $x_0 \in \Omega$  such that

$$M := u(x_0) = \sup_{x \in \Omega} u(x).$$

Set  $\Omega_1 = \{x \in \Omega : u(x) = M\}$  and  $\Omega_2 = \{x \in \Omega : u(x) < M\}$ . The set  $\Omega_1$  is not empty and the set  $\Omega_2$  is open since  $u \in C(\Omega)$ . Consequently  $\Omega_2$  is empty if we can show that  $\Omega_1$  is an open set. Let  $y \in \Omega_1$ , then there is a  $\rho_0 > 0$  such that  $\overline{B_{\rho_0}(y)} \subset \Omega$  and  $u(x) = M$  for all  $x \in B_{\rho_0}(y)$ . If not, then there are  $\rho > 0$ ,  $z \in \Omega$  such that  $|z - y| = \rho$ ,  $0 < \rho < \rho_0$  and  $u(z) < M$ . From Lemma 2.1.1 we have

$$\begin{aligned} M &\leq \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(y)} u(x) dS_x \\ &< \frac{M}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(y)} u(x) dS = M, \end{aligned}$$

which is a contradiction.  $\square$

**Corollary.** *Assume  $\Omega$  is connected and bounded and  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  satisfies  $\Delta u \geq 0$  ( $\Delta u \leq 0$ ) in  $\Omega$ . Then  $u$  achieves its maximum (minimum) on the boundary  $\partial\Omega$ .*

**Corollary.** Assume  $\Omega$  is connected and bounded and  $v, w \in C^2(\Omega) \cap C(\overline{\Omega})$  satisfy  $\Delta v \geq \Delta w$  in  $\Omega$  and  $v \leq w$  on  $\partial\Omega$ . Then  $v \leq w$  in  $\Omega$ .

*Proof.* Exercise.

## 2.2 Subharmonic, superharmonic functions

Sometimes a function  $u \in C^2(\Omega)$  is called subharmonic (superharmonic) in a domain  $\Omega \in \mathbb{R}^n$  if  $\Delta u \geq 0$  ( $\Delta u \leq 0$ ) in  $\Omega$ . It turns out that we can define superharmonic and subharmonic functions if  $u$  merely is in  $C(\Omega)$ .

**Definition.** A function  $u \in C(\Omega)$  is called *subharmonic* (*superharmonic*) in  $\Omega$  if for every ball  $B \subset\subset \Omega$  and every function  $h$  harmonic in  $B$ , that is,  $h \in C^2(B) \cap C(\overline{B})$  and  $\Delta h = 0$  in  $B$ , satisfying  $u \leq h$  ( $u \geq h$ ) on  $\partial B$  we have  $u \leq h$  ( $u \geq h$ ) in  $B$ .

**Corollary.** A function harmonic in  $\Omega$  is both a superharmonic and a subharmonic function. In particular, constants are super- and subharmonic.

**Remark.** A function  $u$  in the class  $C^2(\Omega)$  is subharmonic (superharmonic) if  $\Delta(u - h) \geq 0$  ( $\Delta(u - h) \leq 0$ ) in  $B$  for any harmonic function  $h$  in  $B$  such that  $u \leq h$  ( $u \geq h$ ) on  $\partial B$ .

**Lemma 2.2.1** (Strong maximum principle). Assume  $\Omega$  is connected. If a subharmonic function  $u$  attains its supremum in  $\Omega$ , then  $u \equiv \text{const.}$  in  $\Omega$ , and if a superharmonic function attains its infimum in  $\Omega$ , then  $u = \text{const.}$  in  $\Omega$ .

*Proof.* Consider the case of a subharmonic function. Let  $x^0 \in \Omega$  and

$$M := u(x^0) = \sup_{\Omega} u(x).$$

We will show that

$$\Omega_1 = \{x \in \Omega : u(x) = M\}$$

is an open set. Its not empty since  $x^0 \in \Omega_1$ . Let  $x^1 \in \Omega_1$ , then  $B_{\rho_0}(x^1) \subset \Omega_1$  if  $\rho_0 > 0$  is sufficiently small such that  $B_{\rho_0}(x^1) \subset \Omega$ . If not, then there is a  $\rho$ ,  $0 < \rho \leq \rho_0$  and an  $x^2 \in \partial B_{\rho}(x^1)$  such that  $u(x^2) < M$ . Consider a function  $h$  harmonic in  $B = B_{\rho}(x^1)$  and  $h = u$  on  $\partial B$ . Then, since  $h$  is harmonic in  $B$  and  $u$  is subharmonic in  $\Omega$ ,

$$M \geq \max_{\partial B} u = \max_{\partial B} h \geq h(x) \geq u(x),$$

where  $x \in \overline{B}$ . Consequently  $h(x^1) = M$  since  $u(x^1) = M$ . Thus  $h = \text{const.}$  in  $B$  since the harmonic function  $h$  attains its maximum in  $B$ . Since  $h(x) = u(x)$  on  $\partial\Omega$  we have  $u(x) = M$  for all  $x \in \partial B$ , a contradiction to  $u(x^2) < M$ .  $\square$

The following lemma is a generalization of the comparison principle for  $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$  which says that  $\Delta v \leq \Delta u$  in  $\Omega$  and  $v \geq u$  on  $\partial\Omega$  imply that either  $v > u$  throughout  $\Omega$  or  $v \equiv u$ .

**Lemma 2.2.2.** *Suppose that  $\Omega$  is bounded and connected. Let  $u \in C(\Omega)$  be a subharmonic and  $v \in C(\Omega)$  a superharmonic function with  $u - v \leq 0$  on  $\partial\Omega$ . Then either  $v > u$  throughout  $\Omega$  or  $v \equiv u$ .*

*Proof.* We will show that  $u - v \equiv \text{const.}$  in  $\Omega$  if  $u - v$  attains a nonnegative supremum in  $\Omega$ . Let  $x^0 \in \Omega$  such that

$$M := \sup_{\Omega} (u - v) = (u - v)(x^0) \geq 0.$$

Set  $\Omega_1 = \{x \in \Omega : (u - v)(x) = M\}$ . This set is not empty since  $x^0 \in \Omega_1$ . We will show that  $\Omega_1$  is an open set. Let  $x^1 \in \Omega_1$  and consider a ball  $B_{\rho_0}(x^1) \subset\subset \Omega$ . Then  $B_{\rho_0}(x^1) \in \Omega_1$ . If not, then there is a ball  $B = B_{\rho}(x^1)$ ,  $0 < \rho \leq \rho_0$ , and an  $x^2 \in \partial B$  such that  $(u - v)(x^2) < M$ . Let  $h_1, h_2$  harmonic in  $B$  with  $h_1 = u$  on  $\partial B$  and  $h_2 = v$  on  $\partial B$ . Then, if  $x \in B$ ,

$$\begin{aligned} M &\geq \max_{\partial B} (u - v) = \max_{\partial B} (h_1 - h_2) \\ &\geq h_1(x) - h_2(x) \geq u(x) - v(x). \end{aligned}$$

Set  $x = x^1$ , then by assumption  $u(x^1) - v(x^1) = M$  which implies that the harmonic function  $h_1 - h_2$  attains its maximum in  $B$ . Consequently  $h_1 - h_2 = \text{const.}$  in  $B$ . Thus  $u(x) - v(x) = M$  on  $\partial B$  which is a contradiction to  $u(x^2) < M$ . We have seen that  $u - v \equiv M \geq 0$  in  $\Omega$ . Finally the assumption  $u - v \leq 0$  on  $\partial\Omega$  implies  $u(x) \equiv v(x)$  in  $\Omega$ .  $\square$

Let  $u$  be subharmonic in  $\Omega$ ,  $B \subset\subset \Omega$  a ball and  $\bar{u}$  harmonic in  $B$  such that  $\bar{u} = u$  on  $\partial B$ .

**Definition.** The function

$$U(x) = \begin{cases} \bar{u} & : x \in B \\ u(x) & : x \in \Omega \setminus \overline{B} \end{cases}$$

---

<sup>1</sup>Here  $u - v \leq 0$  on  $\partial\Omega$  means that

$$\limsup_{y \rightarrow x, y \in \Omega, x \in \partial\Omega} (u - v) \leq 0$$

is called *harmonic lifting* of  $u$  in  $B$ .

**Lemma 2.2.3.**  $U$  is subharmonic in  $\Omega$ .

*Proof.* Let  $B' \subset\subset \Omega$  and  $h$  harmonic in  $B'$  with  $h \geq U$  on  $\partial B'$ . We have to show that  $h \geq U$  in  $B'$ . For some of the following notations see Figure 2.1. On  $C_1 = \partial B' \setminus B$  we have  $h \geq U \equiv u$ . On  $C_3 = \partial B' \cap B$  it is, according to

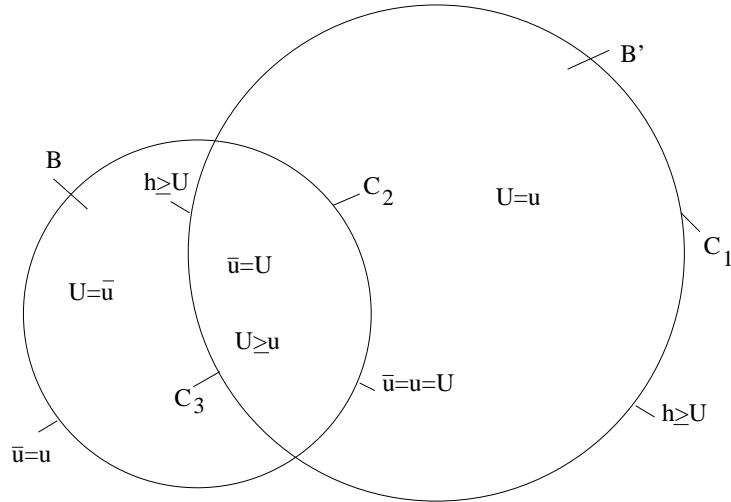


Figure 2.1: Proof of Lemma 2.2.3

the definition of  $\bar{u}$ ,  $h \geq U = \bar{u} \geq u$ . Combining these inequalities, we find that  $h \geq u$  on  $\partial B'$  which implies that  $h \geq u$  in  $B'$ . Then

$$U \leq h \quad \text{in } B' \setminus B \quad (2.3)$$

since  $U \equiv u$  in  $B' \setminus B$ . It remains to show that also  $U \leq h$  in  $B' \cap B$ . On  $\partial(B' \cap B)$  we have  $h \geq U$ , see (2.3) and assumption  $h \geq U$  on  $B'$ . Since  $U \equiv \bar{u}$  in  $B \cap B'$  and  $h$  is harmonic in  $B'$  it follows  $h \geq U$  in  $B \cap B'$ .  $\square$

**Lemma 2.2.4.** Let  $u_1, u_2, \dots, u_N$  be subharmonic in  $\Omega$ . Then

$$u(x) := \max\{u_1(x), \dots, u_N(x)\}$$

is also subharmonic in  $\Omega$ , and if  $u_1, \dots, u_N$  are superharmonic, then  $u(x) := \min\{u_1(x), \dots, u_N(x)\}$  is superharmonic in  $\Omega$ .

