Problem sheet 10

① If $x \perp y$ in an inner product space X, show that
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

▶ $\|x+y\|^2 = \langle x+y, x+y\rangle \overset{(IP1)}{=} \langle x, x+y\rangle + \langle y, x+y\rangle \overset{(IP3)}{=}$
$= \overline{\langle x+y, x\rangle} + \overline{\langle x+y, y\rangle} = \overline{\langle x,x\rangle} + \overline{\langle y,x\rangle} + \overline{\langle x,y\rangle} + \overline{\langle y,y\rangle} =$
$= \langle x,x\rangle + \underbrace{\langle x,y\rangle}_{=0} + \underbrace{\overline{\langle x,y\rangle}}_{=0} + \langle y,y\rangle.$

Since $x \perp y$, then $\langle x,y\rangle = 0$ and $\overline{\langle x,y\rangle} = 0$.

Hence,
$$\|x+y\|^2 = \langle x,x\rangle + \langle y,y\rangle = \|x\|^2 + \|y\|^2.$$
◀

② Show that for a sequence $\{x_n\}_{n\geq 1}$ in an inner product space the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x,x\rangle$ imply the convergence $x_n \to x$.

▶ We need to show that $\|x_n - x\| \to 0$.

$\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \langle x_n, x_n - x\rangle - \langle x, x_n - x\rangle =$
$= \overline{\langle x_n - x, x_n\rangle} - \overline{\langle x_n - x, x\rangle} = \overline{\langle x_n, x_n\rangle} - \overline{\langle x, x_n\rangle} - \overline{\langle x_n, x\rangle} + \overline{\langle x,x\rangle} =$
$= \langle x_n, x_n\rangle - \langle x_n, x\rangle - \overline{\langle x_n, x\rangle} + \langle x,x\rangle =$
$= \|x_n\|^2 - \langle x_n, x\rangle - \overline{\langle x_n, x\rangle} + \|x\|^2.$

We know that $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x,x\rangle$.
Then $\overline{\langle x_n, x\rangle} \to \langle x,x\rangle$ and we have
$$\|x_n - x\|^2 \to \|x\|^2 - \langle x,x\rangle - \langle x,x\rangle + \|x\|^2 =$$
$$= \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 = 0, \quad n\to\infty.$$

Hence, $\|x_n - x\|^2 \to 0 \Rightarrow \|x_n - x\| \to 0 \Rightarrow x_n \to x, \quad n \to \infty.$
◀

(3) Let H be a Hilbert space, $M \subset H$ be a convex subset, and $\{x_n\}_{n\geq 1}$ be a sequence in M such that $\|x_n\| \to d$, where $d = \inf_{x \in M} \|x\|$. Show that $\{x_n\}_{n\geq 1}$ converges in H.
Hint: Use the parallelogram equality.

▶ Take a sequence $\{x_n\}_{n\geq 1}$ in M and show that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence, that is, we need to prove that
$$\|x_n - x_m\| \to 0, \quad n, m \to \infty.$$
Using the parallelogram equality we have
$$\|x_n - x_m\|^2 + \|x_n + x_m\|^2 = 2\left(\|x_n\|^2 + \|x_m\|^2\right). \quad (*)$$
Since M is a convex subset of H, that is,
$$\forall x, y \in M \quad \alpha x + (1-\alpha) y \in M, \ \forall \alpha \in [0;1],$$
then $\forall\, x_n, x_m \in M$
$$\frac{x_n + x_m}{2} \in M, \ \alpha = \frac{1}{2},$$
and
$$\left\| \frac{x_n + x_m}{2} \right\| \geq \inf_{x \in M} \|x\| = d. \quad (**)$$

Then from $(*)$:
$$\|x_n - x_m\|^2 = 2\left(\|x_n\|^2 + \|x_m\|^2\right) - \|x_n + x_m\|^2 =$$
$$= 2\left(\|x_n\|^2 + \|x_m\|^2 - \frac{\|x_n + x_m\|^2}{2}\right) =$$
$$= 2\left(\|x_n\|^2 + \|x_m\|^2 - 2 \cdot \left\|\frac{x_n + x_m}{2}\right\|^2\right) \overset{(**)}{\leq}$$
$$\leq 2\left(\underbrace{\|x_n\|^2}_{\to d} + \underbrace{\|x_m\|^2}_{\to d} - 2d^2\right) \to 0, \quad n, m \to \infty.$$

$\Rightarrow \{x_n\}_{n\geq 1}$ is a Cauchy sequence.

Since H is a Hilbert space, then every Cauchy sequence converges in H.

