Problem sheet 11

① Let T be a bounded linear operator on a Hilbert space H. Show that $(\operatorname{Im} T)^{\perp} = \ker T^*$, where $\operatorname{Im} T = \{Tx, x \in H\}$.

▶ Let

$x \in (\operatorname{Im} T)^{\perp} \Leftrightarrow \langle x, y \rangle = 0 \quad \forall y \in \operatorname{Im} T \Leftrightarrow \langle x, Tz \rangle = 0 \quad \forall z \in H$

$\overset{\text{by the defin.}}{\Leftrightarrow} \quad \langle T^* x, z \rangle = 0 \quad \forall z \in H.$

Since $\langle T^* x, z \rangle = 0$ for $\forall z \in H$, then it is true also for $z = T^* x$.

Hence, we have that $\langle T^* x, \underbrace{T^* x}_{z} \rangle = 0 \Leftrightarrow$

$\Leftrightarrow T^* x = 0 \Leftrightarrow x \in \ker T^*.$

$\Rightarrow (\operatorname{Im} T)^{\perp} = \ker T^*.$ ◀

② Let $k(t,s)$, $t, s \in [0;1]$, be a continuous function and let the operator T act on $L^2[0;1]$ by the formula

$$(Tx)(t) = \int_0^1 k(t,s) x(s)\, ds, \quad t \in [0;1].$$

Find the adjoint operator T^*.

▶ $\langle Tx, y \rangle = \int_0^1 (Tx)(t)\, \overline{y(t)}\, dt = \int_0^1 \left(\int_0^1 k(t,s) x(s)\, ds \right) \overline{y(t)}\, dt =$

$= \int_0^1 \int_0^1 k(t,s) x(s) \overline{y(t)}\, ds\, dt = \int_0^1 \left(\int_0^1 k(t,s) \overline{y(t)}\, dt \right) x(s)\, ds =$

$= \int_0^1 \overline{\overline{\left(\int_0^1 k(t,s) \overline{y(t)}\, dt \right)}} x(s)\, ds = \int_0^1 \overline{\left(\int_0^1 \overline{k(t,s)}\, y(t)\, dt \right)} x(s)\, ds.$

$\Rightarrow T^* y(s) = \int_0^1 \overline{k(t,s)}\, y(t)\, dt.$ ◀

③ Let S, T be bounded linear operators on a normed space X. Show that for every $\lambda \in \rho(S) \cap \rho(T)$ one has

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(S-T)R_\lambda(S).$$

▷ We know that

$$I = (T-\lambda I)\underset{= R_\lambda(T)(T-\lambda I)}{R_\lambda(T)} = (S-\lambda I)R_\lambda(S). \qquad (*)$$

Then

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T) \cdot I - I \cdot R_\lambda(S) \overset{(*)}{=}$$

$$= R_\lambda(T)\underbrace{(S-\lambda I)R_\lambda(S)}_{=I} - \underbrace{R_\lambda(T)(T-\lambda I)}_{=I} \cdot R_\lambda(S) =$$

$$= R_\lambda(T)(S - \lambda I - (T - \lambda I))R_\lambda(S) =$$

$$= R_\lambda(T)(S - \lambda I - T + \lambda I)R_\lambda(S) =$$

$$= R_\lambda(T)(S-T)R_\lambda(S).$$

Hence,

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(S-T)R_\lambda(S).$$

◀

4) Let T be a linear operator on ℓ^2 defined by $Tx = (\xi_2, \xi_1, \xi_3, \xi_4, \xi_5, \dots)$ (permutation of first two components. Find and classify the spectrum of T.

▶ We consider the equation

$$Tx - \lambda x = 0 \qquad (1)$$

and find its non-zero solutions. So, we get

$$(\xi_2, \xi_1, \xi_3, \xi_4, \xi_5, \dots) - \lambda(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \dots) = 0.$$

$$\begin{cases} \xi_2 - \lambda\xi_1 = 0 \\ \xi_1 - \lambda\xi_2 = 0 \\ \xi_3 - \lambda\xi_3 = 0 \\ \xi_4 - \lambda\xi_4 = 0 \\ \dots \end{cases} \Rightarrow \begin{cases} \xi_2 - \lambda\xi_1 = 0 \\ \xi_1 - \lambda\xi_2 = 0 \\ (1-\lambda)\xi_3 = 0 \\ (1-\lambda)\xi_4 = 0 \\ \dots \end{cases}$$

Hence, if $\lambda = 1$, then $x = (0, 0, 1, 0, 0, \dots)$ is a solution to (1). This implies that $\lambda = 1 \in \sigma_p(T)$. Next we consider the system

$$\begin{cases} \xi_2 - \lambda\xi_1 = 0, \\ \xi_1 - \lambda\xi_2 = 0. \end{cases}$$

It has non-zero solutions iff its determinant is equal to zero:

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1.$$

Hence, $\lambda = -1 \in \sigma_p$. We obtained $\sigma_p = \{-1; 1\}$.

