Problem sheet 13

(1) Let $T: \ell^2 \to \ell^2$ be defined by

$$Tx = (\alpha_k \xi_k)_{k\geq 1}, \quad x = (\xi_k)_{k \geq 1},$$

where $\{\alpha_n\}_{n\geq 1}$ is a bounded sequence in $\mathbb{C}$. Prove that T is compact if and only if $\alpha_n \to 0$.

$\Rightarrow$) Let T be compact. We assume that $\alpha_n \not\to 0$. Then $\exists\, \varepsilon > 0$ and a subsequence $\{\alpha_{n_k}\}_{k\geq 1}$ such that

$$|\alpha_{n_k}| \geq \varepsilon \quad \forall k \geq 1.$$

We take $x_k = e_{n_k} = (0, \dots, 0, \underset{n_k\text{-th position}}{1}, 0, \dots)$.

Then the sequence $\{x_k\}_{k\geq 1}$ is bounded because $\|x_k\| = 1 \quad \forall k \geq 1$. But for $k \neq \ell$

$$\|Tx_k - Tx_\ell\|^2 = |\alpha_{n_k}|^2 + |\alpha_{n_\ell}|^2 \geq 2\varepsilon^2.$$

This implies that $\{Tx_k\}_{k\geq 1}$ has no convergent subsequence.

$\Leftarrow$) Let $\alpha_n \to 0$. We consider new operators T_n on ℓ^2 defined as follows

$$T_n x = (\alpha_1 \xi_1, \alpha_2 \xi_2, \dots, \alpha_n \xi_n, 0, 0, \dots).$$

Since the $\operatorname{Im} T_n$ has a finite dimension, then T_n is compact (see Th. 23.6).

Next, we show that $\|T - T_n\| \to 0, \ n \to \infty$.

$$\|Tx - T_n x\|^2 = \|(0, \dots, 0, \alpha_{n+1}\xi_{n+1}, \dots)\|^2 =$$

$$= \sum_{k=n+1}^{\infty} |\alpha_k|^2 |\xi_k|^2 \leq \max_{k\geq n+1} |\alpha_k|^2 \sum_{k=n+1}^{\infty} |\xi_k|^2 \leq$$

$$\leq \max_{k \geq n+1} |\alpha_k|^2 \cdot \|x\|^2.$$

Hence

$$\|T - T_n\| \leq \max_{k\geq n+1} |\alpha_k| \to 0, \quad n \to \infty.$$

By Th. 23.8 T is compact. ◄

② Let $T: D(T) \to \ell^2$ be defined by

$$Tx = (k\xi_k)_{k\geq 1}, \quad x = (\xi_k)_{k\geq 1},$$

where $D(T) \subset \ell^2$ consists of all $x = (\xi_k)_{k\geq 1}$ with only finitely many nonzero terms ξ_k.

(a) Show that is unbounded and not closed. Find the adjoint operator T^* of T.

▶ • Let e_n be a canonical basis in ℓ^2. Then $\|e_n\| = 1$ and $\|Te_n\| = n$. Hence

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = +\infty.$$

So, T is unbounded.

• Take $x_n = (1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, 0, 0, \ldots)$. Then $x_n \to x = (1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, \frac{1}{(n+1)^2}, \ldots)$ because

$$\|x - x_n\|^2 = \sum_{k=n+1}^{\infty} \frac{1}{k^4} \to 0, \quad n \to \infty.$$

Moreover

$$Tx_n = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{n}, 0, \ldots) \to y = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{n}, \tfrac{1}{n+1}, \ldots) \in \ell^2$$

because

$$\|y - Tx_n\|^2 = \sum_{k=n+1}^{\infty} \frac{1}{k^2} \to 0, \quad n \to \infty.$$

But $x \in D(T)$. Hence T is not closed.