*Proof.* Exercise.

**Definition.** Let  $\Omega$  be bounded and  $\phi$  a bounded function on  $\partial\Omega$ . A subharmonic function  $u \in C(\Omega)$  is called *subfunction with respect to  $\phi$*  if  $u \leq \phi$  on  $\partial\Omega$ , and a superharmonic function  $u \in C(\Omega)$  is called a *superfunction with respect to  $\phi$*  if  $u \geq \phi$  on  $\partial\Omega$ .

Here  $u \leq \phi$  on  $\partial\Omega$  means that

$$\limsup_{y \rightarrow x, y \in \Omega, x \in \partial\Omega} u(y) \leq \phi(x).$$

**Lemma 2.2.5.** *Suppose  $\underline{u}$  is a subfunction and  $\bar{u}$  a superfunction with respect to  $\phi$ . Then  $\underline{u} \leq \bar{u}$  in  $\Omega$ .*

*Proof.* Lemma 2.2.2 and

$$\begin{aligned} \limsup_{y \rightarrow x, y \in \Omega, x \in \partial\Omega} (\underline{u}(y) - \bar{u}(y)) &= \limsup_{y \rightarrow x, y \in \Omega, x \in \partial\Omega} (\underline{u}(y) - \phi(x) + \phi(x) - \bar{u}(y)) \\ &\leq \limsup_{y \rightarrow x, y \in \Omega, x \in \partial\Omega} (\underline{u}(y) - \phi(x)) \\ &\quad + \limsup_{y \rightarrow x, y \in \Omega, x \in \partial\Omega} (\phi(x) - \bar{u}(y)) \\ &\leq 0. \end{aligned}$$

□

**Remark.** The set of subfunctions with respect to  $\phi$  and the set of superfunctions with respect to  $\phi$  are not empty since constants  $\leq \inf_{\partial\Omega} \phi$  are subfunctions and constants  $\geq \sup_{\partial\Omega} \phi$  are superfunctions.

Set

$$S_\phi = \{v \in C(\Omega) \text{ subharmonic in } \Omega : v \leq \phi \text{ on } \partial\Omega\}.$$

**Theorem 2.2.1** (Perron, [19]). *The function*

$$u(x) := \sup_{v \in S_\phi} v(x)$$

*is harmonic in  $\Omega$ .*

*Proof.* (i) We have in  $\Omega$  that

$$\inf_{\partial\Omega} \phi \leq u(x) \leq \sup_{\partial\Omega} \phi.$$

To show this inequality, let  $v \in S_\phi$ , then  $v(x) \leq \phi(x) \leq \sup_{\partial\Omega} \phi$  on  $\partial\Omega$ . Since the constant  $\sup_{\partial\Omega} \phi$  is a superfunction with respect to  $\phi$  and  $v$  is a subfunction with respect to  $\phi$  we obtain from Lemma 2.2.5 the inequality  $v(x) \leq \sup_{\partial\Omega} \phi$ ,  $x \in \Omega$ . Consequently  $u(x) \leq \sup_{\partial\Omega} \phi$ ,  $x \in \Omega$ . The other side of the above inequality follows since the constant  $\inf_{\partial\Omega} \phi$  is an element of  $S_\phi$ .

(ii) Let  $y \in \Omega$  be fixed. Then there is a sequence  $v_n \in S_\phi$  with  $\lim_{n \rightarrow \infty} v_n(y) = u(y)$ . Let  $B = B_R(y) \subset\subset \Omega$ ,  $R$  sufficiently small, and let  $V_n$  be the harmonic lifting of  $v_n$  in  $B$ . Then

$$V_n \in S_\phi, \quad (2.4)$$

and

$$\lim_{n \rightarrow \infty} V_n(y) = u(y). \quad (2.5)$$

Proof of (2.4): That  $V_n$  is subharmonic is the assertion of Lemma 2.2.3. Since  $V_n = v_n$  on  $\partial B$ , we have  $V_n \leq \phi$  on  $\partial B$ .

Proof of (2.5): We have

$$v_n(y) \leq V_n(y)$$

since  $v_n = V_n$  on  $\partial B$ ,  $V_n$  is harmonic in  $B$  and  $v_n$  is superharmonic. Then

$$u(y) \leq \liminf_{n \rightarrow \infty} V_n(y).$$

On the other hand, since  $V_n \in S_\phi$ , we have

$$V_n(y) \leq \sup_{v \in S_\phi} v(y) = u(y),$$

which implies that

$$\limsup_{n \rightarrow \infty} V_n(y) \leq u(y).$$

(iii) For every function  $h$  harmonic in  $B$  we have,

$$\sup_{B_\rho(y)} |D^\alpha h| \leq C(\rho, R, \alpha, n) \sup_{B_R(y)} |h|, \quad (2.6)$$

where  $0 < \rho < R$  and the constant  $C$  is finite. This inequality is a consequence of Poisson's formula for the solution of the Dirichlet problem in a ball, see [13, 10, 17], for example. Consequently for each fixed  $\rho$ ,  $0 < \rho \leq R$  there exists a subsequence  $V_{n_k}$  which converges uniformly in  $B_\rho(y)$  to a harmonic function  $v$ . It follows that there is a subsequence of  $V_n$ , denoted

also by  $V_{n_k}$ , which converges uniformly on compact subsets of  $B_R(0)$  to a function  $v$  harmonic in  $B_R(0)$ . We have

$$v(x) \leq u(x), \quad x \in B_R(y) \quad (2.7)$$

since  $V_{n_k}(x) \leq u(x)$  on  $B_R(y)$ , see the definition of  $u(x)$ . At the center  $y$  it is, see (2.5),

$$v(y) = u(y). \quad (2.8)$$

(iv) Claim:  $v(x) = u(x)$ ,  $x \in B$ .

Proof: If not, then there is a  $z \in B$  such that  $v(z) < u(z)$ . Then there exists an  $u_0 \in S_\phi$  with  $v(z) < u_0(z)$ . Set

$$w_k(x) := \max(u_0(x), V_{n_k}(x)).$$

Let  $W_k$  be the harmonic lifting of  $w_k$  in  $B$ . A subsequence of  $W_k$  converges uniformly on each compact subset of  $B$  to a function  $w$  harmonic in  $B$  such that

$$v(x) \leq w(x) \leq u(x), \quad x \in B = B_R(y). \quad (2.9)$$

These inequalities follow since  $w_{n_k}(x) \leq W_{n_k}(x)$ ,  $w_{n_k}(x) \geq V_{n_k}(x)$  and  $W_{n_k}(x) \leq u(x)$ , where  $x \in B$ .

Combining equation (2.8) and inequalities (2.9) we obtain

$$v(y) = w(y) = u(y). \quad (2.10)$$

That is, the harmonic function  $v - w$  is less or equal zero in  $B$  and zero in the interior point  $y \in B$ . The strong maximum principle implies that

$$v(x) = w(x), \quad x \in B. \quad (2.11)$$

According to the assumption we have for a  $z \in B$

$$v(z) < u_0(z).$$

On the other hand, see the definition of  $w_n$  and  $W_n$ , if  $x \in B$  then

$$u_0(x) \leq w_{n_k}(x) \leq W_{n_k}(x),$$

which implies that

$$u_0(x) \leq w(x), \quad x \in B.$$

Summarizing, we have for the particular  $z$  under consideration the inequalities

$$v(z) < u_0(z) \leq w(z),$$

which is a contradiction to (2.11).  $\square$

### 2.3 Boundary behaviour

One of the advantages of Perron's method is that the boundary behaviour of solutions is separated from the existence problem.

**Definition.** A  $C(\overline{\Omega})$ -function  $w = w_\xi$  is called a *barrier* at  $\xi \in \partial\Omega$  relative to  $\Omega$  if

- (i)  $w$  is superharmonic in  $\Omega$ ,
- (ii)  $w > 0$  in  $\overline{\Omega} \setminus \{\xi\}$  and  $w(\xi) = 0$ .

$w$  is called a *local barrier* at  $\xi \in \partial\Omega$  if there is a neighbourhood  $\mathcal{N}$  of  $\xi$  such that  $w$  satisfies (i) and (ii) in  $\Omega \cap \mathcal{N}$  instead in  $\Omega$ .

Let  $w$  be a local barrier at  $\xi \in \partial\Omega$ , then we can define a barrier at  $\xi \in \partial\Omega$  relative to  $\Omega$  as follows. Let  $B = B_R(\xi)$ ,  $R > 0$  sufficiently small such that  $B \subset\subset \mathcal{N}$ , see Figure 2.2. Set

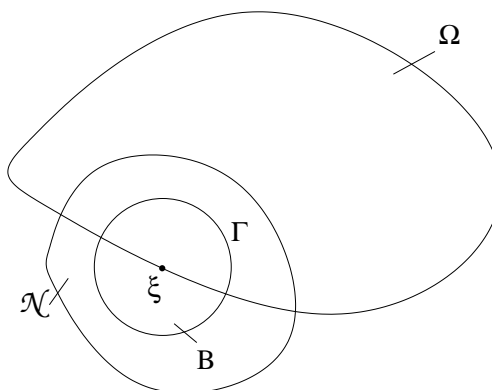


Figure 2.2: Definition of a local barrier

$$m = \inf_{\mathcal{N} \setminus B} w$$

. We have  $m > 0$ , see assumption (ii) in the above definition.