Take $\lambda \notin \sigma_p$ and find $R_\lambda = (T - \lambda I)^{-1}$.

Consider

$$(T - \lambda I)x = Tx - \lambda x = y \in \ell^2.$$

$$\begin{cases} \xi_2 - \lambda\xi_1 = \eta_1 \\ \xi_1 - \lambda\xi_2 = \eta_2 \\ \xi_3 - \lambda\xi_3 = \eta_3 \\ \xi_4 - \lambda\xi_4 = \eta_4 \\ \dots \end{cases} \Rightarrow \begin{cases} \xi_1 = \dfrac{\eta_2 + \lambda\eta_1}{1-\lambda^2} \\ \xi_2 = \dfrac{\eta_1 + \lambda\eta_2}{1-\lambda^2} \\ \xi_3 = \dfrac{1}{1-\lambda}\eta_3 \\ \xi_4 = \dfrac{1}{1-\lambda}\eta_4 \\ \dots \end{cases}$$

So, $R_\lambda y = \left(\dfrac{\eta_2 + \lambda \eta_1}{1-\lambda^2}, \dfrac{\eta_1 + \lambda \eta_2}{1-\lambda^2}, \dfrac{1}{1-\lambda}\eta_3, \dfrac{1}{1-\lambda}\eta_4, \dots \right)$

We estimate

$$\|R_\lambda y\|^2 = \frac{|\eta_2+\lambda\eta_1|^2}{|1-\lambda^2|^2} + \frac{|\eta_1+\lambda\eta_2|^2}{|1-\lambda^2|^2} + \frac{1}{|1-\lambda|^2}|\eta_3|^2 + \frac{1}{|1-\lambda|^2}|\eta_4|^2 + \dots \le$$

$$\le \frac{2|\eta_2|^2 + 2|\lambda||\eta_1|^2}{|1-\lambda^2|^2} + \frac{2|\eta_1|^2 + 2|\lambda||\eta_2|^2}{|1-\lambda^2|^2} + \frac{1}{|1-\lambda|^2}|\eta_3|^2 + \frac{1}{|1-\lambda|^2}|\eta_4|^2 + \dots$$

$$\le \frac{2|\lambda|+2}{|1-\lambda^2|^2}|\eta_1|^2 + \frac{2+2|\lambda|}{|1-\lambda^2|^2}|\eta_2|^2 + \frac{1}{|1-\lambda|^2}|\eta_3|^2 + \frac{1}{|1-\lambda|^2}|\eta_4|^2 + \dots \le$$

$$\le C\left(|\eta_1|^2 + |\eta_2|^2 + \dots\right) = C\|y\|^2,$$

where $C = \max\left\{ \dfrac{2|\lambda|+2}{|1-\lambda^2|^2}, \dfrac{1}{|1-\lambda|^2} \right\} < +\infty$

and we have used the inequality:

$$|a+b|^2 \le \left(|a|+|b|\right)^2 \le 2|a|^2 + 2|b|^2.$$

So, $D(R_\lambda) = \ell^2$ and $\|R_\lambda\| \le C < +\infty$.

Consequently, $\lambda \in \rho(T)$.

Hence, $\rho(T) = \mathbb{C} \setminus \{-1;1\}$, $\sigma_p(T) = \{-1, 1\}$, $\sigma_c(T) = \emptyset$, $\sigma_r(T) = \emptyset$.

(5) Let $X = C[0;\pi]$ and define $T: D(T) \to X$ by $Tx = x''$, where

$$D(T) = \{x \in X : x', x'' \in X,\ x(0) = x(\pi) = 0\}.$$

Show that $\sigma(T)$ is not compact.

