Spherical complexities and closed geodesics

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L-S category and critical points
Theorem (Lusternik-Schnirelmann ’34, Palais ’65)

Let $M$ be a Hilbert manifold and let $f \in C^{1,1}(M)$ be bounded from below and satisfy the Palais-Smale condition with respect to a complete Finsler metric on $M$. Then

$$\# \text{Crit } f \geq \text{cat}(M).$$

Remark

There are various generalisations, e.g. generalized Palais-Smale conditions (Clapp-Puppe ’86), extensions to fixed points of self-maps (Rudyak-Schlenk ’03).
Method of proof of the Lusternik-Schnirelmann theorem

\( f \in C^{1,1}(M) \) bounded from below and satisfies PS condition w.r.t. Finsler metric on \( M \). Put \( f^a := f^{-1}((-\infty, a]) \). Use properties of \( \text{cat}_M(\cdot) \) and minimax methods to show:

- If \( [a, b] \) contains no critical value of \( f \), then
  \[
  \text{cat}_M(f^b) = \text{cat}_M(f^a).
  \]

- If \( c \) is a critical value of \( f \), then
  \[
  \text{cat}_M(f^c) \leq \text{cat}_M(f^{c-\varepsilon}) + \text{cat}_M((\text{Crit } f) \cap f^{-1}(\{c\})).
  \]

Combining these observations yields

\[
\text{cat}_M(f^a) \leq \# ((\text{Crit } f) \cap f^a) \quad \forall a \in \mathbb{R}
\]

and finally the theorem.
Lusternik-Schnirelmann and closed geodesics

Let $M$ be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric (e.g. $F(x, v) = \sqrt{g_x(v, v)}$ for $g$ Riemannian metric),

$$E_F : \Lambda M \to \mathbb{R}, \quad E_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))^2 \, dt.$$  

Here, $\Lambda M := H^1(S^1, M) \cong C^0(S^1, M)$ is a Hilbert manifold locally modelled on $H^1(S^1, \mathbb{R}^{\dim M}) = W^{1,2}(S^1, \mathbb{R}^{\dim M})$.

Then $E_F$ is $C^{1,1}$ and satisfies the PS condition (Mercuri ’77) with

$$\text{Crit } E_F = \{ \text{closed geodesics of } F \} \cup \{ \text{constant loops} \}.$$  

Q: Can we use Lusternik-Schnirelmann theory to obtain lower bounds on

$\#\{ \text{geometrically distinct non-constant closed geodesics of } F \}$?
There are problems:

- Since \(\{\text{constant loops}\} \subset \text{Crit } E_{F}\) with \(E_{F}(\text{const. loop}) = 0\), it holds for each \(a \geq 0\) that \(#((\text{Crit } E_{F}) \cap E_{F}^{a}) = +\infty\).
- \(\text{cat}_{\Lambda M}(\{\text{constant loops}\}) = ?\)
- Critical points of \(E_{F}\) come in \(S^{1}\)-orbits, but \(\text{cat}_{\Lambda M}(S^{1} \cdot \gamma) \in \{1, 2\}\) for each \(\gamma \in \Lambda M\).

**Idea:** Replace \(\text{cat}_{\Lambda M} : \mathcal{P}(\Lambda M) \rightarrow \mathbb{N} \cup \{+\infty\}\) by a different function with similar properties.

There are several similar approaches to \(G\)-invariant functions, e.g. by Clapp-Puppe, Bartsch et al.
Spherical complexities
Definition (A. Schwarz, ’62)

Let \( p : E \to B \) be a fibration. The sectional category or Schwarz genus of \( p \) is given by

\[
\text{secat}(p) = \inf \left\{ r \in \mathbb{N} \mid \exists \bigcup_{j=1}^{r} U_j = B \text{ open cover, } s_j : U_j \to E, \ p \circ s_j = \text{incl}_{U_j} \ \forall j \right\}.
\]

Let \( X \) be a top. space, \( n \in \mathbb{N}_0 \), put \( B_{n+1}X := C^0(B^{n+1}, X) \), \( S_nX := \{ f \in C^0(S^n, X) \mid f \text{ is nullhomotopic} \} \).