**Lemma 2.2.6.** *The function*

$$w_0(x) = \begin{cases} \min(m, w(x)) & : x \in \overline{\Omega} \cap \overline{B} \\ m & : x \in \overline{\Omega} \setminus B \end{cases}$$

is a barrier at  $\xi$  relative to  $\partial\Omega$ .

*Proof.* The property  $w_0 \in C(\overline{\Omega})$  follows since  $w_0 = m$  on  $\Gamma = \partial B \cap \Omega$ . If not, then there is an  $x_0 \in \Gamma$  with  $w(x_0) < m$ , which is a contradiction to the definition of  $m$ . Now we will show that  $w_0$  is superharmonic in  $\Omega$ . This follows since

$$w_0(x) = \begin{cases} w_1 & : x \in \overline{\Omega} \cap \overline{B} \\ w_2 & : x \in \overline{\Omega} \setminus B \end{cases},$$

where  $w_1(x) = \min(m, w(x))$ ,  $w_2 = m$  and  $w_1, w_2$  are superharmonic in  $\overline{\Omega} \cap \overline{B}$  and  $\overline{\Omega} \setminus B$ , respectively, and since  $w_1 = w_2$  on  $\Gamma$ . To show this, consider a ball  $B' \subset\subset \Omega$  located as shown in Figure 2.3. Let  $h$  be harmonic

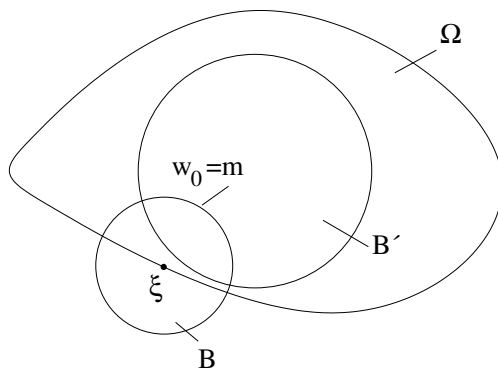


Figure 2.3: A local barrier defines a barrier

in  $B'$  with  $h \leq w_0$  on  $\partial B'$ . We have to show that  $h \leq w_0$  in  $B'$ . Since  $w_0 \leq m$  on  $\partial_1 B' = \partial B' \cap B$  and  $w_0 = m$  on  $\partial_2 B' = \partial B' \setminus \partial_1 B'$ , we have  $w_0 \leq m$  on  $\partial B'$ . Thus the assumption  $h \leq w_0$  on  $\partial B'$  implies that  $h \leq m$  on  $\partial B'$ . In particular  $h \leq w_0$  in  $B' \setminus B$  since  $w_0 = m$  on  $B' \setminus B$ . Finally we have  $h \leq w_0$  in  $B \cap B'$  since  $h \leq w_0$  on  $B' \cap \partial B$  and on  $B \cap \partial B'$ . We recall that  $w_0 = w_1$  in  $\Omega \cap B$  and  $w_1$  is superharmonic in  $\Omega \cap B$ , see Lemma 2.2.4.  $\square$

**Definition.** A boundary point will be called *regular* if there exists a barrier at that point.

**Lemma 2.2.7.** *Let  $u$  be a harmonic function defined in  $\Omega$  by the Perron method with boundary data  $\phi$ . If  $\xi$  is a regular point of  $\partial\Omega$  and if  $\phi$  is continuous at  $\xi$ , then*

$$\lim_{x \rightarrow \xi, x \in \Omega} u(x) = \phi(\xi).$$

*Proof.* Fix  $\epsilon > 0$ . Then there is a  $\delta = \delta(\epsilon) > 0$  such that  $|\phi(x) - \phi(\xi)| < \epsilon$  for all  $x \in \partial\Omega$  satisfying  $|x - \xi| < \delta$ . Set  $M = \sup_{\partial\Omega} |\phi|$ . Let  $w$  be a barrier at  $\xi$ . Then there is a  $k = k(\epsilon) > 0$  such that  $kw(x) > 2M$  if  $|x - \xi| \geq \delta$ .

*Step 1.* We will show that  $\phi(\xi) + \epsilon + kw(x)$  is a superfunction relative to  $\phi$ . We recall that  $w \in C(\overline{\Omega})$ ,  $w$  is superharmonic in  $\Omega$ ,  $w > 0$  in  $\overline{\Omega} \setminus \{\xi\}$  and  $w(\xi) = 0$ . Then

$$\phi(\xi) + \epsilon + kw(x) \geq \phi(x)$$

on  $\partial\Omega$ , since for  $x \in \partial\Omega$  with  $|x - \xi| \geq \delta$  we have

$$\begin{aligned} \phi(\xi) + \epsilon + kw(x) &\geq \phi(\xi) + \epsilon + 2M \\ &\geq \phi(x), \end{aligned}$$

and for  $x \in \partial\Omega$  with  $|x - \xi| < \delta$  we have

$$|\phi(x) - \phi(\xi)| \leq \epsilon$$

since  $|\phi(x) - \phi(\xi)| \leq \epsilon$  if  $x \in \partial\Omega \cap B_\delta(\xi)$  and  $kw(x) \geq 0$ .

*Step 2.* Since

$$u(x) = \sup_{v \in S_\phi} v(x)$$

and  $\phi(\xi) - \epsilon - kw(x)$  is a subfunction relative to  $\phi$ , we have

$$\phi(\xi) - \epsilon - kw(x) \leq u(x)$$

in  $\Omega$ . The function  $\phi(\xi) + \epsilon + kw(x)$  is a superfunction relative to  $\phi$ , see Step 1. Consequently,

$$v(x) \leq \phi(\xi) + \epsilon + kw(x)$$

for all  $v \in S_\phi$ , see Lemma 2.2.5. This inequality implies that

$$u(x) \leq \phi(\xi) + \epsilon + kw(x).$$

Summarizing, we get

$$|u(x) - \phi(\xi)| \leq \epsilon + kw(x).$$

Since  $\lim_{x \rightarrow \xi, x \in \Omega} w(x) = 0$ , we obtain finally

$$\lim_{x \rightarrow \xi, x \in \Omega} u(x) = \phi(\xi).$$

□

**Theorem 2.2.2.** *Let  $\Omega \subset \mathbb{R}^n$  be bounded and connected. Then the Dirichlet problem  $\Delta u = 0$  in  $\Omega$ ,  $u = \phi$  on  $\partial\Omega$ , where  $\phi \in C(\partial\Omega)$  is given, is solvable if and only if the boundary points are all regular.*

*Proof.* (i) Assume all points of  $\partial\Omega$  are regular points. Then the assertion follows from previous Lemma 2.2.7.

(ii) Assume the Dirichlet problem is solvable for all continuous  $\phi \in C(\partial\Omega)$ . Set  $\phi(x) := |x - \xi|$  and consider the Dirichlet problem with the boundary condition  $u(x) = \phi(x)$  on  $\partial\Omega$ . Let  $u_\xi(x)$  be the solution, then  $u_\xi(x)$  is a barrier at  $\xi$ . Consequently, all boundary points are regular. □

### 2.3.1 Examples for local barriers

#### Slit domains in $\mathbb{R}^2$

Let  $\Omega \subset \mathbb{R}^2$  with a slit along the negative  $x$ -axis at  $x = 0$  as indicated in Figure 2.4. Let  $\text{Ln } z := \ln |z| + i\phi$ ,  $-\pi < \phi \leq \pi$ , be a branch of  $\ln z$ . Then

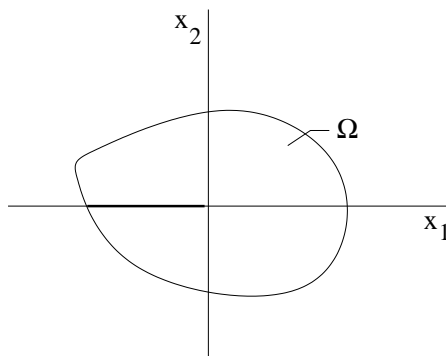


Figure 2.4: A slit domain

$$w := -\text{Re} \left( \frac{1}{\text{Ln } z} \right) = -\frac{\ln r}{\ln^2 r + \phi^2}$$

is a local barrier at  $\xi = 0$ . Here is  $w(0)$  defined as the limit  $w(0) := \lim_{z \rightarrow 0, z \in \Omega} w(z)$ . We have  $w(0) = 0$  and  $w(x) > 0$  in  $\Omega \cap B_R(0)$ . For higher dimensions there are counterexamples. One of them was given by Lebesgue, see [4], Part II, p. 272, for example, which shows that sufficiently sharp cusps are not regular at the tip of the cusp, see Figure 2.5.

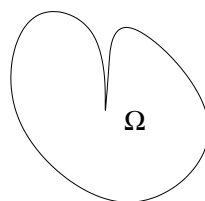


Figure 2.5: A cusp boundary point

**Exterior sphere condition**

We say that  $\Omega$  satisfies the *exterior sphere condition* at  $\xi \in \partial\Omega$  if there is a sphere  $B_R(y) \subset \mathbb{R}^n \setminus \Omega$  such that  $\overline{B_R(y)} \cap \overline{\Omega} = \{\xi\}$ , see Figure 2.6. Let  $\Omega$

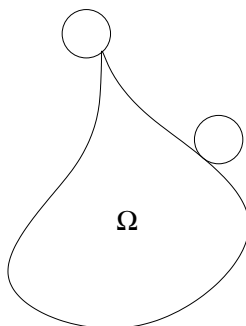


Figure 2.6: Exterior sphere condition

satisfies the exterior sphere condition at  $\xi \in \partial\Omega$ , then

$$w(x) = \begin{cases} R^{2-n} - |x - y|^{2-n} & : n \geq 3 \\ \ln\left(\frac{|x-y|}{R}\right) & : n = 2 \end{cases}$$

is a local barrier at  $\xi$ .