▶ Let us find $\lambda \in \mathbb{R}$ such that $\exists x \in D(T)$ satisfying

$$Tx - \lambda x = 0.$$

This is equivalent to finding a twise continuously differentiable function x such that

$$\begin{cases} x''(t) - \lambda x(t) = 0, & t \in [0;\pi], \quad (5.1) \\ x(0) = x(\pi) = 0. & \quad (5.2) \end{cases}$$

The obtained equation is a linear second-order equation with constant coefficients. To solve it we need to find the roots of characteristic polynomials

$$\mu^2 - \lambda = 0 \Rightarrow \mu = \pm\sqrt{\lambda}.$$

If $\lambda < 0$, then $\mu = \pm i\sqrt{-\lambda}$.

In this case

$$x(t) = C_1 \cos(\sqrt{-\lambda}\, t) + C_2 \sin(\sqrt{-\lambda}\, t)$$

is a general solution to equation (5.1).

We next plug in x into (5.2):

$$\begin{cases} x(0) = C_1 \cos 0 + C_2 \sin 0 = C_1 = 0, \\ x(\pi) = C_2 \sin(\sqrt{-\lambda}\,\pi) = 0 \end{cases}$$

$$\sqrt{-\lambda}\,\pi = \pi n, \quad n \geq 1$$

So, $\lambda_n = -n^2$, $n \geq 1$, and $x_n(t) = C_2 \sin nt$.

Consequently, $\lambda_n = -n^2 \in \sigma_p(T) \subset \sigma(T)$, for all $n \geq 1$.

This implies that $\sigma(T)$ is not bounded.

Hence, $\sigma(T)$ is not compact. ◀

⑥ Let $a(t) = \begin{cases} t & \text{if } t \in [0;1], \\ 1 & \text{if } t \in (1;2]. \end{cases}$

Find and classify the spectrum of the operator $(Tx)(t) = a(t)x(t)$ acting on $C[0;2]$.

▶ $(Tx)(t) = a(t)x(t)$, $t \in [0;2]$, where $a(t) = \begin{cases} t, & t \in [0;1], \\ 1, & t \in (1;2]. \end{cases}$

We start from the equation

(6.1) $(Tx)(t) - \lambda x(t) = (a(t) - \lambda) x(t) = 0$

Consider the following cases

1) $\lambda \notin [0;1] \Rightarrow a(t) - \lambda \neq 0$.

Then (6.1) implies $x(t) = 0 \quad \forall t \in [0;2]$.

2) $\lambda \in [0,1) \Rightarrow a(t) - \lambda = 0 \iff t = \lambda$.

Then (6.1) yields $x(t) = 0 \quad \forall t \in [0,2] \setminus \{\lambda\}$.

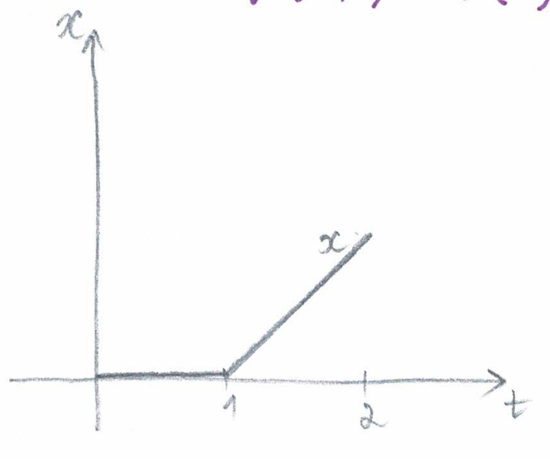
But the continuity of x implies that $x(t) = 0 \quad \forall t \in [0;2]$.

3) $\lambda = 1 \Rightarrow a(t) - \lambda = 0 \iff t \in [1;2]$.

So, for every $x \in C[0;2]$ such that $x(t) = 0$, $t \in [0;1]$ we get $(Tx)(t) - \lambda x(t) = 0$.

In particular, x can be taken as follows:

$$x(t) = \begin{cases} 0, & t \in [0;1], \\ t-1, & t \in (1;2]. \end{cases}$$



Hence, $\sigma_p(T) = \{1\}$.

Let $\lambda \neq 1$. Then $\exists\ R_\lambda = (T-\lambda I)^{-1}$.

