Problem sheet 12

① Show that a bounded self-adjoint linear operator is positive if and only if its spectrum consists of non-negative real values only.

▶ $\Rightarrow$) Let T is positive. Then by the definition

$$\langle Tx, x\rangle \geq 0 \quad \forall x \in H.$$

$$\Rightarrow \quad m = \inf_{\|x\|=1} \langle Tx, x\rangle \geq 0.$$

We know by Th. 21.4 that $\sigma(T)$ is real and $\sigma(T) \subset [m, M]$, $M \geq m \geq 0$ $\Rightarrow$

$\Rightarrow$ spectrum $\sigma(T)$ consists of non-negative real values.

$\Leftarrow$) Let $m \geq 0$, that is, $m = \inf\limits_{\|x\|=1} \langle Tx, x\rangle \geq 0$.

By the Th. 21.4 $m \in \sigma(T)$

$$0 \leq \inf_{\|x\|=1} \langle Tx, x\rangle \in \sigma(T).$$

Take $\forall y \in H$, $y \neq 0$ $(\|y\| \neq 0)$.

Set $x = \frac{y}{\|y\|}$. Then $\|x\| = \frac{\|y\|}{\|y\|} = 1$ and

$$0 \leq \inf_{\|x\|=1} \langle Tx, x\rangle = \inf_{y \in H} \left\langle T\frac{y}{\|y\|}, \frac{y}{\|y\|}\right\rangle \leq \left\langle T\frac{y}{\|y\|}, \frac{y}{\|y\|}\right\rangle =$$

$$= \frac{1}{\|y\|^2}\langle Ty, y\rangle \Rightarrow \langle Ty, y\rangle \geq 0 \quad \forall y \in H.$$

If $y = 0$ then $\|y\| = 0$ and $\langle Ty, y\rangle = 0 \geq 0$.

Hence, $\langle Ty, y\rangle \geq 0$ $\forall y \in H$ $\Rightarrow$ T is positive.

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(2) Let $Y \neq \{0\}$ be a closed subspace of a Hilbert space H which does not coincide with H, and P be the orthogonal projection of H onto Y. Show that for every $\lambda \notin \{0;1\}$,

$$(P-\lambda I)^{-1} = -\frac{1}{\lambda} I + \frac{1}{\lambda(1-\lambda)} P.$$

Find the spectrum of P, spectral family associated with P and write the spectral representation (1) for P.

▶ We first show that

$$(2.1) \quad (P-\lambda I)^{-1} = -\frac{1}{\lambda} I + \frac{1}{\lambda(1-\lambda)} P.$$

We have

$$(P-\lambda I)\left(-\frac{1}{\lambda} I + \frac{1}{\lambda(1-\lambda)} P\right) = P\left(-\frac{1}{\lambda} I\right) + P\left(\frac{1}{\lambda(1-\lambda)} P\right) - \lambda I\left(-\frac{1}{\lambda} I\right) -$$

$$- \lambda I\left(\frac{1}{\lambda(1-\lambda)} P\right) =$$

$$= -\frac{1}{\lambda} P + \frac{1}{\lambda(1-\lambda)} \underbrace{P^2}_{=P} + I - \frac{\lambda}{(1-\lambda)\lambda} P = \Big(\underbrace{-\frac{1}{\lambda} + \frac{1}{\lambda(1-\lambda)} - \frac{\lambda}{\lambda(1-\lambda)}}_{=0}\Big) P + I =$$

$$= I.$$

Similarly

$$\left(-\frac{1}{\lambda} I + \frac{1}{\lambda(1-\lambda)} P\right)(P-\lambda I) = I.$$

This implies equality (2.1).

Hence, for every $\lambda \in \mathbb{C} \setminus \{0;1\}$ the operator

$$R_\lambda = -\frac{1}{\lambda} I + \frac{1}{\lambda(1-\lambda)} P$$

is defined on the whole space H and is bounded. Therefore such λ is a regular value of T. This implies that $\sigma(T) \subseteq \{0;1\}$.

Next, for $\lambda = 1$ and any $x \in Y$, $x \neq 0$, we have

$$Px - \lambda x = x - x = 0.$$

Similarly for $\lambda = 0$ and any $x \in Y^\perp$, $x \neq 0$,

$$Px - \lambda x = Px = 0.$$

Hence, $\sigma(T) = \sigma_p(T) = \{0;1\}$.

