Exam solutions

Each exercise is graded between 0 and 5 points.

1. Let $(X, \mathcal{F})$ be a measurable space and f, g be $\mathcal{F}$-measurable functions. Show that the set $\{x \in X : e^{f(x)} \geq g(x)\}$ belongs to $\mathcal{F}$.

   Solution. The function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by
   $$F(x, y) = e^x - y, \quad (x, y) \in \mathbb{R}^2,$$
   is Borel measurable because it is continuous. Hence, the composition $h(x) := F(f(x), g(x)) = e^{f(x)} - g(x)$, $x \in X$, is an $\mathcal{F}$-measurable function. Therefore, the preimage
   $$h^{-1}([0, \infty)) = \{x \in X : \ h(x) \in [0, \infty)\} = \left\{x \in X : \ e^{f(x)} - g(x) \geq 0\right\}$$
   belongs to $\mathcal{F}$.

2. Construct the σ-algebra on $\mathbb{R}$ generated by the class $\{[0, 2), \ \{3\}\}$.

   Solution. The σ-algebra generated by the class $H := \{[0, 2), \{3\}\}$ is defined as follows:
   $$\sigma(H) = \big\{\emptyset, \mathbb{R}, [0, 2), \{3\}, [0, 2) \cup \{3\}, (-\infty, 0) \cup [2, +\infty), \\ (-\infty, 3) \cup (3, +\infty), (-\infty, 0) \cup [2, 3) \cup (3, +\infty)\big\}.$$

3. Show that the set
   $$G = \{x = (\xi_k)_{k\geq1} \in c_0 : \ \xi_k < 1, \ \forall k \geq 1\}$$
   is open in c_0. Is the set G open in c? Justify your answer.

   Solution. Let $x = (\xi_k)_{k\geq1}$ be an arbitrary element of G. We will find $r > 0$ such that the open ball
   $$B_r(x) = \left\{y = (\eta_k)_{k\geq1} \in c_0 : \ \|x - y\| = \max_{k\geq1} |\xi_k - \eta_k| < r\right\}$$
   is contained in G. This will mean that G is open in c_0. So, we first remark that the sequence $(\xi_k)_{k\geq1}$ converges to 0, by the definition of the space c_0. Consequently, there exists $N \in \mathbb{N}$ such that for all $k \geq N$ it follows that $|\xi_k| < \frac{1}{2}$. In particular, $\xi_k < \frac{1}{2}$ for every $k \geq N$. We define $r := \min\left\{1 - \xi_1, \ldots, 1 - \xi_{N-1}, \frac{1}{2}\right\} > 0$. Then for all $y \in B_r(x)$ we have
   $$\eta_k = \eta_k - \xi_k + \xi_k \leq |\eta_k - \xi_k| + \xi_k \leq \|y - x\| + \xi_k < r + \xi_k.$$
   If $k < N$, then $r + \xi_k \leq 1 - \xi_k + \xi_k = 1$. For $k \geq N$, $r + \xi_k \leq \frac{1}{2} + \xi_k < \frac{1}{2} + \frac{1}{2} = 1$. So, we have obtained that $\eta_k < 1$, $k \geq 1$. This yields that $y \in G$ and, therefore, $B_r(x) \subset G$.

   Hovewer, the set G is not open in c. Indeed, for every $x = (\xi_k)_{k\geq1} \in G$ and $r > 0$ one has that the vector $y = (\xi_k + \frac{r}{2})_{k\geq1}$ belongs to $B_r(x) = \{y \in c : \ \|y - x\| < r\}$, because the sequence $y = (\xi_k + \frac{r}{2})_{k\geq1}$ converges to $\frac{r}{2}$ and $\|y - x\| = \frac{r}{2} < r$. But y is not an element of G because $y \notin c_0$ (the sequence y does not converges to 0).

4. Show that the operator T defined on the space $C[0,1]$ by

$$(Tx)(t) = x(1) - x(t), \quad x \in C[0,1],$$

is linear and find its norm.

Solution. Let $x, y \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}(T(\alpha x + \beta y))(t) &= (\alpha x + \beta y)(1) - (\alpha x + \beta y)(t) = \alpha x(1) + \beta y(1) - \alpha x(t) - \beta y(t) \\ &= \alpha(x(1) - x(t)) + \beta(y(1) - y(t)) = \alpha(Tx)(t) + \beta(Ty)(t).\end{aligned}$$

So, the operator T is linear.