Definition

The \( n \)-spherical complexity of \( X \) is given by

\[
\text{SC}_n(X) := \text{secat}(r_n : B_{n+1}X \to S_nX, \ \gamma \mapsto \gamma|_{S^n}).
\]

For \( A \subset S_nX \) define \( \text{SC}_{n,X}(A) := \text{secat}(r_n|_{r_n^{-1}(A)} : r_n^{-1}(A) \to A) \).
Properties of spherical complexities

**Remark** \( \text{SC}_0(X) = \text{TC}(X) \), the topological complexity of \( X \).

In the following, let \( X \) be a metrizable ANR (e.g. a locally finite CW complex).

**Proposition**

Let \( c_n : X \to S_nX, (c_n(x))(p) = x \) for all \( p \in S^n, x \in X \). Then \( \text{SC}_{n,X}(c_n(X)) = 1 \).

Consider the left \( O(n + 1) \)-actions on \( S_nX \) and \( B_{n+1}X \) by reparametrization, i.e.

\[
(A \cdot \gamma)(p) = \gamma(A^{-1}p) \quad \forall \gamma \in S_nX, \ A \in O(n + 1), \ p \in S^n.
\]

**Proposition** Let \( G \subset O(n + 1) \) be a closed subgroup and \( \gamma \in S_nX \) and let \( G_\gamma \) denote its isotropy group. If \( G_\gamma \) is trivial or \( n = 1 \), then \( \text{SC}_{n,X}(G \cdot \gamma) = 1 \).
Theorem (M., 2019)

Let $G \subset O(n + 1)$ be a closed subgroup, $\mathcal{M} \subset S_n X$ be a $G$-invariant Hilbert manifold, $f \in C^{1,1}(\mathcal{M})$ be $G$-invariant. Let

$$\nu(f, a) := \# \{ \text{non-constant } G\text{-orbits in } \text{Crit } f \cap f^a \}. $$

If

- $f$ satisfies the Palais-Smale condition w.r.t. a complete Finsler metric on $\mathcal{M}$,
- $f$ is constant on $c_n(X)$,
- $G$ acts freely on $(\text{Crit } f) \cap f^a$ or $n = 1$,

then

$$\text{SC}_{n, X}(f^a) \leq \nu(f, a) + 1.$$
Corollary

Let $M$ be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric and $a \in \mathbb{R}$. Let $E_F : \Lambda M \cap S_1M \to \mathbb{R}$ be the restriction of the energy functional of $F$.

Let $\nu(F, a)$ be the number of $SO(2)$-orbits of non-constant contractible closed geodesics of $F$ of energy $\leq a$. Then

$$\nu(F, a) \geq SC_{1,M}(E^a_F) - 1.$$  

If $F$ is reversible, e.g. induced by a Riemannian metric, the same holds for the number of $O(2)$-orbits of contractible closed geodesics.

Remark  The counting does not distinguish iterates of the same prime closed geodesic.
Lower bounds for spherical complexities
Aim  Find "computable" lower bounds on $SC_{n, X}(A)$ using cohomology.

For $X$ top. space, $R$ commutative ring, $I \subset H^*(X; R)$ an ideal, let
$$
\text{cl}(I) := \sup\{ r \in \mathbb{N} | \exists u_1, \ldots, u_r \in I \cap \tilde{H}^*(X; R) \text{ s.t. } u_1 \cup \cdots \cup u_r \neq 0 \}.
$$

Theorem (A. Schwarz, ’62)

Let $p : E \to B$ be a fibration. Then

$$
\text{secat}(p) \geq \text{cl} \left( \text{ker} \left[ p^* : H^*(B; R) \to H^*(E; R) \right] \right) + 1.
$$
Consequences for spherical complexities