We take
$$E_\lambda = \begin{cases} 0, & \lambda < 0, \\ A, & \lambda \in [0;1), \\ I, & \lambda \geq 1, \end{cases}$$
where A is a projection operator on H, and find A from
$$P = \int_{0-}^{1} \lambda \, dE_\lambda = 0 \cdot (A - 0) + 1 \cdot (I - A) = I - A$$
Hence, $A = I - P$, that is, the projection operator onto $Y^\perp$.

It is easily to see that E_λ, $\lambda \in \mathbb{R}$, is a spectral family.

$\blacktriangle$

③ Let an operator $T: \mathbb{C}^3 \to \mathbb{C}^3$ be represented, with respect to a canonical basis, by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the corresponding spectral family. Write the spectral representation (1) for T.

▶ Let us find eigenvalues and eigenvectors.

$$\det(T - \lambda I) = 0, \qquad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda) \cdot \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 1) = 0 \Rightarrow \lambda_1 = -1, \quad \lambda_{2,3} = 1.$$

$\lambda_1 = -1$, $\lambda_{2,3} = 1$ – eigenvalues.

Find eigenvectors from the equation

$$(T - \lambda I)x = 0, \qquad x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

$$\begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0 \iff \begin{cases} -\lambda \xi_1 + \xi_2 = 0 \\ \xi_1 - \lambda \xi_2 = 0 \\ (1-\lambda)\xi_3 = 0 \end{cases}$$

$\underline{\lambda = -1}$

$$\begin{cases} \xi_1 + \xi_2 = 0 \\ \xi_1 + \xi_2 = 0 \\ 2\xi_3 = 0 \end{cases} \iff \begin{cases} \xi_1 = -\xi_2 \\ \xi_3 = 0 \end{cases} \Rightarrow x_1 = \begin{pmatrix} \xi_1 \\ -\xi_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 1}$

$$\begin{cases} -\xi_1 + \xi_2 = 0 \\ \xi_1 - \xi_2 = 0 \\ 0 \cdot \xi_3 = 0 \end{cases} \iff \begin{cases} \xi_1 = \xi_2 \\ \xi_3 \in \mathbb{C} \end{cases} \Rightarrow x_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$

$$x_3 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $e_1 = \dfrac{x_1}{\|x_1\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$;

$$e_2 = \frac{x_2}{\|x_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$

$$e_3 = \frac{x_3}{\|x_3\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$Tx = \lambda_1 < x, e_1 > e_1 + \lambda_2 < x, e_2 > e_2 + \underbrace{\lambda_3}_{=\lambda_2} < x, e_3 > e_3 =$$

$$= \lambda_1 P_1 x + \lambda_2 P_2 x,$$

where

$$P_1 x = < x, e_1 > e_1 = \frac{1}{2} < (\xi_1, \xi_2, \xi_3) \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} > \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} =$$

$$= \frac{\xi_1 - \xi_2}{2} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \underbrace{\frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{P_1} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

$$P_2 x = < x, e_2 > e_2 + < x, e_3 > e_3 = (I - P_1) x = \underbrace{\frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{P_2} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}.$$

Hence,

$$E_\lambda = \begin{cases} 0, & \lambda < -1 \\ P_1, & -1 \le \lambda < 1 \\ I, & \lambda \geq 1 \end{cases} \quad \text{– the spectral family.}$$

The spectral representation (1) for T:

$$T = \int_{-1-0}^{1} \lambda \, dE_\lambda = (-1) \cdot P_1 + 1 \cdot (I - P_1) = -P_1 + I - P_1 = -2P_1 + I.$$

Indeed,

$$-2P_1 + I = -2 \cdot \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = T.$$

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④ Find T_λ^+, $\lambda \in \mathbb{R}$, and the spectral family E_λ, $\lambda \in \mathbb{R}$, associated with operator $T: \ell^2 \to \ell^2$ defined by $Tx = \left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \dots\right)$, $x = (\xi_k)_{k \geq 1}$. Write the spectral representation (1) for T.

▶ We need to find T_λ^+, $\lambda \in \mathbb{R}$.

$T_\lambda^+ = \frac{1}{2}(B_\lambda + T_\lambda)$, where $B_\lambda = (T_\lambda^2)^{\frac{1}{2}}$.

First we need to write $T_\lambda x = (Tx - \lambda x) =$

$= \left((1-\lambda)\xi_1, \left(\frac{1}{2}-\lambda\right)\xi_2, \dots\right)$, $x = (\xi_1, \xi_2, \dots)$.