**Exterior cone condition**

We say that  $\Omega$  satisfies an *exterior cone condition* at  $\xi \in \partial\Omega$  if there is a finite circular cone  $\mathcal{C}$  with the vertex at  $\xi$  such that  $\overline{\mathcal{C}} \cap \overline{\Omega} = \{\xi\}$ , see Figure 2.7. Let  $\xi$  be the origin and assume the exterior cone property is satisfied at  $\xi \in \partial\Omega$ . Then we can find a positive constant  $\lambda$  and a positive function  $f(\theta)$ , where  $\theta$  is the polar angle, such that

$$w = r^\lambda f(\theta),$$

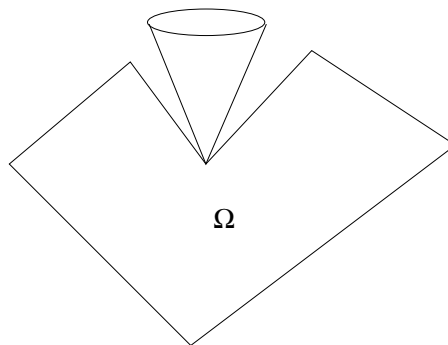


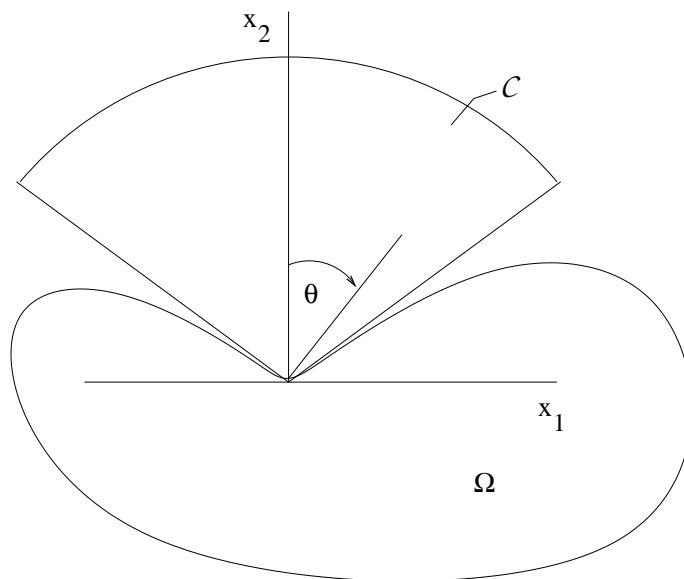
Figure 2.7: Exterior cone condition

$r = |x|$ , is a local barrier at  $\xi$ .

*Two-dimensional domains.* Here we find  $\lambda$  and  $f(\theta)$  as follows. Let

$$\mathcal{C} \subset \{(r, \theta) : r > 0, -\alpha < \theta < \alpha\},$$

see Figure 2.8 Introducing polar coordinates  $(r, \theta)$ , where

Figure 2.8: Local barrier,  $\mathbb{R}^2$ 

$$x_1 = r \sin \theta, \quad x_2 = r \cos \theta,$$

we have

$$\begin{aligned} w(x) &= W(r, \theta) := w(r \cos \theta, r \sin \theta) \\ \Delta w &= \frac{1}{r} \frac{\partial}{\partial r} (r W_r) + \frac{1}{r^2} W_{\theta\theta}. \end{aligned}$$

Consider the ansatz

$$W(r, \theta) = r^\lambda \cos(\mu\theta),$$

where  $\lambda$  and  $\mu$  are positive constants, then

$$\Delta w = r^{\lambda-2}(\lambda^2 - \mu^2) \cos(\mu\theta).$$

Consequently,  $\Delta w \leq 0$  on  $\mathcal{C}$  if  $\lambda \leq \mu$  and  $|\mu\theta| \leq \pi/2$  for all  $\theta$  satisfying  $\alpha < \theta < 2\pi - \alpha$  ( $0 < \alpha < \pi/2$ ).

Then  $w > 0$  if  $r > 0$  and  $\alpha < \theta < 2\pi - \alpha$ . That is,  $w$  is a local barrier at the origin if we choose  $\lambda = \mu$  with a sufficiently small positive  $\mu$ .

*Higher dimensional case.* Let  $\mathcal{M} \subset \partial B_1(O)$  be the manifold as indicated in Figure 2.9. Consider the eigenvalue problem

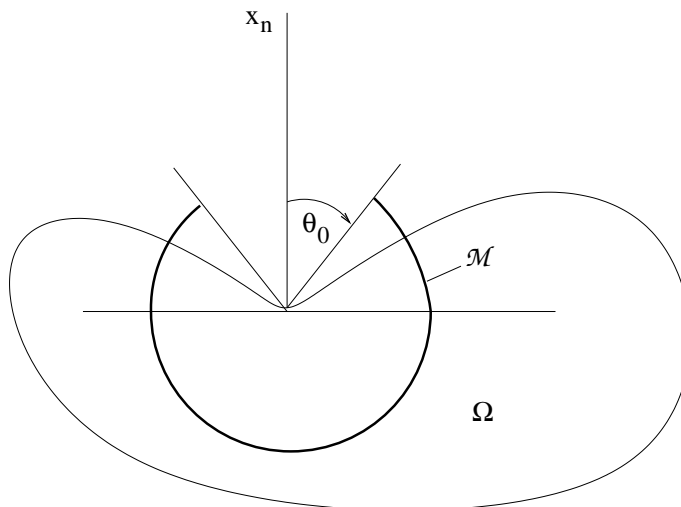


Figure 2.9: Local barrier,  $\mathbb{R}^n$

$$\begin{aligned} -\Delta' v &= \nu^2 v \text{ in } \mathcal{M} \\ v &= 0 \text{ on } \partial\mathcal{M}, \end{aligned}$$

where  $\Delta'$  is the Laplace-Beltrami differential operator on the unit sphere. We recall that in the two-dimensional case

$$\Delta' = \frac{\partial^2}{\partial \theta^2}$$

and in the three-dimensional case

$$\Delta' = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

For the definition of the Laplace-Beltrami operator see for example [2] and for  $n$ -dimensional polar coordinates see [7], Part III, pp. 395, for example.

Let  $\nu_1$  be the first eigenvalue of the above eigenvalue problem. It is known that  $\nu_1$  is positive, a simple eigenvalue and the associated eigenfunction  $v_1$  has no zero in  $\mathcal{M}$ . Thus, we can assume that  $v_1 > 0$  in  $\mathcal{M}$ . Set

$$W = Ar^\kappa v_1 \equiv w(x)$$

where  $A$  and  $\kappa$  are positive constants. Then  $w > 0$  in  $\mathbb{R}^n \setminus \mathcal{C}$  and

$$\Delta w = Ar^{\kappa-2} v_1 (\kappa(\kappa-1) + (n-1)\kappa - \alpha_1),$$

where

$$\alpha_1 = -\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \nu_1^2}.$$

Consequently we have  $\Delta w \leq 0$  in  $\mathbb{R}^n \setminus \mathcal{C}$ , provided  $\kappa > 0$  is sufficiently small.

## 2.4 Generalizations

Perron's method can be applied to the Dirichlet problem for a more general class of linear elliptic equations of second order, see for example [10], pp. 102. The previous discussion in the case of the Laplace equation showed that we need a strong maximum principle, the existence of solutions of the Dirichlet problem on a ball with continuous boundary data and some estimates for the derivatives. Then we are able to show the existence of a solution of the equation in the given domain. The problem of the boundary behaviour requires an additional discussion.

The next chapter concerns the strong maximum principle for a class of linear uniformly elliptic equations of second order, one of the ingredients in Perron's method.

## 2.5 Exercises

1. Let  $\Omega \subset \mathbb{R}^n$  be a connected domain. Consider the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Suppose  $\lambda_0 \geq 0$  is an eigenvalue and  $u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$  an associated eigenfunction satisfying  $u_0(x) \geq 0$  in  $\Omega$ . Show that  $u_0(x) > 0$  in  $\Omega$ .

2. Prove the second corollary to Theorem 2.1.1.
3. Prove Lemma 2.2.4.

## Chapter 3

# Maximum principles

Maximum principles provide powerful tools for linear and nonlinear elliptic equations of second order. In this chapter we consider linear equations.

### 3.1 Basic maximum principles

Set

$$\begin{aligned} Mu &= \sum_{i,j=1}^n a_{ij}(x)D_{ij}u + \sum_{i=1}^n b_i(x)D_iu \\ Lu &= Mu + c(x)u, \end{aligned}$$

where  $a_{i,j}$ ,  $b_i$  and  $c$  are real and defined on a simply connected domain  $\Omega \subset \mathbb{R}^n$ . We assume  $a_{ij} = a_{ji}$ . Let  $\lambda(x)$  be the minimum of the eigenvalues of the symmetric matrix defined by the coefficients  $a_{ij}$  and let  $\Lambda(x)$  be the maximum of these eigenvalues.

**Definition.**  $L$  is called *elliptic* in  $\Omega$  if  $\lambda(x) > 0$  in  $\Omega$ .  $L$  is said to be *strictly elliptic* in  $\Omega$  if  $\lambda(x) \geq \lambda_0 > 0$  in  $\Omega$ , where  $\lambda_0$  is a constant. An elliptic  $L$  is called *uniformly elliptic* if  $\Lambda/\lambda$  is bounded in  $\Omega$ .

In the following we suppose that  $L$  is at least elliptic and for each  $i$

$$\sup_{x \in \Omega} \frac{|b_i(x)|}{\lambda(x)} < \infty. \quad (3.1)$$

**Theorem 3.1.1** (Weak maximum principle). *Let  $L$  be elliptic in the bounded domain  $\Omega$ . Assume a function  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  satisfies  $Mu \geq 0$  ( $Mu \leq 0$ ) in  $\Omega$ . Then the maximum (minimum) of  $u$  on  $\overline{\Omega}$  is achieved on  $\partial\Omega$ .*

*Proof.* Assume initially that  $Mu > 0$  in  $\Omega$ . Then  $u$  cannot achieve an interior maximum since  $\nabla u(x_0) = 0$  at this point where  $u$  achieves its maximum, and since the matrix  $D^2u(x_0) = [D_{ij}u(x_0)]$  is nonpositive (necessary condition of second order). It follows that, see an exercise in [17], for example,

$$Mu(x_0) = \sum_{i,j=1}^n a_{ij}(x_0)D_{ij}u(x_0) \leq 0$$

since the matrix  $[a_{ij}(x_0)]$  is nonnegative (even positive) by assumption. This inequality is a contradiction to our assumption.