The previous theorem, some work and the long exact cohomology sequence of \((S_nX, c_n(X))\) yield:

**Theorem**

Let \(A \subset S_nX\) and let \(\iota : (A, \emptyset) \hookrightarrow (S_nX, c_n(X))\) be the inclusion of pairs. Then

\[
SC_{n,X}(A) \geq \text{cl}\left(\text{im} [\iota^*: H^*(S_nX, c_n(X); R) \to H^*(A; R)]\right) + 1.
\]

**Problem**  The cup product on \(H^*(S_nX, c_n(X); R)\) might be either hard to compute or not that interesting. (E.g. the cup product on \(H^*(LS^2, c_1(S^2); \mathbb{Q})\) vanishes.)

**Idea**  Improve cup length bounds by associating \(\mathbb{N}\)-valued weights to cohomology classes.
Sectional category and fiberwise joins

Given fibrations $p : E \to B$, $p' : E' \to B$, let

$$p \ast p' : E \ast_f E' \to B$$

denote the *fiberwise join of $p$ and $p'*. The fiber over each $b \in B$ is $(E \ast_f E')_b = E_b \ast E'_b$. (* = topological join)

Let $p : E \to B$ be a fibration. Define fibrations $p_r : E_r \to B$, $r \in \mathbb{N}$, recursively by

$$p_1 = p, \ E_1 = E, \quad p_r = p \ast p_{r-1}, \ E_r = E \ast_f E_{r-1}.$$

**Theorem (A. Schwarz, ’62)**

$$\text{secat}(p) = \inf \{ r \in \mathbb{N} | \exists s : B \overset{c_0}{\to} E_r \text{ with } p_r \circ s = \text{id}_B \}.$$
Sectional category weights

Let $p : E \to B$ be a fibration, $R$ be a commutative ring.

**Definition (Farber-Grant 2007; Fadell-Husseini ’92, Rudyak ’99)**

Let $u \in \tilde{H}^\ast(B; R)$, $u \neq 0$. The *weight of $u$ with respect to $p$* is given by $\text{wgt}_p(u) := \sup\{r \in \mathbb{N}_0 \mid p^\ast r u = 0\}$.

**Properties:** Let $u, v \in \tilde{H}^\ast(B; R)$ with $u \neq 0, v \neq 0$.

- If $\text{wgt}_p(u) \geq k$, then $\text{secat}(p) \geq k + 1$.
- $\text{wgt}_p(u \cup v) \geq \text{wgt}_p(u) + \text{wgt}_p(v)$.

Thus, if $k := \text{cl}(\ker p^\ast)$ and $u_1, \ldots, u_k \in \ker p^\ast$ with $u_1 \cup \cdots \cup u_k \neq 0$, then

$$\text{secat}(p) \geq \sum_{j=1}^{k} \text{wgt}_p(u_j) + 1 \geq \text{cl}(\ker p^\ast) + 1.$$
Construction of classes of weight $\geq 1$ for $r_n(\gamma) = \gamma|_{S^n}$

Want to find classes in $\ker [r_n^* : H^*(S_nX; R) \to H^*(B_{n+1}X; R)]$.

**Lemma**

*Let $X$ be a top. space, $R$ be a commutative ring and

$$ev_n : S_nX \times S^n \to X, \quad (\alpha, p) \mapsto \alpha(p).$$

For $k \geq n$ let

$$Z_n : H^k(X; R) \to H^{k-n}(S_nX; R), \quad Z_n(\sigma) = ev_n^*\sigma/[S^n],$$

where $\cdot/[\cdot]$ denotes the slant product. Then $Z_n(\sigma) \in \ker r_n^*$.***

**Remark**  If $n = 1$ and $X$ is simply connected, then

$$Z_1 : \tilde{H}^*(X; R) \to H^{*-1}(LX; R)$$

is injective. (Jones '87)
Construction of classes of weight $\geq 2$

(Generalization of methods from Grant-M. 2018)