We find R_λ from the equation

$$(Tx)(t) - \lambda x(t) = y(t), \quad t \in [0;2].$$

$$\Rightarrow x(t) = \frac{1}{a(t)-\lambda}\, y(t), \quad t \in [0;2].$$

Hence, $(R_\lambda y)(t) = \frac{1}{a(t)-\lambda}\, y(t)$, $t \in [0;1]$,

$$y \in \mathcal{D}(R_\lambda) = \mathrm{Im}(T-\lambda I).$$

If $\lambda \notin [0;1]$, then the function $\frac{1}{a(t)-\lambda}$, $t \in [0;2]$, is a continuous bounded function. Therefore

$$\mathcal{D}(R_\lambda) = C[0;2]$$

and

$$\|R_\lambda y\| = \max_{t\in[0;2]} \left| \frac{1}{a(t)-\lambda}\, y(t) \right| \leq \max_{t\in[0;2]} \frac{1}{|a(t)-\lambda|} \cdot \|y\|.$$

So, R_λ is a bounded linear operator defined on $C[0;2]$. Hence,

$$\rho(T) = \mathbb{C} \setminus [0;1].$$

It remains to consider the case $\lambda \in [0;1)$.

Then for $t=\lambda$ we have

$$(Tx)(t) - \lambda x(t) = (a(t)-\lambda)x(t) = 0 \cdot x(t) = 0.$$

So,

$$\mathcal{D}(R_\lambda) = \mathrm{Im}(T-\lambda I) \subseteq \underbrace{\{y \in C[0;2] : y(\lambda) = 0\}}_{=:A}.$$

But the set A is not dence in $C[0;2]$.

For instance, the function

$$y_0(t) = 1, \quad t \in [0;2].$$

Can not be approximated by functions from A, because

$$\|y_0 - y\| = \max_{t\in[0;2]} |1-y(t)| \geq |1-y(\lambda)| = |1-0| = 1.$$

Hence, $\mathcal{D}(R_\lambda)$ is not dence in $C[0;2]$. This implies $\sigma_r(T) = [0;2]$. Since $\rho(T) \cup \sigma_p(T) \cup \sigma_r(T) = \mathbb{C}$ we get $\sigma_c(T) = \emptyset$. ◀

(7) Let T be the left-shift operator on ℓ^2 defined as follows

$$Tx = (\xi_2, \xi_3, \dots), \quad x = (\xi_k)_{k\geq 1} \in \ell^2.$$

Find the spectrum of T.

▸ We first estimate the norm of T

$$\|Tx\|^2 = |\xi_2|^2 + |\xi_3|^2 + \dots \leq |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 + \dots = \|x\|^2.$$

So, $\|T\| \leq 1$. If $x = (0, 1, 0, 0, \dots)$ we get $\|Tx\|^2 = 1 = \|x\|^2$.

Hence, $\|T\| = 1$.

By Th. 20.7 $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

We next find all λ such that $\exists x \neq 0$ satisfying

$$Tx - \lambda x = 0.$$

$$(\xi_2, \xi_3, \xi_4, \dots) - \lambda(\xi_1, \xi_2, \xi_3, \dots) = 0$$

$$\Rightarrow \begin{cases} \xi_2 - \lambda \xi_1 = 0 \\ \xi_3 - \lambda \xi_2 = 0 \\ \xi_4 - \lambda \xi_3 = 0 \\ \dots\dots \end{cases} \Rightarrow \begin{cases} \xi_2 = \lambda \xi_1 \\ \xi_3 = \lambda \xi_2 = \lambda^2 \xi_1 \\ \xi_4 = \lambda \xi_3 = \lambda^3 \xi_1 \\ \dots\dots \end{cases}$$

We have obtained that

$$x = (\xi_1, \lambda \xi_1, \lambda^2 \xi_1, \dots) = \xi_1 (1, \lambda, \lambda^2, \dots).$$

If $|\lambda| < 1$, then $x \in \ell^2$, because

$$\|x\|^2 = |\xi_1|^2 \sum_{k=0}^{\infty} |\lambda|^{2k} < +\infty \quad \text{as a geometric}$$

series (here $q = |\lambda|^2$).

This implies that

$$\sigma_p(T) \supset \{\lambda \in \mathbb{C} : |\lambda| < 1\}.$$

We remark that $\sigma_p(T) \subseteq \sigma(T)$ and $\sigma(T)$ is a closed set. Hence,

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_p(T) \subset \sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$$

Using the fact that $A \subset B$ implies $\overline{A} \subset \overline{B}$, we obtain

$$\begin{array}{ccccc} \overline{\{\lambda \in \mathbb{C} : |\lambda| < 1\}} & \subset & \overline{\sigma(T)} & \subset & \overline{\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}} \\ \| & & \| & & \| \\ \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} & & \sigma(T) & & \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}. \end{array}$$

Hence, $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.