For a positive sufficiently large constant  $\gamma$  we calculate

$$\begin{aligned} Me^{\gamma x_1} &= (\gamma^2 a_{11} + \gamma b_1)e^{\gamma x_1} \\ &\geq \lambda(\gamma^2 - \gamma c_1)e^{\gamma x_1} > 0. \end{aligned}$$

We recall that  $a_{11} \geq \lambda$  and  $|b_1|/\lambda \leq c_1$ , where  $c_1$  is a positive constant, see assumption (3.1). Consequently for any  $\epsilon > 0$  we have in  $\Omega$

$$M(u + \epsilon e^{\gamma x_1}) > 0.$$

Using the above result, we conclude that

$$\sup_{\Omega} (u + \epsilon e^{\gamma x_1}) = \sup_{\partial\Omega} (u + \epsilon e^{\gamma x_1}).$$

Letting  $\epsilon \rightarrow 0$ , we obtain

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

□

The next theorem is the strong maximum principle. It follows from the boundary point lemma due to E. Hopf [11]. The proof of this lemma needs the previous weak maximum principle. The strong maximum principle is the essential tool to show existence of a solution of the Dirichlet problem via Perron's method.

**Lemma 3.1.1** (E. Hopf, 1952). *Let  $L$  be uniformly elliptic. Assume  $u \in C^2(\Omega)$  satisfies  $Mu \geq 0$  in  $\Omega$ . Let  $x_0 \in \partial\Omega$  and suppose that*

- (i)  $u$  is continuous at  $x_0$ ,
- (ii)  $u(x_0) > u(x)$  for all  $x \in \Omega \cap B_a(x_0)$  for an  $a > 0$ ,
- (iii)  $\partial\Omega$  satisfies the interior sphere condition at  $x_0$ .

*Then the outer normal derivative of  $u$  at  $x_0$ , if it exists, satisfies*

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

*Proof.* Let  $B = B_R(y)$  be the ball related to the interior sphere condition, see Figure 3.1. Consider the function

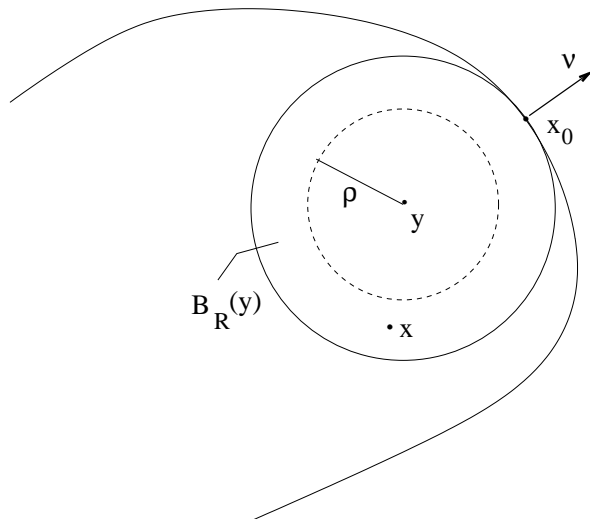


Figure 3.1: Proof of Hopf's boundary point lemma

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2},$$

where  $r = |x - y| > \rho$  and  $\alpha$  is a positive constant which we will determine later. A calculation leads to

$$\begin{aligned} Mv &= e^{-\alpha r^2} \left( 4\alpha^2 \sum_{i,j=1}^n a_{ij}(x_i - y_i)(x_j - y_j) \right. \\ &\quad \left. - 2\alpha \left( \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_i(x_i - y_i) \right) \right) \\ &\geq e^{-\alpha r^2} \left( 4\alpha^2 \lambda(x) r^2 - 2\alpha \left( \sum_{i=1}^n a_{ii} + |b|r \right) \right), \end{aligned}$$

where  $b = (b_1, \dots, b_n)$ . Since by assumption  $a_{ii}/\lambda$  and  $|b|/\lambda$  are bounded, we may choose  $\alpha$  large enough such that  $Mv \geq 0$  in the annular domain  $\mathcal{A} := B_R(y) \setminus B_\rho(y)$ . Since  $u(x) - u(x_0) < 0$  on  $\partial B_\rho(y)$  there is a constant  $\epsilon > 0$  such that  $u(x) - u(x_0) + \epsilon v(x) \leq 0$  on  $\partial B_\rho(y)$ . This inequality is

also satisfied on  $\partial B_R(y)$  by assumption on  $u$  and since  $v = 0$  on  $\partial B_R(y)$ . We have  $M(u(x) - u(x_0) + \epsilon v(x)) = Mu + \epsilon Mv$ . Then the weak maximum principle implies that  $u - u(x_0) + \epsilon v \leq 0$  in  $\mathcal{A}$ . Thus

$$u(x_0) - u(x) \geq -\epsilon(v(x_0) - v(x)),$$

where  $x \in \mathcal{A}$  and on the line defined by  $x_0$  and  $\nu$ . It follows

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\epsilon V'(R),$$

provided the normal derivative exists. Here is

$$V(r) = e^{-\alpha r^2} - e^{-\alpha R^2}.$$

□

**Remark.** If the normal derivative does not exist, then

$$\liminf_{x \rightarrow x_0} \frac{u(x_0) - u(x)}{|x_0 - x|} > 0,$$

where the angle between  $x_0 - x$  and the exterior normal  $\nu$  is less than  $(\pi/2) - \delta$  for a fixed  $\delta > 0$ .

**Corollary.** *Suppose that  $\partial\Omega$  satisfies the interior sphere condition at  $x_0 \in \partial\Omega$ ,  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  satisfies  $Lu \geq 0$  and  $u(x_0) > u(x)$  in  $\Omega \cap \mathcal{U}$ , where  $\mathcal{U}$  is a neighbourhood of  $x_0$ . If additionally  $c \leq 0$  in  $\Omega \cap \mathcal{U}$  and  $u(x_0) > 0$  then  $\partial u / \partial \nu(x_0) > 0$ , provided the normal derivative exists.*

*Proof.*  $Mu = Lu - cu \geq 0$  in  $\Omega \cap \mathcal{V}$ , where  $\mathcal{V}$  is a neighbourhood of  $x_0$ . □

In generalization to the strong maximum principle for  $\Delta$  we have

**Theorem 3.1.2** (Strong maximum principle). *Let  $L$  be uniformly elliptic. Assume  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  satisfies  $Mu \geq 0$  ( $Mu \leq 0$ ) in a connected domain  $\Omega$ , not necessarily bounded. Then if  $u$  achieves its supremum (infimum) in the interior of  $\Omega$ ,  $u$  is a constant.*

*Proof.* Consider the case of a maximum. Assume  $u$  is not constant and achieves its maximum  $m$  in the interior of  $\Omega$ . Set  $\Omega_1 = \{x \in \Omega : u(x) = m\}$  and  $\Omega_2 = \{x \in \Omega : u(x) < m\}$ . By assumption  $\Omega_1$  is not empty. We will show that  $\Omega_1$  is open. Then  $\Omega_2 = \emptyset$  since we suppose that  $\Omega$  is connected.

Let  $x^1 \in \Omega_1$ . Consider a ball  $B = B_{2\rho_0}(x^1) \subset\subset \Omega$ . If  $\Omega_1$  is not open, then there is an  $x^2 \in B_{\rho_0}(x^1)$  such that  $u(x^2) < m$ . Consequently there is a ball  $B_\rho(x^2)$ , where  $0 < \rho \leq |x^2 - x^1|$ , and  $u(x) < m$  in  $B_\rho(x^2)$  and there is an  $x^3 \in \partial B_\rho(x^2)$  such that  $u(x^3) = m$ . See Figure 3.2 for notations. Hopf's

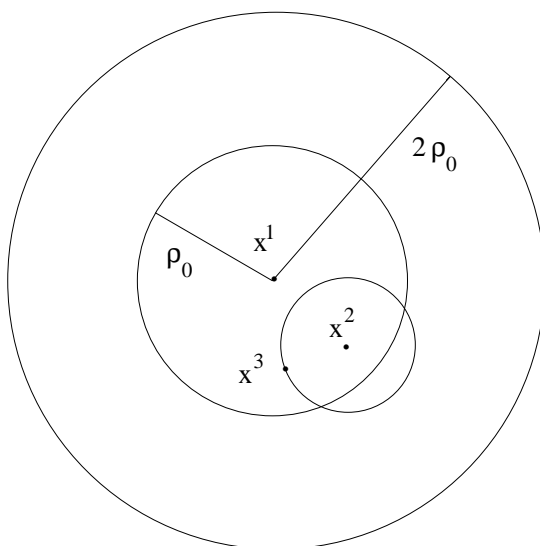


Figure 3.2: Proof of the strong maximum principle

lemma says that  $\partial u / \partial \nu(x^3) > 0$ , where  $\nu$  is the exterior normal derivative on  $x^3 \in \partial B_\rho(x^2)$  at  $u(x^3)$  which is a contradiction to the fact that  $u$  attains an interior maximum at  $x^3$ .  $\square$

In many cases the assumption  $c \equiv 0$  in  $\Omega$  is not satisfied. If  $c(x) \leq 0$  in  $\Omega$ , then we have the following corollary to the previous theorem. If  $c(x)$  is positive on a subset of  $\Omega$ , then the situation is more complicated. In this case one studies an associated eigenvalue problem.

