Spherical complexities and closed geodesics

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L-S category and critical points
The Lusternik-Schnirelmann category of a space

**Definition**

For a topological space $X$ and $A \subset X$ put

$$\text{cat}_X(A) := \inf \left\{ r \in \mathbb{N} \mid \exists \bigcup_{j=1}^{r} U_j \supset A \text{ open cover, } \right. \\
\left. \quad \text{s.t. } U_j \hookrightarrow X \text{ nullhomotopic } \forall j \in \{1, 2, \ldots, r\} \right\}.$$  

\[ \text{cat}(X) := \text{cat}_X(X) \] is the *Lusternik-Schnirelmann category* of $X$.

- $\text{cat}(X)$ is a homotopy invariant of $X$.
- $\text{cat}(X)$ is hard to compute explicitly.
Theorem (Lusternik-Schnirelmann ’34, Palais ’65)

Let $M$ be a Hilbert manifold and let $f \in C^{1,1}(M)$ be bounded from below and satisfy the Palais-Smale condition with respect to a complete Finsler metric on $M$. Then

$$\# \text{Crit } f \geq \text{cat}(M).$$
Method of proof of the Lusternik-Schnirelmann theorem

$f \in C^{1,1}(M)$ bounded from below and satisfies PS condition w.r.t. Finsler metric on $M$. Put $f^a := f^{-1}((\infty, a])$. Use properties of $\text{cat}_X$ and minimax methods to show:

- If $[a, b]$ contains no critical value of $f$, then
  \[ \text{cat}_M(f^b) = \text{cat}_M(f^a). \]

- If $c$ is a critical value of $f$, then
  \[ \text{cat}_M(f^c) \leq \text{cat}_M(f^{c-\epsilon}) + \text{cat}_M(\text{Crit } f \cap f^{-1}([c])). \]

Combining these observations yields

\[ \text{cat}_M(f^a) \leq \#(\text{Crit } f \cap f^a) \quad \forall a \in \mathbb{R} \]

and finally the theorem.
Lusternik-Schnirelmann and closed geodesics

Let $M$ be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric (e.g. $F(x, v) = \sqrt{g_x(v, v)}$ for $g$ Riemannian metric),

$\Lambda M := H^1(S^1, M) = W^{1,2}(S^1, M)(\cong C^0(S^1, M))$,

$E_F : \Lambda M \to \mathbb{R}, \quad E_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))^2 \, dt$.

Then $\Lambda M$ is a Hilbert manifold, $E_F$ is $C^{1,1}$ and satisfies the PS condition (Mercuri, '77).

$\text{Crit } E_F = \{\text{closed geodesics of } F\} \cup \{\text{constant loops}\}$.

Q: Can we use Lusternik-Schnirelmann theory to obtain lower bounds on
$\#\{\text{geometr. distinct non-constant closed geodesics of } F\}$?
Problems with the LS-approach and closed geodesics

There are problems:

• Since \( \{ \text{constant loops} \} \subset \text{Crit } E_F \), it holds for each \( a \geq 0 \) that \( \#(\text{Crit } E_F \cap \Lambda M^a) = +\infty \).

• \( \text{cat}_{\Lambda M}(\{ \text{constant loops} \}) = ? \)

• Critical points of \( E_F \) come in \( S^1 \)-orbits, but \( \text{cat}_{\Lambda M}(S^1 \cdot \gamma) \in \{1, 2\} \) for each \( \gamma \in \Lambda M \).

**Idea:** Replace \( \text{cat}_{\Lambda M} : \mathcal{P}(\Lambda M) \to \mathbb{N} \cup \{+\infty\} \) by a different function with similar properties.

There are several similar approaches to \( G \)-invariant functions, e.g. by Clapp-Puppe, Bartsch et al.
Spherical complexities
Definition of spherical complexities (M., 2019)

Let $X$ top. space, $n \in \mathbb{N}_0$, $B_{n+1}X := C^0(B^{n+1}, X)$, $S_nX := \{f \in C^0(S^n, X) | f \text{ is nullhomotopic}\}$.

**Definition**

Let $A \subset S_nX$. A sphere filling on $A$ is a continuous map $s : A \to B_{n+1}X$ with $s(\gamma)|_{S^n} = \gamma$ for all $\gamma \in A$.

$$SC_{n,X}(A) := \inf \left\{ r \in \mathbb{N} \left| \exists \bigcup_{j=1}^{r} U_j \supset A \text{ open cover, sphere fillings } s_j : U_j \to B_{n+1}X \forall j \in \{1, 2, \ldots, r\} \right. \right\} \in \mathbb{N} \cup \{\infty\}.$$

Call $SC_n(X) := SC_{n,X}(S_nX)$ the $n$-spherical complexity of $X$.