Let us compute the norm of T. For any $x \in C[0,1]$ we estimate

$$\begin{aligned}\|Tx\| &= \max_{t\in[0,1]} |x(1) - x(t)| \le \max_{t\in[0,1]} (|x(1)| + |x(t)|) \\ &= |x(1)| + \max_{t\in[0,1]} |x(t)| \le \|x\| + \|x\| = 2\|x\|.\end{aligned}$$

Thus, $\|T\| \le 2$. We next take $x(t) = -1 + 2t$, $t \in [0,1]$. Since the graph of the function x is a segment that connects the points $(0,-1)$ and $(1,1)$, we trivially have

$$\|x\| = \max_{t\in[0,1]} |-1+2t| = 1.$$

Since $(Tx)(t) = x(1) - x(t) = 1 - (-1+2t) = 2 - 2t$, $t \in [0,1]$, the graph of Tx is a segment connectiong the points $(0,2)$ and $(1,0)$. Therefore, we can conclude that

$$\|Tx\| = \max_{t\in[0,1]} |2 - 2t| = 2.$$

Hence, $\|T\| = 2$.

5. Check if the linear functional

$$f(x) = \sum_{k=1}^{\infty} \frac{\xi_k}{\sqrt{k}}, \quad x = (\xi_k)_{k\ge1} \in l^{\frac{4}{3}},$$

is continuous on $l^{\frac{4}{3}}$.

Solution. To check that f is continuous, it is enough to show that f is a bounded linear functional. We take $p = \frac{4}{3}$. Then for $q = 4$, $\frac{1}{p} + \frac{1}{q} = 1$. Using the Hölder inequality, we obtain

$$|f(x)| = \left|\sum_{k=1}^{\infty} \frac{\xi_k}{\sqrt{k}}\right| \le \sum_{k=1}^{\infty} \frac{|\xi_k|}{\sqrt{k}} \le \left(\sum_{k=1}^{\infty} |\xi_k|^{\frac{4}{3}}\right)^{\frac{3}{4}} \left(\sum_{k=1}^{\infty} \frac{1}{(\sqrt{k})^4}\right)^{\frac{1}{4}} = \|x\|_{l^{\frac{4}{3}}} \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{4}}.$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, we have that f is bounded and, hence, continuous.

Another way to solve this exercise is to conclude that the element $y = \left(\frac{1}{\sqrt{k}}\right)_{k\ge1}$ belongs to the dual space $l^4 = (l^{\frac{4}{3}})'$ because $\sum_{k=1}^{\infty} \frac{1}{(\sqrt{k})^4} < +\infty$. This, immediately gives that f is continuous.

6. Let T be a linear operator on l^2 defined by

$$Tx = (\xi_1, 2\xi_2, \xi_3, 2\xi_4, \xi_5, 2\xi_6, \xi_7, \dots), \quad x = (\xi_k)_{k\geq 1} \in l^2,$$

(even coordinates are multiplied by 2). Find and classify the spectrum of T.

Solution. We first find all $\lambda \in \mathbb{C}$ for which the operator $T - \lambda I$ is not injective. Consider the equation

$$Tx - \lambda x = (\xi_1 - \lambda\xi_1, 2\xi_2 - \lambda\xi_2, \xi_3 - \lambda\xi_3, 2\xi_4 - \lambda\xi_4, \dots) = 0. \tag{1}$$

Hence, for $\lambda = 1$ the vector $x = (1, 0, 0, 0, \dots) \in l^2$ is a nontrivial solurion to (1), and for $\lambda = 2$ the vector $x = (0, 1, 0, 0, \dots) \in l^2$ is a nontrivual solution to (1). Consequently, the points $\lambda = 1$ and $\lambda = 2$ belong to the point spectrum $\sigma_p(T)$. If $\lambda \notin \{1, 2\}$, then obviously (1) has only zero solution. Hence, we can conclude that there exists the inverse operator $R_\lambda = (T - \lambda I)^{-1}$ with $\mathcal{D}(R_\lambda) = \operatorname{Im}(T - \lambda I)$ for all $\lambda \notin \{1, 2\}$. Moreover, for every $y = (\eta_k)_{k\geq 1} \in \mathcal{D}(R_\lambda)$

$$R_\lambda y = \left(\frac{\eta_1}{1-\lambda}, \frac{\eta_2}{2-\lambda}, \frac{\eta_3}{1-\lambda}, \frac{\eta_4}{2-\lambda}, \dots\right).$$