**Corollary.** *Let  $\Omega$  be a connected domain, not necessarily bounded. Suppose  $L$  is uniformly elliptic and  $c(x) \leq 0$  in  $\Omega$ . Assume  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  satisfies  $Lu \geq 0$  ( $Lu \leq 0$ ). Then if  $u$  achieves its **positive** supremum (**negative** infimum) in the interior of  $\Omega$ ,  $u$  is a constant.*

*Proof.* Consider the case of a maximum. Set  $m = \sup_{\Omega} u(x)$  and  $\Omega_1 = \{x \in \Omega : u(x) = m\}$ ,  $\Omega_2 = \{x \in \Omega : u(x) < m\}$ . By assumption  $\Omega_1$  is not empty. We show that  $\Omega_1$  is an open set. Let  $x^1 \in \Omega_1$ . Then there is ball  $B_\rho(x^1) \subset \Omega$

where  $u$  is nonnegative. Thus

$$Mu \equiv Lu - c(x)u \geq 0$$

in  $B_\rho(x^1)$ . The above strong maximum principle (Theorem 3.1.2) says that  $u(x) = m$  for all  $x \in B_\rho(x^1)$ .  $\square$

From this corollary follows a result important for many applications:

**Theorem 3.1.3** (Comparison principle). *Let  $\Omega$  be a bounded and connected domain. Suppose  $L$  be uniformly elliptic and  $c(x) \leq 0$  in  $\Omega$ . Assume  $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$  and satisfy  $Lu \geq Lv$  in  $\Omega$  and  $u \leq v$  on  $\partial\Omega$ . Then  $u \leq v$  in  $\Omega$ .*

*Proof.* Set  $w = u - v$ . Then  $Lw \geq 0$  in  $\Omega$  and  $w \leq 0$  on  $\partial\Omega$ . From the above corollary we see that  $w$  can not achieve a positive maximum in  $\Omega$ .  $\square$

### 3.1.1 Directional derivative boundary value problem

As an application of the previous corollary we consider a generalization of the Neumann problem. Let  $\Omega \subset \mathbb{R}^n$  be a bounded and connected domain, and assume  $\partial\Omega$  is sufficiently smooth. Consider

$$Lu = f, \quad \text{in } \Omega \tag{3.2}$$

$$\frac{\partial u}{\partial \alpha} = \phi \quad \text{on } \partial\Omega, \tag{3.3}$$

where  $f, \phi$  are given and sufficiently regular, and the direction  $\alpha$  is not tangential on  $\partial\Omega$  at each point of  $\partial\Omega$ , see Figure 3.3.

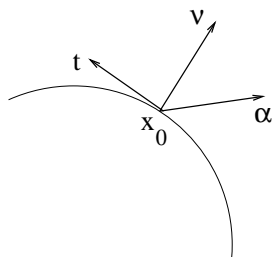


Figure 3.3: Directional derivative boundary value problem

**Proposition 3.1.1.** *Suppose that  $c \leq 0$  in  $\Omega$  and let  $u_1, u_2 \in C^2(\Omega) \cap C^1(\bar{\Omega})$  are solutions of (3.2), (3.3). Then  $u_1 - u_2 \equiv \text{const.}$  in  $\Omega$ .*

*Proof.* Set  $u = u_1 - u_2$ , then  $Lu = 0$  in  $\Omega$  and  $\partial u/\partial\alpha = 0$  on  $\partial\Omega$ . Suppose that  $u$  is not constant, then we can assume  $\max_{\overline{\Omega}} u > 0$ . This maximum is achieved at  $x_0 \in \partial\Omega$  and  $u(x_0) > u(x)$  for all  $x \in \Omega$ , see the corollary to the strong maximum principle. For the tangential derivative we have  $(\partial u/\partial t)(x_0) = 0$ , and the by assumption  $(\partial u/\partial\alpha)(x_0) = 0$ . Thus

$$\begin{aligned} (\partial u/\partial\nu)(x_0) &= a(x_0)(\partial u/\partial t)(x_0) + b(x_0)(\partial u/\partial\alpha)(x_0) \\ &= 0, \end{aligned}$$

which is a contradiction to the corollary to the Hopf boundary point lemma which says that  $\partial u/\partial\nu > 0$  at  $x_0$ .  $\square$

### 3.1.2 Behaviour near a corner

Set  $Lu = \sum_{i,j=1}^2 a_{ij}(x)u_{x_i x_j}$  and consider the Dirichlet problem

$$Lu = f \text{ in } \Omega \tag{3.4}$$

$$u = 0 \text{ on } \partial\Omega, \tag{3.5}$$

where  $f$  is given. Suppose that the boundary of  $\Omega \subset \mathbb{R}^2$  has a corner. Without loss of generality we suppose that the corner is the origin. Set  $\Omega_\rho = \Omega \cap B_\rho(0)$ . Then we assume that there is a  $\rho > 0$  such that in  $\Omega_\rho$  we have  $a_{ij} = a_{ji}$ ,  $L$  is uniformly elliptic,  $a_{ij} \in C^\alpha(\overline{\Omega_\rho})$ . An appropriate rotation with center at the origin and a stretching of the axis transforms (3.4), (3.5) into a Dirichlet problem where  $a_{ij}(0,0) = \delta_{ij}$ . Here we denote the transformed coefficients  $a_{ij}(C^{-1}y)$  and the transformed right hand side  $f(C^{-1}y)$  by  $a_{ij}$  and  $f$ , respectively, again, as well as  $y$  by  $x$ . Also the new domain is denoted by  $\Omega$  again. The new interior angle  $\omega$  can be calculated from the original interior angle  $\gamma$  and the original coefficients  $[a_{ij}(0,0)]$ , see an exercise. After this mapping we arrived at the problem

$$Lu = \Delta u + \sum_{i,j=1}^2 (a_{ij}(x) - \delta_{ij}) u_{x_i x_j} = f \text{ in } \Omega_\rho \tag{3.6}$$

$$u = 0 \text{ on } \partial\Omega \cap \overline{\Omega_\rho}, \tag{3.7}$$

where

$$|a_{ij}(x) - \delta_{ij}| \leq c|x|^\alpha.$$

Suppose that  $\Omega_\rho$  is contained in a domain, see Figure 3.4, defined by  $0 < r < \rho$  and  $\theta_1(\rho) \leq \theta \leq \theta_2(\rho)$ , where  $r = (x_1^2 + x_2^2)^{1/2}$ . Set

$$\omega = \omega(\rho) = \theta_2(\rho) - \theta_1(\rho)$$

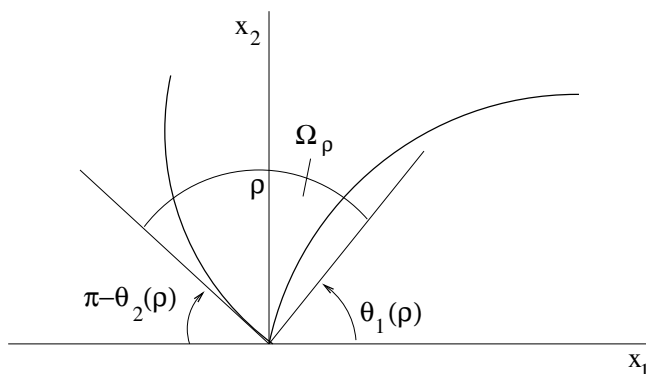


Figure 3.4: Corner domain

and consider the function

$$v(\theta) = \sin \left( \left( \frac{\pi}{\omega} - h(\epsilon) \right) (\theta - \theta_1(\rho) + \epsilon) \right),$$

where  $0 < \epsilon \leq \epsilon_0$ ,  $\epsilon_0$  sufficiently small, and  $h(\epsilon) = 2\pi\epsilon/\omega^2$ . There is an  $\epsilon_0 > 0$  and a positive constant  $c$ , independent of  $\epsilon$  such that  $v(\theta) \geq c\epsilon$  for all  $0 < \epsilon \leq \epsilon_0$  and  $\theta_1 \leq \theta \leq \theta_2$  (exercise). Set

$$\kappa = \frac{\pi}{\omega} - h(\epsilon)$$

and consider the function

$$\begin{aligned} W(r, \theta) &= Ar^{\kappa-\eta}v(\theta) \\ &= w(x_1, x_2), \end{aligned}$$

where  $A$  and  $\eta$  are constants,  $0 < \eta < \kappa$ . In polar coordinates we have

$$\Delta w = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial W}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2}.$$

Then

$$\Delta w = Ar^{\kappa-\eta-2} ((\kappa - \eta)^2 - \kappa^2) v$$

and

$$|w_{x_i x_j}| \leq c|A|r^{\kappa-\eta-2}.$$

The constant  $c$  here and in the following formulae are independent of  $r$  and  $\theta$ . Thus,

$$Lw = Ar^{\kappa-\eta-2} ((\kappa - \eta)^2 - \kappa^2) v + O(Ar^{\kappa-\eta-2+\alpha}).$$

Suppose that the constant  $A$  is positive, then from the above considerations it follows that for given  $\delta > 0$ ,  $0 < \delta \leq \delta_0$ ,  $\delta_0$  sufficiently small, there are positive constants  $c(\delta)$  and  $\rho = \rho(\delta)$  such that

$$Lw \leq -Ac(\delta)r^{(\pi/\omega)-\delta-2}$$

in  $\Omega_\rho$ .

**Proposition 3.1.2.** *Assume the right hand side  $f$  of (3.4) satisfies*

$$|f| \leq cr^{(\pi/\omega)-2-\delta+\tau}$$

in  $\Omega_{\rho_0}$ ,  $\rho_0 > 0$ , for a  $\tau > 0$ . Assume  $u \in C^2(\Omega_{\rho_0})$  and  $\sup_{\Omega_{\rho_0}} |u(x)| < \infty$ . Then for given (small)  $\epsilon > 0$  there exists positive constants  $c(\epsilon)$  and  $\rho(\epsilon) > 0$  such that

$$|u| \leq c(\epsilon)|x|^{(\pi/\omega)-\epsilon}$$

in  $\Omega_{\rho(\epsilon)}$ .

*Proof.* We have  $Lw \leq Lu$  in  $\Omega_\rho$  and  $w \geq u$  on  $\partial\Omega_\rho$ , provided  $\rho > 0$  is sufficiently small.  $\square$

**Remark 1.** The *additional* assumption that  $u$  remains bounded up to the corner is essential for the previous proposition since there exists also solutions which are unbounded near the corner. An example is the boundary value problem  $\Delta u = 0$  in  $\Omega_\alpha$ ,  $u = 0$  on  $\partial\Omega_\alpha$ , where  $\Omega_\alpha$  is the sector defined by  $r > 0$  and  $0 < \theta < \alpha$ , where  $0 < \alpha < 2\pi$ . Solutions are given by

$$u(x) = r^{\pi k/\alpha} \sin((\pi/\alpha)k\theta),$$

where  $k \in \{\pm 1, \pm 2, \dots\}$ . For a class of quasilinear *nonuniformly* boundary value problems the behaviour of the solution near the corner does not require such an additional assumption near the boundary, see [8, 16, 3, 18]. The behaviour follows from the problem itself. The reason for this striking difference is that essential information is lost through the linearization.