Then
$$T_\lambda^2 x = \left((1-\lambda)^2 \xi_1, \left(\frac{1}{2}-\lambda\right)^2 \xi_2, \dots\right)$$
and
$$B_\lambda x = (T_\lambda^2)^{\frac{1}{2}} = \left(|1-\lambda|\xi_1, \left|\frac{1}{2}-\lambda\right|\xi_2, \dots\right)$$
Hence, the positive part of T_λ^+ is defined as follows
$$T_\lambda^+ = \frac{1}{2}(B_\lambda + T_\lambda) = \frac{1}{2}\left(|1-\lambda|\xi_1 + (1-\lambda)\xi_1, \left|\frac{1}{2}-\lambda\right|\xi_2 + \left(\frac{1}{2}-\lambda\right)\xi_2, \dots\right)$$
$$= \frac{1}{2}\left(\left(|1-\lambda| + (1-\lambda)\right)\xi_1, \left(\left|\frac{1}{2}-\lambda\right| + \left(\frac{1}{2}-\lambda\right)\right)\xi_2, \dots\right) =$$
$$= \left((1-\lambda)^+\xi_1, \left(\frac{1}{2}-\lambda\right)^+\xi_2, \dots\right),$$
where $s^+ = \begin{cases} s, & s \geq 0 \\ 0, & s < 0. \end{cases}$

Let $\lambda \geq 1$ then $T_\lambda^+ x = (0, 0, 0, \dots)$

$\frac{1}{2} \leq \lambda < 1$ then $T_\lambda^+ x = ((1-\lambda)\xi_1, 0, 0, \dots)$

$\frac{1}{3} \leq \lambda < \frac{1}{2}$ then $T_\lambda^+ x = \left((1-\lambda)\xi_1, \left(\frac{1}{2}-\lambda\right)\xi_2, \dots\right)$

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$\frac{1}{k+1} \leq \lambda < \frac{1}{k}$ then $T_\lambda^+ x = \left((1-\lambda)\xi_1, \left(\frac{1}{2}-\lambda\right)\xi_2, \dots, \left(\frac{1}{k}-\lambda\right)\xi_k, 0, 0, \dots\right)$

If $\lambda \leq 0$ then $T_\lambda^+ x = \left((1-\lambda)\xi_1, \left(\frac{1}{2}-\lambda\right)\xi_2, \dots, \left(\frac{1}{k}-\lambda\right)\xi_k, \dots\right) =$
$$= T_\lambda x, \quad x = (\xi_1, \xi_2, \dots).$$

So,
$$T_\lambda^+ x = \begin{cases} T_\lambda x, & \lambda \le 0 \\ \left((1-\lambda)\xi_1, (\tfrac{1}{2}-\lambda)\xi_2, \dots, (\tfrac{1}{k}-\lambda)\xi_k, 0, 0, \dots \right), & \tfrac{1}{k+1} \le \lambda < \tfrac{1}{k},\ k=1,2,\dots \\ 0, & \lambda \ge 1. \end{cases}$$

Then $\operatorname{Ker} T_\lambda^+ = \{x : T_\lambda^+ x = 0\}$ is equal for

$\lambda \ge 1$ $\quad \operatorname{Ker} T_\lambda^+ = \{x : T_\lambda^+ x = 0\} = l_2;$

$\frac{1}{2} \le \lambda < 1$ $\quad \operatorname{Ker} T_\lambda^+ = \{x : T_\lambda^+ x = 0\} = \{x : (1-\lambda)\xi_1 = 0\} =$
$\quad = \{(0; \xi_2, \xi_3, \dots)\};$

$\frac{1}{3} \le \lambda < \frac{1}{2}$ $\quad \operatorname{Ker} T_\lambda^+ = \{x : (1-\lambda)\xi_1 = 0$ and $(\frac{1}{2}-\lambda)\xi_2 = 0\} =$
$\quad = \{(0, 0, \xi_3, \xi_4, \dots)\};$

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$\frac{1}{k+1} \le \lambda < \frac{1}{k}$ $\quad \operatorname{Ker} T_\lambda^+ = \{(0, 0, \dots, 0, \xi_{k+1}, \xi_{k+2}, \dots)\};$

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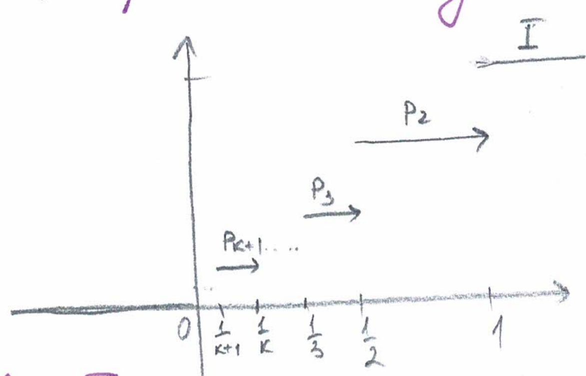
$\lambda \le 0$ $\quad \operatorname{Ker} T_\lambda^+ = \{(0, 0, 0, \dots)\}.$