If $p : E \to B$ is a fibration, then $p_2 : E \ast f E \to B$ is constructed as a homotopy pushout of a pullback (double mapping cylinder):

$$
\begin{array}{ccc}
Q & \overset{f_2}{\longrightarrow} & E \\
\downarrow f_1 & & \downarrow p \\
E & \overset{p}{\longrightarrow} & B \\
\end{array}
\quad \text{pullback}

\begin{array}{ccc}
Q & \overset{f_2}{\longrightarrow} & E \\
\downarrow f_1 & & \downarrow p \\
E & \longrightarrow & E \ast f E \\
\end{array}
\quad \text{homotopy pushout}

\begin{array}{ccc}
P & \longrightarrow & E \\
\downarrow p & & \downarrow p \\
E & \longrightarrow & B \\
\end{array}
\quad p_2
$$
As a homotopy pushout of a pullback, it has a Mayer-Vietoris sequence:

\[ \ldots \rightarrow H^{k-1}(Q; R) \xrightarrow{\delta} H^k(E \ast f E; R) \xrightarrow{\oplus_{i=1}^2} H^k(E; R) \rightarrow \ldots \]

Want to find \( u \in H^k(B; R) \) with \( p_2^*u = 0 \). If \( u \in \ker p^* \), then \( p_2^*u \in \text{im} \delta \). Try to find \( \alpha_u \in H^{k-1}(Q; R) \) with

\[ \delta(\alpha_u) = p_2^*u, \]

find conditions that imply \( \alpha_u = 0 \).
Back to our setting: Here $p = r_n : B_{n+1}X \to S_nX$, $\gamma \mapsto \gamma|_{S^n}$, and the pullback is

$$Q = \{(\gamma_1, \gamma_2) \in (B_{n+1}X)^2 \mid \gamma_1|_{S^n} = \gamma_2|_{S^n}\} \cong C^0(S^{n+1}, X),$$

hence for $E_2 := B_{n+1}X \wedge B_{n+1}X$ the Mayer-Vietoris sequence has the form

$$\cdots \to H^{k-1}(C^0(S^{n+1}, X); R) \xrightarrow{\delta} H^k(E_2; R) \to \bigoplus_{i=1}^2 H^k(B_{n+1}X; R) \to \cdots$$

**Q:** If $u \in H^k(X; R)$, $Z_n(u) \neq 0 \in H^{k-n}(S_nX; R)$, what is

$$\alpha Z_n(u) \in H^{k-n-1}(C^0(S^{n+1}, X); R) ?$$
Construction of classes of weight $\geq 2$, cont.

**Lemma**

Let $u \in H^k(X; \mathbb{Q})$, $k \geq n + 1$. Then

$$\alpha Z_n(u) = e_{n+1}^* u / [S^{n+1}],$$

where $e_{n+1} : C^0(S^{n+1}, X) \times S^{n+1} \to X$, $e_{n+1}^*(\gamma, p) = \gamma(p)$.

Use lemma and Mayer-Vietoris sequence to show:

**Theorem**

Let $u \in H^k(X; \mathbb{Q})$ with $Z_n(u) \neq 0$. If $f^* u = 0$ for all $f : S^{n+1} \times P \to X$, where $P$ is any closed oriented manifold with $\dim P = k - n - 1$, then

$$\text{wgt}(Z_n(u)) \geq 2.$$
Consequences for topological complexity (the case $n = 0$)

**Theorem (Grant-M. ’18, M. ’19)**

Let $M$ be a closed or. manifold with $\dim M \geq 3$. If there exists $u \in H^2(M; \mathbb{Q})$ with $f^*u = 0$ for all $f \in C^0(T^2, M)$, then $TC(M) \geq 6$.

**Theorem (M. ’19)**

Let $M$ be an even-dim. closed or. manifold. If $TC(M) \leq 4$, then $M$ is dominated by a manifold $P \times S^1$, i.e. there exists a degree-1 map $P \times S^1 \to M$. Here, $P$ is a closed or. manifold with $\dim P = \dim M - 1$.