**Remark 2.** Asymptotic expansions near a corner yield more precise behaviour near the corner, see [15] for a class of uniformly quasilinear elliptic Dirichlet problems. In general, the expansion, depends on the solution considered, in contrast to some nonuniformly problems, see [16].

### 3.1.3 An apriori estimate

Consider for given bounded functions  $f, \Phi$  defined on  $\bar{\Omega}$  and on  $\partial\bar{\Omega}$ , respectively, the Dirichlet problem

$$Lu = \sum_{i,j=1}^n a_{ij}(x)D_{ij}u + \sum_{i=1}^n b_i(x)u_{x_i} + c(x) = f(x) \quad \text{in } \Omega \quad (3.8)$$

$$u = \Phi \quad \text{on } \partial\Omega \quad (3.9)$$

where  $a_{i,j}, b_i$  and  $c$  are real and defined on a simply connected and bounded domain  $\Omega \subset \mathbb{R}^n$ . We assume  $a_{ij} = a_{ji}$ ,  $L$  is strictly elliptic and

$$\sup_{\Omega} |b_i(x)| < \infty$$

for every  $i = 1, \dots, n$ . Let  $K$  be a bound of  $b_1$  and set

$$\alpha = \frac{1}{\lambda_0} \left( K + (K^2 + 4\lambda_0)^{1/2} \right), \quad (3.10)$$

where the positive constant  $\lambda_0$  is a lower bound of the minimum of the eigenvalues of the matrix  $[a_{ij}(x)]$ . Set  $d = \text{diam } \Omega$ .

**Proposition 3.1.3.** *Suppose that  $c(x) \leq 0$  in  $\Omega$  and let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  be a solution of (3.8), (3.9). Then*

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |\Phi| + \left( e^{\alpha d} - 1 \right) \sup_{\Omega} |f|.$$

*Proof.* Suppose that  $\Omega$  is contained in the strip defined by  $0 \leq x_1 \leq d$ . Consider the function

$$g(x) = e^{\alpha d} - e^{\alpha x_1},$$

where  $\alpha$  is positive constant which will be determined later. We have  $g(x) \geq 0$  if  $x \in \Omega$ . We get

$$\begin{aligned} Lg &= -(a_{11}\alpha^2 + b_1\alpha)e^{\alpha x_1} + cg \\ &\leq (\lambda_0\alpha^2 + K\alpha)e^{\alpha x_1} \\ &\leq -e^{\alpha x_1} \\ &\leq -1, \end{aligned}$$

provided that  $\alpha$  is large enough. We can choose  $\alpha$  given by (3.10). Set

$$h = \sup_{\partial\Omega} |\Phi| + g(x) \sup_{\Omega} |f|.$$

Then

$$\begin{aligned} Lh &= Lg \sup_{\Omega} |f| + c \sup_{\partial\Omega} |\Phi| \\ &\leq -\sup_{\Omega} |f|. \end{aligned}$$

We recall that  $c \leq 0$  on  $\Omega$ . Summarizing, we have

$$\begin{aligned} Lh &\leq -\sup_{\Omega} |f| \quad \text{in } \Omega \\ h &\geq \sup_{\Omega} |\Phi| \quad \text{on } \partial\Omega. \end{aligned}$$

Set  $v = u - h$ , then

$$\begin{aligned} Lv &= f - Lh \geq f + \sup_{\Omega} |f| \geq 0 \quad \text{in } \Omega \\ v &= \Phi - \sup_{\partial\Omega} |\Phi| \leq 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The comparison principle says that  $v \leq 0$  in  $\Omega$ . The same argument leads to the inequality  $u \geq -h$  in  $\Omega$  if we set  $v = -u - h$ . Then

$$\begin{aligned} Lv &= -f - Lh \geq -f + \sup_{\Omega} |f| \geq 0 \quad \text{in } \Omega \\ v &= -\Phi - h \leq -\Phi - \sup_{\partial\Omega} |\Phi| \leq 0 \quad \text{on } \partial\Omega. \end{aligned}$$

□

### 3.2 A discrete maximum principle

To simplify the presentation we consider here a subclass of elliptic boundary value problems in a domain  $\Omega \in \mathbb{R}^2$ , see [12], pp. 458, for example. Set

$$Mu = \Delta u + b_1(x, y)u_x + b_2(x, y)u_y.$$

Suppose that there is a constant  $K$  such that

$$\sup_{\Omega} (|b_1(x, y)| + |b_2(x, y)|) \leq K. \quad (3.11)$$

Let

$$Lu = Mu + c(x, y)u,$$

where  $c(x, y)$  is defined on  $\overline{\Omega}$ . Then we consider for given  $f$  defined on  $\Omega$ , and  $\Phi$  defined on  $\partial\Omega$ , the Dirichlet problem

$$Lu = f \text{ in } \Omega \quad (3.12)$$

$$u = \Phi \text{ on } \partial\Omega. \quad (3.13)$$

Let  $h > 0$  be a (small) constant. Then we define an associated difference operator to  $Mu$  by

$$M_h u = \frac{u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - 4u(x, y)}{h^2} \\ + b_1(x, y) \frac{u(x+h, y) - u(x-h, y)}{2h} + b_2(x, y) \frac{u(x, y+h) - u(x, y-h)}{2h}.$$

We have  $\lim_{h \rightarrow 0} M_h u = Mu$  on every compact subdomain of  $\Omega$ , provided  $u \in C^2(\Omega)$ .

Set  $P_0 = (x, y)$  and assume  $P_0 \in \Omega$ . The four points  $P_{01} = (x+h, y)$ ,  $P_{02} = (x, y+h)$ ,  $P_{03} = (x-h, y)$ ,  $P_{04} = (x, y-h)$  are called  $h$ -neighbourhood of  $P_0$  or a *star* around  $P_0$ . Consider the intersection of  $\Omega$  with an  $h$ -net  $N_h$  of  $\mathbb{R}^2$  defined by

$$N_h(x, y) = \{(x+lh, y+kh) \in \mathbb{R}^2 : k, l = 0, \pm 1, \pm 2, \dots\},$$

where  $(x, y) \in \mathbb{R}^2$  is given. Let  $\Omega_h$  be the set of the points  $P_i$  of  $\Omega \cap N_h$  such that each point of the star associated to  $P_i$  are contained in  $\overline{\Omega}$ . The set of all star points not in  $\Omega_h$  is denoted by  $\partial\Omega_h$ , see Figure 3.5. Then we define

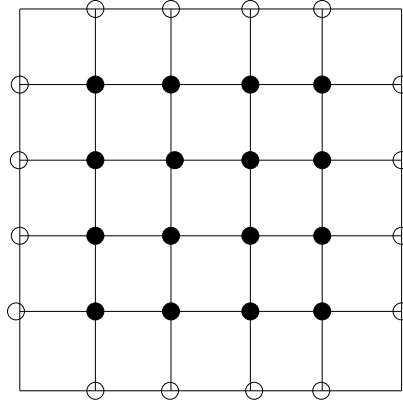


Figure 3.5: Definition of  $\Omega_h$  and  $\partial\Omega_h$

$$\overline{\Omega_h} = \Omega_h \cup \partial\Omega_h.$$

Suppose that  $\Omega_h$  is connected in the sense that for given  $P, R \in \Omega_h$  there exists points  $Q_1, Q_2, \dots, Q_s \in \Omega_h$ , where  $Q_1 = P$  and  $Q_s = R$ , such that  $Q_j$  is a point of the  $h$ -neighbourhood of  $Q_{j-1}$ ,  $j = 1, \dots, s+1$ . Moreover it is assumed that  $h > 0$  is sufficiently small such that

$$hK < 1. \quad (3.14)$$

**Theorem 3.2.1.** *Let  $u$  be defined on  $\overline{\Omega_h}$ . Suppose that  $M_h u \geq 0$  in  $\Omega_h$  and  $u$  attains its maximum in  $\Omega_h$ . Then  $u$  is constant on  $\overline{\Omega_h}$ .*

*Proof.* Let  $P_i \in \Omega_h$ . From the definition of  $M_h u$  and the assumption  $M_h u \geq 0$  we obtain that

$$u(P_i) \leq \sum_{j=1}^4 \lambda_{ij} u(P_{ij}),$$

where  $P_{ij}$  are the points of the  $h$ -neighbourhood of  $P_i$  and

$$\begin{aligned} \lambda_{i1} &= \frac{1}{4} \left( 1 + \frac{hb_1(P_i)}{2} \right), & \lambda_{i2} &= \frac{1}{4} \left( 1 + \frac{hb_2(P_i)}{2} \right) \\ \lambda_{i3} &= \frac{1}{4} \left( 1 - \frac{hb_1(P_i)}{2} \right), & \lambda_{i4} &= \frac{1}{4} \left( 1 - \frac{hb_2(P_i)}{2} \right). \end{aligned}$$

We have  $\sum_{j=1}^4 \lambda_{ij} = 1$  and, see (3.14),  $\lambda_{ij} > 0$ . Assume

$$m := \max_{\overline{\Omega_h}} u(x) = u(P_i).$$

Then  $u = m$  in all points of the  $h$ -neighbourhood of  $P_i$ . Since  $\Omega_h$  is connected, the theorem is shown.  $\square$

Set  $L_h u = M_h u + c(x)u$ . The following corollary is the discrete version of the corollary to the above Theorem 3.2.