Hence, $\{x_n\}_{n\geq 1}$ converges in H. ▲

(4) Let $\{e_1, \dots, e_n\}$ be an orthonormal set in an inner product space X, where n is fixed. Let $x \in X$ be any fixed element and $y = \beta_1 e_1 + \dots + \beta_n e_n$. Then $\|x-y\|$ depends on $\beta_1, \dots, \beta_n$. Show by direct calculation that $\|x-y\|$ is minimum if and only if $\beta_k = \langle x, e_k \rangle$, $k = 1, \dots, n$.

▶ Let $y_0 = \langle x, e_1 \rangle e_1 + \dots + \langle x, e_n \rangle e_n$, that is, $\beta_k = \langle x, e_k \rangle$.

1) Let us show that $x - y_0 \perp e_k \quad \forall k = 1, \dots, n$.

$$\langle x - y_0, e_k \rangle = \langle x, e_k \rangle - \langle y_0, e_k \rangle =$$

$$= \langle x, e_k \rangle - \langle \underbrace{\sum_{i=1}^{n} \langle x, e_i \rangle e_i}_{=y}, e_k \rangle =$$

$$= \langle x, e_k \rangle - \sum_{i=1}^{n} \langle x, e_i \rangle \underbrace{\langle e_i, e_k \rangle}_{= \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}} =$$

$$= \langle x, e_k \rangle - \langle x, e_k \rangle \underbrace{\langle e_k, e_k \rangle}_{=1} = 0 \implies x - y_0 \perp e_k, \ \forall k = 1, \dots, n.$$

2) Next let us show that $x - y_0 \perp y \quad \forall y = \sum_{j=1}^{n} \beta_j e_j$. (*)

Indeed,

$$\underline{\langle x - y_0, y \rangle} = \langle x, y \rangle - \langle y_0, y \rangle = \langle x, \sum_{j=1}^{n} \beta_j e_j \rangle - \langle \sum_{i=1}^{n} \langle x, e_i \rangle e_i, \sum_{j=1}^{n} \beta_j e_j \rangle =$$

$$= \langle \sum_{j=1}^{n} \beta_j e_j, x \rangle - \sum_{i=1}^{n} \langle \langle x, e_i \rangle e_i, \sum_{j=1}^{n} \beta_j e_j \rangle =$$

$$= \sum_{j=1}^{n} \bar{\beta}_j \langle x, e_j \rangle - \sum_{i,j=1}^{n} \langle x, e_i \rangle \bar{\beta}_j \langle e_i, e_j \rangle =$$

$$= \sum_{j=1}^{n} \bar{\beta}_j \langle x, e_j \rangle - \sum_{j=1}^{n} \langle x, e_j \rangle \bar{\beta}_j = 0. \implies x - y_0 \perp y, \ y = \sum_{j=1}^{n} \beta_j e_j.$$

3) Prove that $\|x-y\|$ is minimum iff $y = y_0$. We have that

$\|x - y\| = \|x - y_0 + y_0 - y\|$.

Since $y_0 - y$ can be written as (*), then $x - y_0 \perp y_0 - y$.

Then

$$\|x-y\|^2 = \|x - y_0 + y_0 - y\|^2 \overset{\text{Pythagorean relation}}{=} \|x - y_0\|^2 + \|y_0 - y\|^2.$$

$\|x - y_0\|^2$ does not depend on y

$\|y_0 - y\|^2$ is nonnegative function of y.

Then $\|x-y\|^2$ reaches minimum iff

$$\|y_0 - y\|^2 = 0 \iff y = y_0 \implies \beta_k = \langle x, e_k \rangle \ \forall k = 1, \dots, n.$$

◀

⑤ Let $x_1(t)=t^2$; $x_2(t)=t$, $x_3(t)=1$. Orthonormalize x_1, x_2, x_3 in this order in the space $L^2[-1;1]$.

▶ Gram-Schmidt procedure:

- $e_1 = \frac{x_1}{\|x_1\|}$, where $x_1(t)=t^2$,
$$\|x_1\|^2 = \int_{-1}^{1}(x_1(t))^2dt = \int_{-1}^{1}(t^2)^2dt = \int_{-1}^{1}t^4dt =$$
$$= \frac{t^5}{5}\Big|_{-1}^{1} = \frac{2}{5}. \Rightarrow \|x_1\| = \sqrt{\frac{2}{5}}.$$

$$e_1 = \frac{t^2}{\sqrt{\frac{2}{5}}} = \sqrt{\frac{5}{2}}\,t^2.$$

- $e_2 = \frac{v_2}{\|v_2\|}$, where $v_2(t) = x_2(t) - \langle x_2, e_1\rangle e_1$, $\|v_2\|^2 = \int_{-1}^{1} v_2^2(t)dt$

$$\langle x_2, e_1\rangle = \langle t, \sqrt{\tfrac{5}{2}}t^2\rangle = \int_{-1}^{1} t\cdot\sqrt{\tfrac{5}{2}}t^2\,dt = 0$$