• We compute the adjoint operator T^*. We take $y = (\eta_k)_{k\geq 1} \in \ell^2$ and consider for $x = e_n \in D(T)$

$$\langle Tx, y\rangle = \sum_{k=1}^{\infty} k\xi_k \overline{\eta_k} = n\overline{\eta_n} = \langle e_n, y^*\rangle = \langle x, y^*\rangle.$$

Hence

$$\langle e_n, y^*\rangle = n\overline{\eta_n} \quad \forall n. \text{ Consequently, we get}$$

$$y^* = (k\eta_k)_{k\geq 1}.$$

But y^* must belong to ℓ^2. So, we get

$$\sum_{k=1}^{\infty} k^2 |\eta_k|^2 < +\infty.$$

Hence,

$$D(T^*) \subseteq \{y : \sum_{k=1}^{\infty} k^2 |\eta_k|^2 < +\infty\}.$$

We also remark that for every y satisfying $\sum_{k=1}^{\infty} k^2 |\eta_k|^2 < +\infty$ we have

$$\langle Tx, y\rangle = \langle x, y^*\rangle, \text{ where } y^* = (k\eta_k)_{k\geq 1} \in \ell^2.$$

Hence,

$$D(T^*) = \{y : \sum_{k=1}^{\infty} k^2 |\eta_k|^2 < +\infty\} \text{ and } T^* y = (k\eta_k)_{k\geq 1}.$$

In particular, we get $T^* \neq T$ because $D(T) \neq D(T^*)$.

▲

(b) Show that T is closable and find its closure $\overline{T}$. Find the adjoint operator $\overline{T}^*$ of $\overline{T}$.

▶ We first show that T is symmertic.

Let $x, y \in D(T)$. Then

$$\langle Tx, y\rangle = \sum_{k=1}^{\infty} k \xi_k \overline{\eta_k} = \sum_{k=1}^{\infty} \xi_k \overline{k\eta_k} = \langle x, Ty\rangle.$$

Hence, T is closable. We next find the closure of. We take

$$\overline{T}x := (k\xi_k)_{k\geq 1}$$

$$D(\overline{T}) := \{x \in \ell^2 : \sum_{k=1}^{\infty} k^2 |\xi_k|^2 < +\infty\}.$$

Since $\overline{T} = T^*$, $\overline{T}$ is closed.

We next show that there exists no closed operator $\tilde{T} \neq \overline{T}$ such that

$$T \subset \tilde{T} \subset \overline{T}.$$

Let us assume that such an operator $\tilde{T}$ exists.
Then

$$D(\tilde{T}) \not\supseteq D(\overline{T}).$$

Consequently, $\exists x \in D(\overline{T}) \setminus D(\tilde{T})$.
Then

$$\sum_{k=1}^{\infty} k |\xi_k|^2 < +\infty.$$

We take

$$x_n := (\xi_1, \dots, \xi_n, 0, 0, \dots) \in D(T).$$

Then

$$x_n \to x \quad \text{in } \ell^2 \text{ because}$$

$$\|x - x_n\|^2 = \sum_{k=n+1}^{\infty} |\xi_k|^2 \longrightarrow 0, \ n \to \infty.$$

Moreover

$$T x_n = \tilde{T} x_n = \overline{T} x_n \longrightarrow y = \overline{T} x \in \ell^2,$$

because

$$\|\overline{T} x - T x_n\|^2 = \sum_{k=n+1}^{\infty} k^2 |\xi_k|^2 \longrightarrow 0, \ n \to \infty.$$

Then we have that

$$\tilde{T} x_n \to y \in \ell^2.$$

Since $\tilde{T}$ is closed, we get $x \in D(\tilde{T})$ and $\tilde{T} x = y$.
We get the contradiction to the assumption that $x \notin D(\tilde{T})$.

Let us compute $\overline{T}^*$. Since $T \subset \overline{T}$ we have

$$\overline{T}^* \subset T^* = \overline{T}.$$

From the another hand side, by Th. 24.2 b)

$$T \subset T^{**} = (\overline{T})^*.$$

So, we obtained

$$T \subset (\overline{T})^* \subset \overline{T}.$$

According to Th. 24.10 $\overline{T}^*$ is closed.
But we have proved before that there exists no closed operator $\tilde{T}$ satisfying

$$T \subset \tilde{T} \subset \overline{T}$$

except $\tilde{T} = \overline{T}$.

Hence, $(\overline{T})^* = \overline{T}$.

Thus, $\overline{T}$ is self-adjoint.