**Remark** $SC_0(X) = TC(X)$, the topological complexity of $X$.

(Farber, ’03)
Properties of spherical complexities

Let $X$ be a metrizable ANR (e.g. a locally finite CW complex).

**Proposition** Let $c_n : X \to S_nX$, $(c_n(x))(p) = x$ for all $p \in S^n$, $x \in X$. Then $SC_{n,X}(c_n(X)) = 1$.

Consider the left $O(n+1)$-actions on $S_nX$ and $B_{n+1}X$ by reparametrization, i.e.

$$(A \cdot \gamma)(p) = \gamma(A^{-1}p) \quad \forall \gamma \in S_nX, \ A \in O(n+1), \ p \in S^n.$$ 

**Proposition** Let $G \subset O(n+1)$ be a closed subgroup and $\gamma \in S_nX$ and let $G_{\gamma}$ denote its isotropy group. If $G_{\gamma}$ is trivial or $n = 1$, then $SC_{n,X}(G \cdot \gamma) = 1$.

**Proof for $G_{\gamma}$ trivial:** Take $\beta \in B_{n+1}X$ with $\beta|_{S^n} = \gamma$, put

$$s : G \cdot \gamma \to B_{n+1}X, \quad s(A \cdot \gamma) = A \cdot \beta \ \forall A \in G.$$
A Lusternik-Schnirelmann-type theorem for $\text{SC}_n$

**Theorem (M., 2019)**

Let $G \subset O(n+1)$ be a closed subgroup, $\mathcal{M} \subset S_nX$ be a $G$-invariant Hilbert manifold, $f \in C^{1,1}(\mathcal{M})$ be $G$-invariant. Let

$$\nu(f, a) := \#\{\text{non-constant } G\text{-orbits in } \text{Crit } f \cap f^a\}.$$

If

- $f$ satisfies the Palais-Smale condition w.r.t. a complete Finsler metric on $\mathcal{M}$,
- $f$ is constant on $c_n(X)$,
- $G$ acts freely on $\text{Crit } f \cap f^a$ or $n = 1$,

then

$$\text{SC}_{n,X}(f^a) \leq \nu(f, a) + 1.$$
Lower bounds and cohomology
Spherical complexities and sectional category

**Aim**  Find "computable" lower bounds on $SC_{n,X}(A)$.

**Method**  Put spherical complexities in a bigger framework and use results by A. S. Schwarz from a more general context.

**Definition (A. Schwarz, ’62)**

Let $p : E \to B$ be a fibration. The sectional category of $p$ is given by

$$
\text{secat}(p) = \inf \left\{ r \in \mathbb{N} \mid \exists \bigcup_{j=1}^{r} U_j = B \text{ open cover}, s_j : U_j \xrightarrow{co} E, p \circ s_j = \text{incl}_{U_j} \quad \forall j \in \{1, 2, \ldots, r\} \right\}.
$$

Here: $SC_n(X) = \text{secat} \left( r_n : B_{n+1}X \to S_nX, \gamma \mapsto \gamma|_{S^n} \right)$,

$$
SC_{n,X}(A) \geq \text{secat} \left( r_n|_{r_n^{-1}(A)} : r_n^{-1}(A) \to A \right) \quad \forall A \subset S_nX.
$$
Sectional categories and cup length

Theorem (Lusternik-Schnirelmann ’34)

Let $X$ be a topological space, $R$ a commutative ring. Then

$$\text{cat}(X) \geq \text{cup-length}(H^*(X; R)) + 1.$$ 

Theorem (A. Schwarz, ’62)

Let $p : E \to B$ be a fibration. Then

$$\text{secat}(p) \geq \text{cup-length} \left( \ker [p^* : H^*(B; R) \to H^*(E; R)] \right) + 1.$$ 

Improve bounds using weights $\text{wgt}_p : \tilde{H}^*(B; R) \to \mathbb{N}$. If $u_1, \ldots, u_k \in \ker p^*$ with $u_1 \cup \cdots \cup u_k \neq 0$, then

$$\text{secat}(p) \geq \sum_{i=1}^{k} \text{wgt}_p(u_i) + 1.$$ 

(Fadell-Husseini ’92, Rudyak ’99, Farber-Grant ’07).
Consequences for spherical complexities