Then $v_2(t) = t - 0\cdot\sqrt{\frac{5}{2}}t^2 = t$,
$$\|v_2\|^2 = \int_{-1}^{1}t^2dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{2}{3} \Rightarrow \|v_2\| = \sqrt{\frac{2}{3}}.$$

$$e_2 = \frac{t}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}\,t.$$

- $e_3 = \frac{v_3}{\|v_3\|}$, where $v_3(t) = x_3(t) - \langle x_3, e_1\rangle e_1 - \langle x_3, e_2\rangle e_2$.

$$\langle x_3, e_1\rangle = \int_{-1}^{1} 1\cdot\sqrt{\tfrac{5}{2}}\,t^2\,dt = \sqrt{\tfrac{5}{2}}\cdot\frac{t^3}{3}\Big|_{-1}^{1} = \frac{2}{3}\sqrt{\frac{5}{2}}.$$

$$\langle x_3, e_2\rangle = \int_{-1}^{1} 1\cdot\sqrt{\tfrac{3}{2}}\,t\,dt = 0.$$

$$v_3(t) = 1 - \frac{2}{3}\sqrt{\frac{5}{2}}\cdot\underbrace{\sqrt{\frac{5}{2}}t^2}_{e_1} - 0\cdot\sqrt{\frac{3}{2}}t = 1 - \frac{5}{3}t^2.$$

$$\|v_3\|^2 = \int_{-1}^{1}\left(1-\frac{5}{3}t^2\right)^2dt = \int_{-1}^{1}\left(1 - \frac{10}{3}t^2 + \frac{25}{9}t^4\right)dt = 2\int_{0}^{1}\left(1 - \frac{10t^2}{3} + \frac{25}{9}t^4\right)dt =$$
$$= 2\left(t - \frac{10t^3}{9} + \frac{5t^5}{9}\right)\Big|_0^1 = 2\left(1 - \frac{10}{9} + \frac{5}{9}\right) = 2\cdot\frac{9-10+5}{9} = \frac{8}{9} \Rightarrow$$
$$\Rightarrow \|v_3\| = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}.$$

$$e_3 = \frac{1-\frac{5}{3}t^2}{\frac{2\sqrt{2}}{3}} = \frac{3-5t^2}{2\sqrt{2}} = \frac{\sqrt{2}\,(3-5t^2)}{4}.$$ ◀

⑥ Let M be a total set in an inner product space X. If $\langle v,x\rangle = \langle w,x\rangle$ for all $x \in M$, show that $v=w$.

► We have that $\langle v,x\rangle = \langle w,x\rangle$ $\forall x\in M$, that is

$$\langle v-w, x\rangle = 0 \quad \forall x \in M. \qquad (*)$$

Let us show that

$$\langle v-w, x\rangle = 0 \quad \forall x \in \operatorname{span} M \qquad (**)$$

Take $x \in M$. Then $x = \sum_{i=1}^{n} \alpha_i e_i$ and

$$\langle v-w, x\rangle = \langle v-w, \sum_{i=1}^{n} \alpha_i e_i\rangle =$$

$$= \langle v, \sum_{i=1}^{n} \alpha_i e_i\rangle - \langle w, \sum_{i=1}^{n} \alpha_i e_i\rangle =$$

$$= \sum_{i=1}^{n} \bar{\alpha}_i \underbrace{\langle v, e_i\rangle}_{\overset{(*)}{=}\langle w, e_i\rangle \text{, because } e_i \in M} - \sum_{i=1}^{n} \bar{\alpha}_i \langle w, e_i\rangle = \sum_{i=1}^{n} \bar{\alpha}_i \langle w, e_i\rangle - \sum_{i=1}^{n} \bar{\alpha}_i \langle w, e_i\rangle = 0.$$

Since M is a total set in an inner product space X, that is,

$$\overline{\operatorname{span} M} = X,$$

then $\forall\, v-w \in X$ $\exists x_n \in \operatorname{span} M$ s.t. $x_n \to v-w$, $n\to\infty$.