Hence,
$$\operatorname{Ker} T_\lambda^+ = \begin{cases} \{(0,0,0,\dots)\}, & \lambda \le 0 \\ \{x \in l^2 : (0, 0, \dots, 0, \xi_{k+1}, \xi_{k+2}, \dots)\}, & \tfrac{1}{k+1} \le \lambda < \tfrac{1}{k},\ k=1,2,\dots \\ l^2, & \lambda \ge 1. \end{cases}$$

By ~~the~~ Th. 22.8 the family $\{E_\lambda, \lambda \in \mathbb{R}\}$, where E_λ is the projection of H onto $\operatorname{Ker} T_\lambda^+$ is spectral family associated with the operator T.



$$E_\lambda = \begin{cases} 0, & \lambda \le 0 \\ P_{k+1}, & \tfrac{1}{k+1} \le \lambda < \tfrac{1}{k},\ k=1,2,\dots \\ I, & \lambda \ge 1, \end{cases}$$

$$P_{k+1} x = (0, 0, \dots, 0, \xi_{k+1}, \xi_{k+2}, \dots).$$

The spectral representation (1) for T:
$$\int_{-\infty}^{\infty} \lambda \, dE_\lambda x = 1 \cdot (I - P_2)x + \tfrac{1}{2}(P_2 - P_3)x + \dots + \tfrac{1}{k}(P_k - P_{k+1})x + \tfrac{1}{k+1}(P_{k+1} - P_{k+2}) + \dots =$$
$$= \sum_{k=1}^{\infty} \frac{1}{k}(P_k - P_{k+1})x.$$

Since $\quad P_k x = (0, 0, \dots, 0, \xi_k, \xi_{k+1}, \xi_{k+2}, \dots)$

$\qquad\quad P_{k+1} x = (0, 0, \dots, 0, 0, \xi_{k+1}, \xi_{k+2}, \dots),$

then $\quad (P_k - P_{k+1}) x = (0, 0, 0, \dots, \xi_k, 0, 0, \dots)$

and

$$\sum_{k=1}^{\infty} \frac{1}{k} (P_k - P_{k+1}) x = \frac{1}{1}(\xi_1, 0, 0, \dots) + \frac{1}{2}(0, \xi_2, 0, 0, \dots) + \\ + \frac{1}{3}(0, 0, \xi_3, 0, 0, \dots) + \dots =$$

$$= \left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \dots\right) = Tx.$$

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(5) For the multiplication operator $T: L^2[0;1] \to L^2[0;1]$ defined by

$$(Tx)(t) = tx(t), \ t \in [0;1],$$

compute $\sin T$.

▶ We know that for a continuous function $f: [m, M] \to \mathbb{R}$ one can define the operator (which is bounded and self-adjoint) $f(T)$ by the equality

$$f(T) := \int_{m-0}^{M} f(\lambda)\, dE_\lambda.$$

We need to find $\sin T$, where $(Tx)(t) = tx(t)$, $t \in [0;1]$.

$$\sin T = \int_0^1 \sin\lambda \, dE_\lambda.$$

To find the precise form of $\sin T$ we compute

$$<\sin T x, y> = \int_0^1 \sin\lambda \, d<E_\lambda x, y>, \text{ where}$$

$$<E_\lambda x, y> = \int_0^1 E_\lambda x(t)\, \overline{y(t)}\, dt = \int_0^1 \mathbb{I}_{[0,\lambda]}(t)\, x(t)\, \overline{y(t)}\, dt =$$

$$= \int_0^\lambda x(t)\, \overline{y(t)}\, dt, \qquad (*)$$

because $(E_\lambda x)(t) = \mathbb{I}_{[0,\lambda]}(t)\, x(t)$ (from lecture 22).

So,

$$<\sin T x, y> = \int_0^1 \sin\lambda \, d<E_\lambda x, y> \overset{(*)}{=}$$

$$= \int_0^1 \sin\lambda \, d\left(\int_0^\lambda x(t)\,\overline{y(t)}\, dt\right) = \int_0^1 \underbrace{\sin\lambda \ x(\lambda)}_{(Sx)(t)} \overline{y(\lambda)}\, d\lambda =$$

$$= <Sx, y>.$$

Hence, $((\sin T)x)(t) = \sin t \cdot x(t)$

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