**Proof.**

Assume $M$ is not dominated by ... Let $n := \dim M$, $u \neq 0 \in H^n(M; \mathbb{Q})$, $\bar{u} := 1 \times u - u \times 1 \in H^n(M \times M; \mathbb{Q})$. Then $wgt(\bar{u}) \geq 2$ by the assumptions. Since $n$ is even, $\bar{u}^2 = -2u \times u \neq 0$, hence $wgt(\bar{u}^2) \geq 4$, so $TC(M) \geq 5$.  

$\square$
Results on closed geodesics
Theorem (Lusternik-Fet, ’51, for Riemannian manifolds)

Every Finsler metric on a closed manifold admits a non-constant closed geodesic.

Definition  Two closed geodesics $\gamma_1, \gamma_2 : S^1 \to X$ are positively distinct if either $\gamma_1(S^1) \neq \gamma_2(S^1)$ or $\exists A \in O(2) \setminus SO(2)$ with $\gamma_1 = A \cdot \gamma_2$.

- Bangert-Long, 2007: every Finsler metric on $S^2$ has two positively distinct ones
- Rademacher, 2009: every bumpy Finsler metric on $S^n$ has two positively distinct ones
- etc., Long-Duan 2009 for $S^3$, Wang 2019 for pinched metrics on $S^n$, ...
New results using spherical complexities

**Theorem (M., 2020)**

Let $M$ be a closed oriented manifold, $F : TM \to [0, +\infty)$ be a Finsler metric of reversibility $\lambda$ and flag curvature $K$. Let $\ell_F > 0$ be the length of the shortest non-const. closed geodesic of $F$.

a) If $M = S^{2d}$, $d \geq 2$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda} \sqrt{g_1}$, then $F$ admits two pos. distinct closed geodesics of length $< 2\ell_F$. ($g_1 = \text{round metric of constant curvature 1}$)

b) If $M = \mathbb{C}P^n$ or $M = \mathbb{H}P^n$, $n \geq 3$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda} \sqrt{g_1}$, then $\exists$ two pos. distinct closed geodesics of length $< 2\ell_F$.

c) If $M = S^{2d+1}$, $d \in \mathbb{N}$, $\frac{\lambda^2}{(1+\lambda)^2} < K \leq 1$ and $F \leq \frac{(k+1)(1+\lambda)}{m\lambda} \sqrt{g_1}$, then $F$ admits $\lceil \frac{2m}{k} \rceil$ pos. distinct closed geodesics of length $< (k + 1)\ell_F$. 
Let $\iota_a : E_F^a \hookrightarrow \Lambda M$ be the inclusion. For parts a) and b):

- If $\gamma$ is a closed geodesic of length $\ell_F$, then its iterates satisfy $E_F(\gamma^k) \geq E_F(\gamma^2) = 4\ell_F^2 \forall k \geq 2$. Thus, if $a < 4\ell_F^2$ and $\nu(F, a) \geq 2$, then $E_F^a$ contains two distinct closed geodesics.

- If $u \in H^*(\Lambda M; R)$ satisfies $\iota_a^* u \neq 0$, then $\nu(F, a) \geq \text{wgt}(u)$. Thus, it suffices to find such $u$ with $\text{wgt}(u) \geq 2$ and $\iota_a^* u \neq 0$, where $a < 4\ell_F^2$.

- For classes of fixed degree, positive curvature bounds provide such energy bounds.
Perspectives and possible applications

- Equivariant versions, use richer ring structure in $H_{S^1}^*(LM, M; \mathbb{Q})$
- Applicable in greater generality to periodic Reeb orbits on contact manifolds? (generalizing closed geodesics on $T^1M$)
- Higher-dimensional applications, i.e. for $SC_{n,M}$ if $n > 1$?
- Any ideas? Contact me!
Thank you for your attention!

talk based on:


(slides at http://www.math.uni-leipzig.de/~mescher)

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