We remark that $\mathcal{D}(R_\lambda) = l^2$. Indeed, for every $y = (\eta_k)_{k\geq 1} \in l^2$, we take

$$x = \left(\frac{\eta_1}{1-\lambda}, \frac{\eta_2}{2-\lambda}, \frac{\eta_3}{1-\lambda}, \frac{\eta_4}{2-\lambda}, \dots\right).$$

Then $x \in l^2$, because

$$\begin{aligned} \|x\|^2 &= \frac{|\eta_1|^2}{|1-\lambda|^2} + \frac{|\eta_2|^2}{|2-\lambda|^2} + \frac{|\eta_3|^2}{|1-\lambda|^2} + \frac{|\eta_4|^4}{|2-\lambda|^2} + \dots \\ &\leq \max\left\{\frac{1}{|1-\lambda|^2}, \frac{1}{|2-\lambda|^2}\right\} \|y\|^2 < \infty, \end{aligned} \tag{2}$$

and $Tx - \lambda x = y$. So, $y \in \operatorname{Im}(T - \lambda I) = \mathcal{D}(R_\lambda)$. Hence, $\mathcal{D}(R_\lambda) = l^2$ for all $\lambda \in \mathbb{C} \setminus \{1, 2\}$. Similarly as in (2), we obtain

$$\|R_\lambda y\|^2 \leq \max\left\{\frac{1}{|1-\lambda|^2}, \frac{1}{|2-\lambda|^2}\right\} \|y\|^2.$$

So, R_λ is a bounded linear operator on l^2. Thus, $\mathbb{C} \setminus \{1, 2\} \subset \rho(T)$. Consequently, $\rho(T) = \mathbb{C} \setminus \{1, 2\}$, $\sigma_p(T) = \{1, 2\}$, $\sigma_c(T) = \emptyset$ and $\sigma_r(T) = \emptyset$.

7. Let $T : \mathcal{D}(T) \to L^2[0, +\infty)$ be defined by

$$(Tx)(t) = \sqrt{t}x(t), \quad t \geq 0, \quad x \in \mathcal{D}(T),$$

where $\mathcal{D}(T) = \left\{x \in L^2[0, +\infty) : \int_0^\infty t|x(t)|^2 dt < +\infty\right\}$. Check whether T is closed.

Solution. We will show that the operator T is closed. We take any sequence $\{x_n\}_{n\geq 1}$ from $\mathcal{D}(T)$ such that $x_n \to x$ and $Tx_n \to y$ in $L^2[0, \infty)$.

We know that the convergence

$$\|x_n - x\|^2 = \int_0^\infty |x_n(t) - x(t)|^2 dt \to 0, \quad n \to \infty,$$

implies that $|x_n - x|^2 \xrightarrow{\lambda} 0$, $n \to \infty$ (see Problem sheet 5, Ex. 6), where λ denotes the Lebesgue measure on $[0, \infty)$. Hence $x_n \xrightarrow{\lambda} x$. Thus, there exists a subsequence $\{n_k\}_{k \geq 1}$ such that $x_{n_k} \to x$ a.e. on $[0, \infty)$ as $k \to \infty$, that is, there exists a Borel set $\Phi \subset [0, \infty)$ with $\lambda(\Phi^c) = 0$ and $x_{n_k}(t) \to x(t)$, $k \to \infty$, for all $t \in \Phi$. Consequently, $(Tx_{n_k})(t) = \sqrt{t} x_{n_k}(t) \to \sqrt{t} x(t) =: \tilde{x}(t)$, $k \to \infty$, for all $t \in \Phi$. Hence, $Tx_{n_k} \to \tilde{x}$ a.e. on $[0, \infty)$ as $k \to \infty$.

Similarly, one can conclude that $Tx_n \xrightarrow{\lambda} y$, $n \to \infty$. Thus, there exists a susequence $\{n_{k_j}\}_{j \geq 1}$ of $\{n_k\}_{k \geq 1}$ such that $Tx_{n_{k_j}} \to y$ a.e. on $[0, \infty)$ as $j \to \infty$. This implies that

$$Tx_{n_{k_j}} \to \tilde{x} \text{ a.e. and } Tx_{n_{k_j}} \to y \text{ a.e.}$$

as $j \to \infty$. By the uniqueness of the limit, $\tilde{x} = y$ a.e. So, $\tilde{x}(t) = \sqrt{t} x(t) = y(t)$ for almost all $t \in [0, \infty)$. Therefore, $x \in \mathcal{D}(T)$, because $\int_0^\infty t |x(t)|^2 dt = \int_0^\infty |y(t)|^2 dt < +\infty$ and $y = Tx$.