We have that

$$\underbrace{\langle v-w, x_n\rangle}_{\overset{\text{by } (**)}{=}\, 0} \xrightarrow[n\to\infty]{} \langle v-w, v-w\rangle$$

$\overset{\text{by defin. of inn. product}}{\Rightarrow}$ $v-w=0 \Rightarrow v=w$. ◄

7. Let $\{e_k, k \geq 1\}$ be an orthonormal sequence in a Hilbert space H. Show that if
$$x = \sum_{k=1}^{\infty} \alpha_k e_k, \quad y = \sum_{k=1}^{\infty} \beta_k e_k,$$
then
$$\langle x, y \rangle = \sum_{k=1}^{\infty} \alpha_k \bar{\beta}_k$$
and the series converges absolutely.

▶ 1) Let us show that $\sum_{k=1}^{\infty} \alpha_k \bar{\beta}_k$ converges absolutely, that is $\sum_{k=1}^{\infty} |\alpha_k \bar{\beta}_k| < \infty$.

We know by Th. 18.9 that the series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges iff the series $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges in $\mathbb{R}$.

Since $\sum_{k=1}^{\infty} \alpha_k e_k$ and $\sum_{k=1}^{\infty} \beta_k e_k$ converge, then $\sum_{k=1}^{\infty} |\alpha_k|^2$ and $\sum_{k=1}^{\infty} |\beta_k|^2$ converge in $\mathbb{R}$.

From the Cauchy-Schwarz inequality
$$\sum_{k=1}^{\infty} |\alpha_k \bar{\beta}_k| \leq \underbrace{\left(\sum_{k=1}^{\infty} |\alpha_k|^2\right)^{1/2}}_{<\infty} \underbrace{\left(\sum_{k=1}^{\infty} |\bar{\beta}_k|^2\right)^{1/2}}_{|\bar{\beta}_k| = |\beta_k|, \ <\infty} < \infty$$

$\Rightarrow \sum_{k=1}^{\infty} \alpha_k \bar{\beta}_k$ converges absolutely.

2) We need to show that $\langle x, y \rangle = \sum_{k=1}^{\infty} \alpha_k \bar{\beta}_k$.

Set $x_n = \sum_{k=1}^{n} \alpha_k e_k$, $y_n = \sum_{j=1}^{n} \beta_j e_j$.

Since $\sum_{k=1}^{\infty} \alpha_k e_k$ and $\sum_{j=1}^{\infty} \beta_j e_j$ converge, then $x_n \to x = \sum_{k=1}^{\infty} \alpha_k e_k$,
$y_n \to y = \sum_{j=1}^{\infty} \beta_j e_j$.

Compute
$$\langle x_n, y_n \rangle = \left\langle \sum_{k=1}^{n} \alpha_k e_k, \sum_{j=1}^{n} \beta_j e_j \right\rangle = \sum_{k=1}^{n} \alpha_k \left\langle e_k, \sum_{j=1}^{n} \beta_j e_j \right\rangle =$$
$$= \sum_{k=1}^{n} \alpha_k \overline{\left\langle \sum_{j=1}^{n} \beta_j e_j, e_k \right\rangle} = \sum_{k,j=1}^{n} \alpha_k \bar{\beta}_j \overline{\langle e_j, e_k \rangle} = \sum_{k,j=1}^{n} \alpha_k \bar{\beta}_j \langle e_k, e_j \rangle =$$
$$= \sum_{k=1}^{n} \alpha_k \bar{\beta}_k \langle e_k, e_k \rangle = \sum_{k=1}^{n} \alpha_k \bar{\beta}_k.$$

We have that $x_n \to x$, $y_n \to y$ in H. Then (by Lemma 17.6)
$$\langle x_n, y_n \rangle \to \langle x, y \rangle, \quad n \to \infty,$$
that is,
$$\lim_{n \to \infty} \langle x_n, y_n \rangle = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k \bar{\beta}_k = \sum_{k=1}^{\infty} \alpha_k \bar{\beta}_k = \langle x, y \rangle.$$
◀

(8) Let $\{e_k, k\geq 1\}$ be an orthonormal sequence in a Hilbert space H, and let $M = \text{span}\{e_k, k\geq 1\}$. Show that for any $x\in H$ we have that $x \in \bar{M}$ if and only if x can be represented as

$$x = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k.$$

▶ We need to show that $x\in \bar{M} \iff x = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$.

$\Rightarrow$) Let $x \in \bar{M}$. Then $\exists x_n \in M$ s.t. $x_n \to x$, $n\to\infty$.

Since $x_n \in M$, then $x_n = \sum_{k=1}^{\infty} \alpha_k^n e_k$, where $\alpha_k^n \neq 0$, $k = 1, \dots, l_n$
$\alpha_k^n = 0$, $k \geq l_n+1$.