The previous theorem, some work and the long exact cohomology sequence of \((S_nX, c_n(X))\) yield:

**Theorem**

Let \(A \subset S_nX\) and let \(\iota : (A, \emptyset) \hookrightarrow (S_nX, c_n(X))\) be the inclusion of pairs. Then

\[
SC_{n,X}(A) \geq \text{cup-length}\left(\text{im } [\iota^* : H^*(S_nX, c_n(X); R) \to H^*(A; R)]\right) + 1.
\]

*(plus improvements using weights)*
Results on closed geodesics
Main estimate for the number of closed geodesics

Let $M$ be a closed manifold, $F : TM \rightarrow [0, +\infty)$ be a Finsler metric and let $E_F : \Lambda M \cap S_1M \rightarrow \mathbb{R}$ be its energy functional.

**Theorem**

Let $\nu(F, a)$ be the number of SO(2)-orbits of non-constant contractible closed geodesics of $F$ of energy $\leq a$. Then

$$\nu(F, a) \geq SC_{1, M}(E^a_F) - 1.$$ 

If $F$ is reversible, i.e. if $F_x(v) = F_x(-v) \forall (x, v) \in TM$, the same holds for the number of O(2)-orbits of contractible closed geodesics.

**Remark** The counting does not distinguish iterates of the same prime closed geodesic.
Results on closed geodesics

Theorem (Lusternik-Fet, ’51, for Riemannian manifolds)

Every Finsler metric on a closed manifold admits a non-constant closed geodesic.

Definition  Two closed geodesics $\gamma_1, \gamma_2 : S^1 \to X$ are positively distinct if either $\gamma_1(S^1) \neq \gamma_2(S^1)$ or $\exists A \in O(2) \setminus SO(2)$ with $\gamma_1 = A \cdot \gamma_2$.

- Bangert-Long, 2007: every Finsler metric on $S^2$ has two positively distinct ones
- Rademacher, 2009: every bumpy Finsler metric on $S^n$ has two positively distinct ones (generic condition)
- etc., Long-Duan 2009 for $S^3$, Wang 2019 for pinched metrics on $S^n$, ...
New results using spherical complexities

Theorem (M., 2020)

Let $M$ be a closed oriented manifold, $F : TM \to [0, +\infty)$ be a Finsler metric of reversibility $\lambda$ and flag curvature $K$. Let $\ell_F > 0$ be the length of the shortest non-const. closed geodesic of $F$.

a) If $M = S^{2d}$, $d \geq 2$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda} \sqrt{g_1}$, then $F$ admits two pos. distinct closed geodesics of length $< 2\ell_F$. ($g_1 = \text{round metric of constant curvature 1}$)

b) If $M = S^{2d+1}$, $d \in \mathbb{N}$, $\frac{\lambda^2}{(1+\lambda)^2} < K \leq 1$ and $F \leq \frac{(k+1)(1+\lambda)}{m\lambda} \sqrt{g_1}$, then $F$ admits $\lceil \frac{2m}{k} \rceil$ pos. distinct closed geodesics of length $< (k+1)\ell_F$.

c) If $M = \mathbb{C}P^n$ or $M = \mathbb{H}P^n$, $n \geq 3$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda} \sqrt{g_1}$, then $\exists$ two pos. distinct closed geodesics of length $< 2\ell_F$. 
Method of proof of results for closed geodesics

For parts a) and c):

- Use lower bounds by cup length and cohomology weights, ring structure on $H^*(\Lambda M; \mathbb{Q})$ is well-known for these spaces (Vigué-Poirrier/Sullivan ’76).

- Use algebraic topology to establish criteria on cohomology classes to have weight two.

- Use energy bounds provided by conditions on $F$ to show that a class of weight two is supported on $E_{F}^{<4\ell_{F}^2}$.

- Conclude that $E_{F}^{<4\ell_{F}^2}$ contains two closed geodesics, distinctness follows from energy bound.
Perspectives and possible applications

• Equivariant versions, use richer ring structure in $H^*_{S^1}(LM, M; \mathbb{Q})$

• Applicable in greater generality to periodic orbits of Reeb flows on contact manifolds? (generalizing geodesic flows on $T^1M$)

• Higher-dimensional applications, i.e. for $SC_{n,M}$ if $n > 1$?

• Any other ideas? Contact me!
Thank you for your attention!

talk based on:


(slides at http://www.math.uni-leipzig.de/~mescher)

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