**Corollary.** *Suppose that  $c(x) \leq 0$  on  $\Omega_h$  and  $u$  defined on  $\overline{\Omega_h}$  satisfies  $L_h u \geq 0$  on  $\Omega_h$ . Then if  $u$  achieves its nonnegative supremum  $m$  in  $\Omega_h$ ,  $u$  is constant on  $\overline{\Omega_h}$ .*

*Proof.* Set  $\Omega_{h,1} = \{x \in \Omega_h : u(x) = m\}$  and  $\Omega_{h,2} = \{x \in \Omega_h : u(x) < m\}$ . By assumption  $\Omega_{h,1}$  is not empty. The set  $\Omega_{h,1}$  is open in the sense that if  $P_i \in \Omega_{h,1}$  then  $u(P_{il}) = m$  for every point  $P_{il}$ ,  $l = 1, 2, 3, 4$ , of the associated

star to  $P_i$ . This follows from the inequality

$$u(P_i) \leq \sum_{j=1}^3 \lambda_{ij} u(P_{ij}) + \frac{h^2}{4} c(P_i) u(P_i).$$

Thus  $u = m$  on  $\overline{\Omega_h}$  since  $\Omega_h$  is connected by assumption.  $\square$

From this corollary it follows

**Theorem 3.2.2** (Comparison principle). *Let  $\Omega_h$  be a bounded and connected domain. Suppose that  $c(x) \leq 0$ . Assume  $u, v$  are defined on  $\overline{\Omega_h}$  and satisfy  $L_h u \geq L_h v$  in  $\Omega_h$  and  $u \leq v$  on  $\partial\Omega_h$ . Then  $u \leq v$  on  $\overline{\Omega}$ .*

*Proof.* Set  $w = u - v$ . Then  $L_h w \geq 0$  in  $\Omega_h$  and  $w \leq 0$  on  $\partial\Omega_h$ . From the above corollary we see that  $w$  can not achieve a nonnegative maximum in  $\Omega$ .  $\square$

Suppose that  $\Omega_h$  is bounded and connected. We consider the discrete Dirichlet problem

$$L_h u = f \text{ in } \Omega_h \tag{3.15}$$

$$u = \Phi \text{ on } \partial\Omega_h, \tag{3.16}$$

where  $f$  is defined in  $\Omega_h$  and  $\Phi$  on  $\partial\Omega_h$ . Assume  $c \leq 0$  and  $h$  is sufficiently small such that the inequality (3.14) is satisfied.

**Corollary.** *There exists a unique solution of the discrete Dirichlet problem (3.15), (3.16).*

*Proof.* The Dirichlet problem defines a linear system of  $N$  equations in  $N$  unknowns. From the comparison principle it follows that there is at most one solution. From the linear algebra it is known that uniqueness implies existence.  $\square$

**Proposition 3.2.1** (A priori estimate). *Assume  $u$  is a solution of the discrete Dirichlet problem (3.15), (3.16), where  $c(x) \leq 0$ . Then*

$$\max_{\Omega_h} |u| \leq \max_{\partial\Omega_h} |\Phi| + c \max_{\Omega_h} |f|,$$

where the constant  $c$  is independent of  $u$  and  $h$ , and  $0 < h < h_0$ ,  $h_0$  sufficiently small, see the following proof for an explicit  $h_0$ .

*Proof.* The proof is the same as the proof of the apriori estimate of Theorem 3.4. Concerning notations in the following formula see the proof of this theorem. For a sufficiently large  $\alpha$  we have

$$\begin{aligned} L_h(e^{\alpha d} - e^{\alpha x}) &= -\alpha^2 e^{\alpha x} \left( \frac{\sinh(\alpha h/2)}{\alpha h/2} \right)^2 \\ &\quad - b_1(x) \alpha e^{\alpha x} \frac{\sinh(\alpha h)}{\alpha h} + c(x) (e^{\alpha d} - e^{\alpha x}) \\ &\leq -e^{\alpha x} \left( \alpha^2 \left( \frac{\sinh(\alpha h/2)}{\alpha h/2} \right)^2 + b_1 \alpha \frac{\sinh(\alpha h)}{\alpha h} \right). \end{aligned}$$

Suppose that  $\alpha h \leq 1$ , then

$$\begin{aligned} L_h(e^{\alpha d} - e^{\alpha x}) &\leq -e^{\alpha x} (\alpha^2 - K\alpha \cosh(1)) \\ &\leq -1, \end{aligned}$$

if we take an  $\alpha$  which satisfies

$$\alpha \geq (K/2) \cosh(1) + \sqrt{((K/2) \cosh(1))^2 + 1}.$$

We recall that we assume that  $\Omega$  is bounded and contained in the strip  $0 < x_1 < d$ , where  $d$  is the diameter of  $\Omega$ .  $\square$

The following result says that the solution of the discrete Dirichlet problem is an approximation of the solution of the original problem, provided that this solution is sufficiently smooth,

**Corollary.** *Suppose that  $u \in C^3(\overline{\Omega})$  is a solution of the continuous Dirichlet problem (3.12), (3.13), where  $c(x) \leq 0$  and  $\Phi$  is defined on a boundary strip and in  $C^1$  in the closed strip. Let  $u_h$  be a solution of the discrete problem (3.15), (3.16). Then*

$$\max_{\Omega_h} |u(x) - u_h(x)| \leq ch,$$

where the constant  $c$  is independent of  $h < h_0$ ,  $h_0$  sufficiently small.

*Proof.* Here we make the additional assumption that  $\partial\Omega_h \subset \partial\Omega$ , see Figure 3.5 for an example. The proof of the general case is left as an exercise. The assumption on  $u$  implies that

$$|L_h u - Lu| \leq ch. \quad (3.17)$$

Let  $U$  be defined on  $\overline{\Omega_h}$ . Then we have from the above apriori estimate that

$$\max_{\Omega_h} |U| \leq \max_{\partial\Omega_h} |U| + c \max_{\Omega_h} |L_h U|.$$

Set  $U = u - u_h$ , then

$$\max_{\Omega_h} |u - u_h| \leq c \max_{\Omega_h} |L_h u_h - L_h u|$$

since  $u - u_h = 0$  on  $\partial\Omega_h$ . Finally, we have on  $\Omega_h$  that

$$\begin{aligned} L_h u_h - L_h u &= L_h u_h - Lu + Lu - L_h u \\ &= Lu - L_h u \end{aligned}$$

since  $L_h u_h = f$  and  $Lu = f$  on  $\Omega_h$ . Then the estimate of the corollary follows from the estimate (3.17).  $\square$

### 3.3 Exercises

1. Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain and assume  $u \in C(\overline{\Omega})$  and

$$\sup_{\Omega} (u + \epsilon e^{\gamma x_1}) = \sup_{\partial\Omega} (u + \epsilon e^{\gamma x_1})$$

for each  $\epsilon$ . Show that

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

2. Let  $[a_{ij}]$  be a real regular symmetric matrix in  $\mathbb{R}^n$ . Find a regular Matrix  $C$  in  $\mathbb{R}^n$  such that

$$\sum_{i,j=1}^n a_{ij} u_{x_i x_j} = \sum_{i=1}^n U_{y_i y_i},$$

where  $U(y) := u(C^{-1}y)$ .

*Hint:* Let  $Z_1, Z_2$  be an orthonormal system of eigenvectors to  $[a_{ij}]$  to the eigenvalues  $\lambda_1, \lambda_2$ , respectively. Set  $B = (Z_1, Z_2)$ , then

$$C = \begin{pmatrix} 1/\sqrt{\lambda_1} & 0 \\ 0 & 1/\sqrt{\lambda_2} \end{pmatrix} B^T.$$

3. Let  $\gamma$  be the interior angle of a sector with its corner at the origin in  $\mathbb{R}^2$ . Calculate the interior angle  $\omega$  of the sector transformed by the above mapping.
4. Let  $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$  be a solution of

$$\begin{aligned} \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} &= f \quad \text{in } \Omega \\ u &= \Phi \quad \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega \in \mathbb{R}^2$  and  $f, \Phi$  are given. Suppose the origin is a corner of  $\Omega$  with interior angle  $\gamma$ ,  $0 < \gamma < \pi$ . Show that  $\omega$ , see the previous exercise, is the opening angle of the surface  $\mathcal{S}$  defined by  $z = u(x_1, x_2)$ , over the origin, see Figure 3.6.

*Hint:*

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = \sum_{i,j=1}^2 a_{ij}(x) u_{x_i x_j},$$

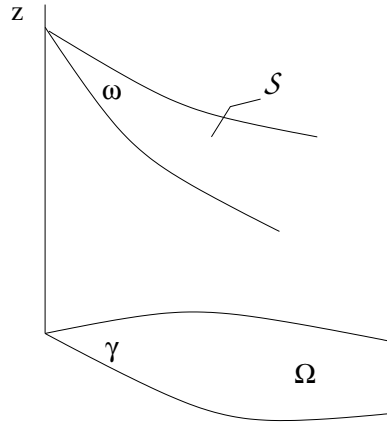


Figure 3.6: Transformed opening angle

where

$$a_{ij} = \frac{1}{(1 + |\nabla u|^2)^{3/2}} \left( \delta_{ij} - \frac{u_{x_i} u_{x_j}}{1 + |\nabla u|^2} \right).$$

5. Let  $\Omega_\alpha$  be the sector in  $\mathbb{R}^2$  defined by  $r > 0$  and  $0 < \theta < \alpha$ , where  $0 < \alpha < 2\pi$ . Set  $\Omega_{\alpha,\rho} = \Omega_\alpha \cap B_\rho(0)$ . Suppose that  $u \in C^2(\overline{\Omega_{\alpha,\rho}} \setminus \{0\})$  and  $\sup_{\Omega_{\alpha,\rho}} |u| < \infty$  is a solution of  $\Delta u = 0$  in  $\Omega_{\alpha,\rho}$ ,  $u = 0$  on  $(\partial\Omega_\alpha \cap B_\rho(0)) \setminus \{0\}$ .

Show that there is a constant  $c$  such that

$$|u| \leq c|x|^{\pi/\alpha}$$

in  $\Omega_{\alpha,\rho}$ .

*Hint:* Choose the comparison function  $W = Ar^{\pi/\alpha} \sin(\pi\theta/\alpha)$  and show that  $-AW \leq u \leq AW$  on  $\Omega_\alpha \cap \partial B_\rho(0)$  provided the positive constant  $A$  is sufficiently large.

6. Let  $m, \lambda_j, c_j$  be real numbers satisfying  $c_j \leq m, \lambda_j > 0$  and  $\sum_{j=1}^4 \lambda_j = 1$ . Show that  $m \leq \sum_{j=1}^4 \lambda_j c_j$  implies  $c_j = m$ .
7. Prove the corollary to Proposition 3.3.

*Hint:* The inclusion  $\partial\Omega_h \subset \partial\Omega$  is not assumed. The result follows since for given  $x \in \partial\Omega_h$  there exists an  $x^1 \in \partial\Omega$  such that  $|x - x^1| < h$ .

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