We know that $x_n \to x$, $n\to\infty$ $\Rightarrow$ x_n is a Cauchy sequence, that is, $\|x_n - x_m\| \to 0$, $n, m\to\infty$.

$$\|x_n - x_m\| = \left\|\sum_{k=1}^{\infty}\alpha_k^n e_k - \sum_{k=1}^{\infty}\alpha_k^m e_k\right\| = \left\|\sum_{k=1}^{\infty}(\alpha_k^n - \alpha_k^m) e_k\right\| =$$

$$= \sum_{k=1}^{\infty}(\alpha_k^n - \alpha_k^m)^2 \to 0,\ n, m\to\infty.$$

For $\alpha_n = (\alpha_k^n) \in l_2$ $\quad \|\alpha_n - \alpha_m\|_{l_2}^2 = \sum_{k=1}^{\infty}(\alpha_k^n - \alpha_k^m)^2 \to 0 \Rightarrow$

$\Rightarrow \{\alpha_n\}_{n\geq 1}$ is a Cauchy sequence in l_2 (l_2 is a Hilbert space) $\Rightarrow \{\alpha_n\}_{n\geq 1}$ converges in l_2.

$\Rightarrow \exists \alpha = (\alpha_1, \dots, \alpha_k, \dots) : \alpha_n \to \alpha$, $n\to\infty$, in l_2.

$$\Rightarrow \underbrace{\sum_{k=1}^{\infty}(\alpha_k^n - \alpha_k)^2}_{=\|\alpha_n - \alpha\|_{l_2}} \to 0,\ n\to\infty.$$

Take $y = \sum_{k=1}^{\infty}\alpha_k e_k$. Since $\alpha \in l_2$, then $\|\alpha\|^2 = \sum_{k=1}^{\infty}\alpha_k^2 < \infty$

and $\sum_{k=1}^{\infty}\alpha_k e_k$ converges. (by Th. 18.9)

$$\|x_n - y\|^2 = \left\|\sum_{k=1}^{\infty}(\alpha_k^n - \alpha_k) e_k\right\|^2 = \sum_{k=1}^{\infty}(\alpha_k^n - \alpha_k)^2 \to 0 \Rightarrow$$

$\Rightarrow x_n \to y$, $n\to\infty$.

We have that $x_n \to x$, $n\to\infty$, and $x_n \to y$, $n\to\infty$

$\Rightarrow x = y = \sum_{k=1}^{\infty}\alpha_k e_k$.

Since $\sum_{k=1}^{\infty}\alpha_k e_k$ converges, then $\alpha_k = \langle x, e_k\rangle$ (by Th. 18.3)

Hence, $x = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$.

$\Leftarrow$) Let $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$.

(*) Set $x_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$. $\Rightarrow x_n \in$ $M := \text{span}\{e_k, k \geq 1\}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, e_k \rangle e_k = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k = x.$$

We have that $x_n \in M$, $x_n \to x$, $n \to \infty$ $\Rightarrow x \in \overline{M}$.

The second method of solution

$$x \in \overline{M} \Leftrightarrow x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$$

$\Rightarrow$) Let $x \in \overline{M}$.

We need to show that $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$.

Take $y = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$.

For $\forall x \in H$ the series $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ converges (by Th. 18.9)

It is enough to show that $x = y$.

First we show that $x - y \perp e_i$, $\forall i = 1, \dots, n$.

$$\langle x-y, e_i \rangle = \langle x, e_i \rangle - \langle y, e_i \rangle = \langle x, e_i \rangle - \langle \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k, e_i \rangle =$$

$$= \langle x, e_k \rangle - \sum_{k=1}^{\infty} \langle x, e_k \rangle \underbrace{\langle e_k, e_i \rangle}_{=\begin{cases} 1, & k=i \\ 0, & k \neq i \end{cases}} = \langle x, e_k \rangle - \langle x, e_k \rangle = 0.$$

$\Rightarrow x - y \perp e_i \Rightarrow x - y \perp M \Rightarrow x - y \perp \overline{M} \Rightarrow x - y \in \overline{M}^{\perp}$.

Since M is a linear spase, then $\overline{M}$ is also a linear space. Thus, $x - y \in \overline{M}$, because $x, y \in \overline{M}$.

Since $x - y \in \overline{M} \cap \overline{M}^{\perp} = \{0\}$, then $x - y = 0 \Rightarrow$

$$\Rightarrow x = y = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k.$$

$\Leftarrow$) as (*)